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THE MATHEMATICS PROGRAM IN THE COLLEGE OF THE UNIVERSITY OF CHICAGO*

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1. **Introduction.** Some three years ago I was given the opportunity to address members of the Association on the subject of the mathematics program in the College of the University of Chicago.† That report was made when the staff concerned with the program was small, when the program itself was hardly more than a year old, and when the aims of the program were only dimly seen. The staff has since grown in number from three to twelve or fourteen, the program is about to enter its fifth year, and the aims have become clearer—although they still constitute the chief battleground over which the staff struggles, week by week, in staff meetings. It is hoped that the present report on work which has been accomplished in the intervening years and on plans being made for the future, with some indication of areas of success and failure, and some information about what has been done to meet the failures, will be of interest to teachers of mathematics and perhaps even of use to those who are planning or are already engaged in similar enterprises elsewhere.

2. **The Chicago program of liberal education.** The aims of the College mathematics program at Chicago are intimately bound with those of the College program as a whole. Stated briefly, the College is engaged in presenting a four-year program of liberal education. On the assumption that the faculty is a better judge than the student of what constitutes a liberal education, the program is a required one, common for all students. And on the assumption that the liberally educated person should have learned something about each of the great areas of human thought, the student is required to show competence in the knowledge, methods, and skills relevant to the humanities (including English and a foreign language), the social sciences, the biological and physical sciences, and mathematics.

The student is admitted to the College on the basis of psychological tests and tests in reading and writing skills, to which it is hoped that there will soon be added a test in quantitative skills. He may apply for admission after two, three, or four years of high school, or after a year of college elsewhere. Once admitted, he is faced with the problem of passing fourteen six-hour comprehensive examinations covering the general areas mentioned above. From some of them he may be excused on the basis of placement tests which he takes directly following his admission; upon completing the others he is awarded the Bachelor of Arts degree. He normally prepares for four of the examinations each year, although he may take any of them at any time he feels he is adequately prepared for them.

In such a program there is no problem of designing and presenting several

* Presented at the summer meeting of the Mathematical Association of America, New Haven, Connecticut, September 2, 1947.

† Mathematics in a Liberal Education, this MONTHLY, vol. 52 (1945), pp. 132–137.

courses in mathematics at the first-year college level: one—presumably terminal—for students of the humanities, another for physical science majors, a third for engineering students, and so on. This fact appears to me to constitute a distinct advantage. For I am convinced of the desirability of a single course at the level concerned, and deplore current attempts to construct different courses for different purposes. A liberal education is presumably the kind of education everyone ought to have, regardless of the career he expects to enter. If this be true, then any mathematics course designed as an integral part of such an education should be one that every student ought to take, whether or not he plans to take further courses in mathematics. This is one of the assumptions on which the Chicago program is based, and the greater part of the remainder of this paper will consist of a description of the kind of course in mathematics which has emerged from this assumption and concomitant assumptions.

Actually, two one-year courses will be discussed. The first, Mathematics 1, is a required part of the College program. It is ordinarily taken in the first year of the College, and serves as a springboard for a three-year sequence in the natural sciences. The second, Mathematics 2, is designed for those students who wish to pursue the subject further. It is not an elective in the sense that it may be used to satisfy the requirements for the bachelor's degree: these requirements may be met only by passing the fourteen comprehensive examinations mentioned above, one of which is Mathematics 1. Students who wish to make plans for specialized work in one area or another, however, are permitted to take advanced courses for which they are qualified concurrently with the specified courses in the College program.

3. **Mathematics 1, the required general course.** What ought to go into a one-year course in mathematics to be taken by *all* students—students who, it may be assumed, have learned something of the ABC's of the subject in their two or more years of high school algebra and geometry? This question can be answered only after a prior question has been discussed: What, if anything, in mathematics is appropriate to a liberal education? I raised and attempted to answer this question in my earlier report. Rather than to repeat myself here, I should like to quote from W. L. Schaaf, who came close to my convictions in the matter ten years ago when he wrote, in the pages of this MONTHLY:*

> The distinctive feature of mathematics, one that sets it apart from all other domains of human achievement, is that it exemplifies, or, more strictly, it is, a *unique* style of thinking. . . . The uniqueness of mathematics as a mode of thought results from the following features: (1) the formulation of generalizations, (2) the method of postulational thinking, and (3) the ceaseless quest for greater rigor.

From this view of the place of mathematics in liberal education one may conclude that the student should be taught to think deductively, to know what a deductive system is, to understand the relation between an abstract deductive system and its models, or concrete interpretations, and to have some apprecia-

* Vol. 44 (1937), p. 447.

tion of what rigor is and how it may be achieved. Add to this the fact that he should learn, both for the skills themselves and for the part they will play in his later courses in science, how to understand and to deal with problems of quantity and space, and the content of a year course in mathematics appropriate to a program of liberal education becomes fairly clear: it should include at least the study of logic, algebra, and geometry. As the staff is now teaching Mathematics 1, about one fifth of the course is devoted to logic, two fifths to algebra, and two fifths to analytic geometry and trigonometry.

In the earlier versions of our course the approach to logic was Aristotelian, but we soon found—and it is not surprising that we did so—that this approach is inadequate even for the elementary mathematics we attempt to develop. We defended it at the time on the grounds that many of our students were deathly afraid of symbolism in any form and needed the reassurances to be gained by emphasizing such everyday phenomena as verbal propositions and the syllogism. Upon cautiously experimenting with fundamental ideas from modern symbolic logic, however, we discovered that, properly introduced to them, the student can take in his stride variables and constants, propositions and relations between them, propositional functions and quantifiers, sets and relations between them, undefined terms and rules of definition, and unproved propositions and rules of proof. This revelation has prompted us to push even further in a comprehensive revision of our text which is now being prepared by the staff for the academic year 1948–49.*

This excursion into the fundamental concepts of modern symbolic logic culminates in a careful study of the commutative group as an example of a deductive system. The group was chosen in preference to other possible examples, such as the theory of relations, or the propositional calculus, or certain simple geometric systems, for a number of reasons. The group is adequate in that it exemplifies almost everything which has gone before: sets, operations, relations, quantifiers, undefined terms, definitions, postulates, theorems, proofs, and models. It is simple in that the undefined terms and postulates are few in number, and the proofs easy to follow. It is meaningful in that the postulates and theorems find immediate application in the work of the course which follows. Finally its models are both numerous and significant, and offer excellent opportunities to study the relation between an abstract system and its concrete interpretations.

* In outline form, this revision will include the following topics: 1. Theory of Classes (general concept of class or set, universe, null class, subsets, sets of ordered pairs, complement, union, intersection, relations between classes, variables); 2. Relation and Function (relation as a set of ordered pairs, as a correspondence, as a table, function, unary and binary operations); 3. Propositions (general concept, negation, conjunction, disjunction, implication, equivalence); 4. Propositional Functions and Quantifiers (propositional functions of one variable, sentential conjunctions of propositional functions, universal and existential quantifiers, negation, sets and propositional functions, propositional functions of two variables, mixed quantifiers); 5. Propositional Calculus (symbolism, truth tables and universal propositions, special universal propositions); 6. Deductive systems (undefined terms, definitions, postulates and theorems, simple example of a deductive system, model of a deductive system, rules of proof, applications to simple proofs, indirect proof.)

It may appear that I have dwelt overlong on the material which occupies only the first fifth of the year. But it is primarily this material, together with its later applications, which makes Mathematics 1 unique among first-year college courses, and which the staff has found most difficult to organize and present. There is no question in our minds of its importance as groundwork for the rest of the course. For it is here that the student first learns the rudiments of scientific discourse, and without clear ideas about deductive systems he will gain little from the discussions of real numbers and analytic geometry, which are later developed as such systems.

The second part of the course has already been referred to as "algebra," although it is more algebra in the elementary textbook sense than in the strict sense, being a combination of the study of real numbers and real-valued functions. The development of real numbers, which is postulational in nature, may be described briefly as follows: the postulates for addition and multiplication, for subtraction, for division, the order postulates, and the continuity postulate are introduced successively, with pauses for assimilation of the postulates and their resulting theorems between them. It is assumed—and properly, we have found—that the student is sufficiently aware of the common properties of natural numbers, integers, and rational and real numbers to make them available for purposes of motivation and exemplification, although other models of the system at each stage of its development are also introduced and discussed. It should be apparent that this approach to real numbers offers ample opportunity to spend what time may be necessary in drill work on the elementary operations. The complex numbers are constructed from the reals (this development gives the student a view of the alternative, constructive approach to number systems) and their properties discussed briefly.

In that part of the "algebra" which is strictly analytic, the notions of real-valued variable and function are easily grasped by the student after the introduction he has had to the general concepts of variable and function in the first part of the course. Here the work continues with the construction of functional relations from given data, the use of functional notation, a variety of classifications of functions, and the solutions of equations as zeros of functions.

In the light of its experience the staff proposes to institute a number of changes in the approach to both real numbers and functions. In the first place, it seems desirable to have the student spend more time on the study of groups, and to exploit his increased power in that area by working into the real numbers through, successively, the additive group, the multiplicative group, and the field. Again, once the field has been achieved, it appears well to introduce the notion of function—integral and rational—and to say something here about the solution of inequalities, which are studied in connection with the order postulates that follow. The rest of the analytic material of the so-called "algebra" section could then be properly postponed until the study of analytic geometry.

Originally, the course next devoted a few weeks to the study of plane geometry. Analytic geometry was then developed in essentially the historical manner

as a combination of algebra and geometry. Having brought the work in logic and algebra to a satisfactorily respectable degree of rigor, however, the staff found, somewhat to its dismay, that it could not achieve that same degree of rigor in synthetic plane geometry without getting into involved and time-consuming considerations which promised to be irksome to both student and instructor. After a vain search for a solution to this problem, it was tentatively decided that the student's previous training in geometry could be counted upon for intuitive interpretations of geometric concepts and that geometry might best be constructed from scratch in the modern analytic fashion—by defining point as an ordered pair of real numbers, line as a set of points satisfying a linear equation, and so on. It is intended to explore this approach further in the proposed revision, although some members of the staff are conducting what may amount to a filibuster to convince the rest of their colleagues of the desirability of an approach through vectors. In any event, it seems unlikely that the study of analytic geometry will be carried much beyond a comprehensive treatment of the straight line and circle. In the future, as now, the remaining few weeks of the course will probably be devoted to the introduction and study of the trigonometric functions and their properties. It should be emphasized, however, that the treatment of trigonometry is one of which the analyst, and not the surveyor, would approve.

The majority of the students in the College take no mathematics beyond the course which has just been described. For such students it may be thought that the course is inadequate in its failure to include a number of topics for which various authors have argued with varying degrees of fervor. Although I cannot agree with such authors as the one who would include number theory and permutations so that the student will be able to face later in life the problems he will find, respectively, in current popular magazines and in trying to arrange his dinner guests, I am quite ready to admit the desirability of giving the terminal student some idea of such topics as the calculus and statistical analysis. The difficulty, of course, is that of finding time for acceptable developments of these topics in a year course which meets four hours per week. The tendency at Chicago is to sacrifice breadth of coverage of many topics for depth of understanding of a few fundamental ones. The choice of those which constitute Mathematics 1 is defended on the grounds that any student who has successfully fulfilled the course requirements will be able to pursue, independently and with understanding, any further topics in mathematics in which he may have a genuine but nonprofessional interest.

4. Mathematics 2, the optional continuing course. It is apparent that Chicago students who desire to continue their formal work in mathematics need some instruction designed to bridge the gap between the somewhat unconventional work of Mathematics 1 and the more advanced courses in calculus and post-calculus subjects offered by the Department of Mathematics in the (graduate) Division of the Physical Sciences. The College has accepted the responsibility of providing such continuing instruction in Mathematics 2. In terms of

both time and number of students, the staff has had less experience with this course than with Mathematics 1,* but its experience is sufficient for the formulation of some tentative conclusions about what can and cannot be done with students who have met the established prerequisite for entrance into the course, namely, a grade of middle C or higher in Mathematics 1.

Briefly described, the first third of Mathematics 2 is devoted to an extension of some of the material, initiated in Mathematics 1, of college algebra, trigonometry, and analytic geometry; and the last two thirds to the calculus and to further work in analytic geometry. This description is disarming in the sense that the topics considered are more sophisticated, and the approach more rigorous, than is ordinarily the case in courses in the subjects named. Thus, for example, not only are the postulates which characterize the real numbers reviewed, with careful considerations of such new concepts as absolute value and the related inequalities, but the integers as well are characterized, with careful considerations of alternative formulations of the principle of finite induction and a wealth of applications of the same. Again, abstract vector spaces are introduced and studied in connection with polar coordinates and the polar representation of complex numbers. The study of the calculus is conducted on a similarly mature level. Here the emphasis is on a clear understanding of and ability to work with the concepts and properties of function, limit, continuity, derivative, and definite integral. It is no small satisfaction to the staff, for example, to find that students of the age of high school seniors can be taught to understand clearly the relations between existence of a limit, continuity, and differentiability.

Although the staff hopes ultimately to have as complete a text for Mathematics 2 as it has for Mathematics 1,† it has at present only a set of notes, amounting to an extended outline, for the work of the first part of the course. Notes of this kind lend themselves easily to yearly revision, and can be expanded to textbook form after ideas concerning the most effective material to include in the course have crystallized. The remainder of the course at present consists of adaptations of material from Randolph and Kac's excellent text, *Analytic Geometry and Calculus*, the only serious drawback of which, for our purposes, is the fact that it was not written as an actual sequel to the material which precedes it in our two-year program.

5. **Conclusion.** Here, then, for whatever information, encouragement, and use it may be to teachers of mathematics, is a brief description of the present state of the mathematics program at the College of the University of Chicago. After four years, difficulties and shortcomings still plague us, and will probably continue to do so for years to come. But I am convinced that our enthusiastic staff, having already demonstrated how much more can be done with the ele-

* Mathematics 1 is now (1947–48) in its fifth year and is being taken by some 450 students; Mathematics 2 is in its second year, and is being taken by some 75 students.

† The text for Mathematics 1, prepared by the author of this paper with the assistance of other members of the College mathematics staff, is Fundamental Mathematics, Chicago, University of Chicago Bookstore, 1946 (2nd ed.)

mentary student than is ordinarily expected of him, will ultimately achieve a program which will be meaningful both in the sense of liberal education and in the sense of mathematical education. Indeed, in my first report I argued boldly for the meaningfulness of our program within the framework of a liberal education, and somewhat more timidly for its meaningfulness for the continuing student. I have since heard from a surprising number of mathematicians who feel that the program is a fine one for the continuing student, but much too difficult for the purposes of liberal education. This reaction I should like to regard both as an indication of vigor in the health of our program and as a confirmation of the thesis that one course for all students, not different courses for different groups of students, is appropriate to the level of instruction concerned.

AN ANALLAGMATIC CUBIC

C. E. NOBLE, Neenah, Wisconsin

1. Introduction. This paper is devoted to the study of the cubic

$$(1)\qquad \begin{gathered} LX(Y^2+Z^2)+MY(Z^2+X^2)+NZ(X^2+Y^2)+KXYZ=0, \quad \text{or} \\ \sum LX(Y^2+Z^2)+KXYZ=0. \end{gathered}$$

The cubic (1) is invariant under the isogonal transformation,

$$x':y':z' = \frac{1}{x}:\frac{1}{y}:\frac{1}{z},$$

a quadratic transformation of the plane, which sends the point P having trilinear coördinates x, y, z into the point P' having trilinear coördinates $1/x$, $1/y$, $1/z$ [1]. Cubics invariant under the isogonal transformation are often called anallagmatic cubics [2, 5]. Neuberg [5] stated in 1923 that he was not aware of any geometric interpretation of the cubic (1). In 1930 Goormaghtigh [3] presented a geometric interpretation of a special case of cubic (1). To the writer's knowledge, no additional information concerning this cubic has appeared in the literature. In this discussion a number of geometric interpretations of cubic (1) are presented together with many of its properties.

2. Geometric interpretations. We shall find it convenient to refer to the cubic (1) as the *plus-sign* cubic to distinguish it from the *minus-sign* cubic

$$(2)\qquad \sum LX(Y^2-Z^2)=0,$$

the other anallagmatic cubic in the triangle. The cubic (2) has been widely discussed in the literature [2, 5].

Throughout this paper, P and P' represent two isogonally conjugate points having trilinear coördinates x, y, z and $1/x$, $1/y$, $1/z$, respectively. The angles,

the lengths of the sides, and the area of the reference triangle are denoted by A, B, C; a, b, c; and S, respectively. The points of intersection of the lines AP, BP, CP and AP', BP', CP' with the opposite sides of the reference triangle are designated by the letters A_p, B_p, C_p and A_p', B_p', C_p', respectively. Generally, the coördinates x_1, y_1, z_1 of a point Q are considered to be proportional to the distances of Q from the sides of the reference triangle. If we insist that x_1, y_1, z_1 represent the actual distances of Q from the sides of the reference triangle, then x_1, y_1, z_1 satisfy the relation $2S=ax_1+by_1+cz_1$ and are called *actual coördinates.*

Several geometric interpretations of the plus-sign cubic will now be given as theorems, beginning with the special case derived by Goormaghtigh [3].

THEOREM 1. *If the areas of triangles $A_pB_pC_p$ and $A_p'B_p'C_p'$ are equal, then the locus of P and P' is a plus-sign or minus-sign cubic.*

If x, y, z are the actual coördinates of P, then the actual coördinates of A_p, A_p', and so on, are respectively $[0, 2Sy/(by+cz), 2Sz/(by+cz)]$, $[0, 2Sz/(bz+cy), 2Sy/(bz+cy)]$, and so on. The conditions that the areas of triangle $A_pB_pC_p$ and $A_p'B_p'C_p'$ be equal is that

$$(by + cz)(cz + ax)(ax + by) = \pm (bz + cy)(cx + az)(ay + bx). \tag{3}$$

If the plus sign is taken on the right, equation (3) becomes the equation of the minus-sign cubic

$$\sum a(b^2 - c^2)x(y^2 - z^2) = 0. \tag{4}$$

If the minus sign on the right is taken, the equation (3) becomes

$$\sum a(b^2 + c^2)x(y^2 + z^2) + 4abcxyz = 0, \tag{5}$$

which is the equation of the special plus-sign cubic obtained by Goormaghtigh. This special case of the plus-sign cubic appears to be the only one in the literature with the exception of the degenerate locus made up of a line and a conic which are isogonally conjugate, namely

$$(lx + my + nz)(lyz + mzx + nxy) = 0, \quad \text{or}$$
$$\sum mnx(y^2 + z^2) + (l^2 + m^2 + n^2)xyz = 0.$$

In a similar manner it may be shown that if the areas of triangles $A_pB_p'C_p'$ and $A_p'B_pC_p$, $A_p'B_pC_p'$ and $A_pB_p'C_p$, or $A_p'B_p'C_p$ and $A_pB_pC_p'$ are required to be equal, then the locus of P and P' is a plus-sign or minus-sign cubic. The locus of P and P' is also a plus-sign or minus-sign cubic if we equate the areas of triangles $\overline{A}_p\overline{B}_p'\overline{C}_p'$ and $\overline{A}_p'\overline{B}_p\overline{C}_p$, $\overline{A}_p'\overline{B}_p\overline{C}_p'$ and $\overline{A}_p\overline{B}_p'\overline{C}_p$, or $\overline{A}_p'\overline{B}_p'\overline{C}_p$ and $\overline{A}_p\overline{B}_p\overline{C}_p'$, where $\overline{A}_p$ and $\overline{A}_p'$ are the harmonic conjugates of A_p and A_p' with respect to B and C, and $\overline{B}_p$, $\overline{B}_p'$, $\overline{C}_p$ and $\overline{C}_p'$ are analogous points.

THEOREM 2. *If the six point conic $A_pB_pC_pA_p'B_p'C_p'$ passes through a seventh point, then the locus of P and P' is a plus-sign cubic.*

Through the six points A_p, B_p, C_p, A_p', B_p', C_p' passes a conic whose equation is

$$(6)\qquad \sum x(y^2+z^2)YZ - xyz(X^2+Y^2+Z^2) = 0,$$

where X, Y, Z are the running coördinates. If this system of conics is required to pass through a fixed point, say $Q(X_1, Y_1, Z_1)$, then the locus of the points P and P' is the plus-sign cubic

$$(7)\qquad \sum Y_1Z_1x(y^2+z^2) - (X_1^2+Y_1^2+Z_1^2)xyz = 0.$$

It is interesting to observe that if the cubic (7) is forced to pass through the incenter $(1, 1, 1)$ of triangle ABC, then the fixed point Q must lie on the inscribed conic

$$X^2+Y^2+Z^2-2YZ-2ZX-2XY = 0.$$

Similarly, if the cubic (7) passes through an excenter, say $(1, 1, -1)$, then the point Q must lie on the inscribed conic

$$X^2+Y^2+Z^2+2YZ+2ZX-2XY = 0.$$

THEOREM 3. *If the six point conic $A_pB_pC_pA_p'B_p'C_p'$ be a rectangular hyperbola, then the locus of P and P' is a plus-sign cubic.*

The condition on the coefficients of the conic (6) in order that it be a rectangular hyperbola is

$$(8)\qquad \sum \cos A\, x(y^2+z^2) + 3xyz = 0,$$

which shows that the locus of P and P' is then a plus-sign cubic.

Similarly, a plus-sign cubic is obtained as the locus of P and P' if the six point conic $\overline{A}_p\overline{A}_p'B_pB_p'C_pC_p'$ (or $A_pA_p'\overline{B}_p\overline{B}_p'C_pC_p'$, or $A_pA_p'B_pB_p'\overline{C}_p\overline{C}_p'$) is required either to pass through a seventh point or to be a rectangular hyperbola.

The conic passing through $\overline{A}_p$, $\overline{A}_p'$, $\overline{B}_p$, $\overline{B}_p'$, $\overline{C}_p$, $\overline{C}_p'$ degenerates into the trilinear polars of P and P' with respect to the triangle ABC. These trilinear polars are, respectively,

$$yzX+zxY+xyZ = 0 \quad \text{and} \quad xX+yY+zZ = 0.$$

If these two lines are perpendicular, then the locus of the points P and P' is the plus-sign cubic

$$(9)\qquad \sum \cos A\, x(y^2+z^2) - 3xyz = 0.$$

If these two lines meet in a point on the line

$$(10)\qquad LX+MY+NZ = 0,$$

the locus of P and P' is the minus-sign cubic

$$(2)\qquad \sum Lx(y^2-z^2) = 0.$$

If the line (10) is the infinite line, we have the condition that the trilinear polars of P and P' be parallel, in which case the locus of P and P' is

$$\sum \sin Ax(y^2 - z^2) = 0. \tag{11}$$

THEOREM 4. *The locus of the foci of inscribed conics tangent to a given line is a plus-sign cubic.*

Casey [1] shows that the equation of a conic inscribed in triangle ABC with the point $F(x, y, z)$ as focus may be expressed in the form

$$\sin A_1(X/x)^{1/2} + \sin B_1(Y/y)^{1/2} + \sin C_1(Z/z)^{1/2} = 0, \tag{12}$$

where A_1, B_1, C_1 are respectively the angles subtended from the point F by the sides BC, CA, AB of triangle ABC.

Since $\sin A_1 = x2S/[(x^2+z^2+2xz \cos B)(x^2+y^2+2xy \cos C)]^{1/2}2R$, where R is the circumradius of ABC, and $\sin B_1$ and $\sin C_1$ are equal to similar quantities, equation (12) may be written as

$$\sum [x(y^2 + z^2 + 2yz \cos A)X]^{1/2} = 0. \tag{13}$$

The condition that the inscribed conic (13) be tangent to the line

$$\frac{X}{L} + \frac{Y}{M} + \frac{Z}{N} = 0 \tag{14}$$

is that

$$\sum Lx(y^2 + z^2) + 2(L \cos A + M \cos B + N \cos C)xyz = 0. \tag{15}$$

Thus, the locus of the focus F is a plus-sign cubic. Since the foci of inscribed conics are isogonally conjugate points, the other focus $F'(1/x, 1/y, 1/z)$ of the conic (13) also lies on the cubic (15).

THEOREM 5. *The locus of the foci of inscribed conics whose conjugate axes are of constant length is a plus-sign cubic.*

If the actual coördinates of the focus F of a conic inscribed in triangle ABC are x, y, z, then the actual coördinates of the other focus F' are $(2Syz/U, 2Szx/U, 2Sxy/U)$, where $U = ayz + bzx + cxy$. It is generally known that the square of the semi-conjugate axis of a conic is equal to the product of the distances of the foci of the conic to any line tangent to the conic [6]. Taking one of the sides of triangle ABC as the tangent to the inscribed conic and $1/k$ as the square of the semi-conjugate axis, we obtain the relation

$$xyz2S/(ayz + bzx + cxy) = 1/k. \tag{16}$$

Introducing $(ax+by+cz)/2S$ into equation (16) in order to make it homogeneous, we get the equation of the locus of the foci of a family of inscribed conics whose conjugate axes are constant to be

$$\sum bcx(y^2 + z^2) + (a^2 + b^2 + c^2 - 4S^2k)xyz = 0. \tag{17}$$

THEOREM 6. *The locus of the foci of inscribed conics whose centers lie on a given line is a plus-sign cubic.*

If the coördinates of the foci F and F' of an inscribed conic are the same as given above, then the coördinates of the midpoint F'' of the line segment FF' are x_1, y_1, z_1, where

$$x_1 = \frac{2axyz + x^2(bz + cy) + yz(by + cz)}{2U}.$$

If we require F'', which is the center of the inscribed conic having the points F and F' as foci, to lie on the line

$$LX + MY + NZ = 0,$$

then the foci F and F' travel over the locus

$$\sum (cM + bN)x(y^2 + z^2) + 2(aL + bM + cN)xyz = 0. \tag{18}$$

If the center F'' is required to lie on the line

$$\begin{aligned} Xa(-aL + bM + cN) + Yb(aL - bM + cN) \\ + Zc(aL + bM - cN) = 0, \end{aligned} \tag{19}$$

then the locus of the foci F and F' is the cubic (15). Thus, when an inscribed conic remains tangent to the line (14), the locus of its foci is the cubic (15), and the locus of its center is the line (19).

THEOREM 7. *If the locus of the point of intersection of the tangents at P and P' to the circumconic passing through P and P' be a straight line, then the locus of P and P' is a plus-sign cubic.*

The equation of the circumconic passing through the points $P(x, y, z)$ and $P'(1/x, 1/y, 1/z)$ is

$$\sum x(y^2 - z^2)YZ = 0.$$

The tangents to this conic at the points P and P' meet in the point T whose coördinates are $x(y^2+z^2)$, $y(z^2+x^2)$, $z(x^2+y^2)$. If T lies on the line

$$LX + MY + NZ = 0,$$

then P and P' lie on a plus-sign cubic whose equation is

$$\sum Lx(y^2 + z^2) = 0. \tag{20}$$

THEOREM 8. *If P and P' are conjugate points with respect to a given conic, the locus of P and P' is a plus-sign cubic.*

The condition that the point $P'(1/x, 1/y, 1/z)$ lie on the first polar of the point $P(x, y, z)$ with respect to the general conic

$$DX^2 + EY^2 + FZ^2 + 2LYZ + 2MZX + 2NXY = 0$$

is the plus-sign cubic

$$\sum Lx(y^2 + z^2) + Kxyz = 0, \tag{1}$$

where $K = D + E + F$.

3. Other properties. It is readily seen that the equation (1) is invariant under the isogonal transformation. The cubic curve represented by equation (1) passes through the vertices of the reference triangle ABC and meets the sides of triangle ABC in the points in which the line

$$\frac{X}{L} + \frac{Y}{M} + \frac{Z}{N} = 0 \tag{21}$$

cuts these sides. Since this cubic is invariant under the isogonal transformation, it follows that it is tangent to the isogonal conjugate of line (21) at the vertices; that is, it is tangent at the vertices to the circumconic

$$\frac{YZ}{L} + \frac{ZX}{M} + \frac{XY}{N} = 0.$$

In fact, the cubic is tangent at the vertices to any curve whose isogonal conjugate passes through the points in which the line (21) cuts the sides of triangle ABC.

The polars of $P(x, y, z)$ with respect to the conics

$$\begin{gathered} KX^2 + 2LYZ + 2MZX + 2NXY = 0, \\ X^2 - Y^2 = 0, \qquad Y^2 - Z^2 = 0 \end{gathered} \tag{22}$$

are, respectively,

$$\begin{gathered} X(Kx + Mz + Ny) + Y(Lz + Nx) + Z(Ly + Mx) = 0, \\ Xx - Yy = 0, \qquad Yy - Zz = 0. \end{gathered} \tag{23}$$

These three polars meet in a point, say $\overline{P}$, if

$$\sum Lx(y^2 + z^2) + Kxyz = 0. \tag{1}$$

The cubic (1) is called the Jacobian of the three conics (22) [4]. It may readily be shown that the cubic (1) is the Jacobian of any three conics, not passing through the same four points, of the family

$$KX^2 + 2LYZ + 2MZX + 2NXY + \mu(X^2 - Y^2) + \nu(Y^2 - Z^2) = 0. \tag{24}$$

The coördinates of $\overline{P}$, the point of intersection of the polars (23), are $1/x$, $1/y$, $1/z$. Hence, $\overline{P}$ lies on the Jacobian (1) and is identical with P'. The points P and $\overline{P}$ are called conjugate points on the Jacobian. Thus, the isogonally conjugate points P and P' on the cubic (1) are conjugate points on this cubic considered as the Jacobian of three conics of the family (24). This property of the cubic leads to the following theorem.

THEOREM 9. *If the points P and P', Q and Q' are pairs of isogonally conjugate points on the cubic* (1), *then the points of intersection of the lines PQ and $P'Q'$, PQ' and $P'Q$ are isogonally conjugate points on this cubic.*

This theorem results from the fact that if two pairs of opposite vertices, P and P', Q and Q', of a quadrilateral are conjugate points with respect to a conic, then the third pair, R and R', are also conjugate points with respect to the conic [4]. Hence, if we are given three pairs of isogonally conjugate points on the cubic (1), we may, in general, construct an infinite number of isogonally conjugate points on the cubic by use of the ruler only.

If we allow the point Q to approach the point P along the cubic, then the point Q' approaches the point P' along the cubic. In the limit the following result is obtained.

THEOREM 10. *The tangents to the cubic* (1) *at the two isogonally conjugate points P and P' on the cubic meet in a point R which lies on the cubic, and the line PP' meets the cubic in the point R', the isogonal conjugate of R.*

We may easily show that the circumconic which passes through any two isogonally conjugate points P and P' on the cubic (1) meets the cubic a sixth time in the point R which is the point of intersection of the tangents to the cubic at the points P and P'.

In general, the cubic (1) is non-singular. However, it has a double point at the incenter of triangle ABC when it passes through this point. For, if the cubic (1) passes through the incenter, it is necessary that $K+2L+2M+2N=0$. This is the condition that the cubic have a double point at the incenter. Similarly, if cubic (1) passes through any one of the excenters of triangle ABC, it possesses a double point at that particular excenter.

For the special case in which $|L|=|M|=|N|$ in equation (1), we see that $(0, M, -N)$, $(L, 0, -N)$, $(L, -M, 0)$ are inflection points of (1). No generality is lost by taking $L=M=N=1$. For this case, the polar conic of the inflection point $(1, -1, 0)$ is

$$X^2 - Y^2 + Z(X - Y)(K - 2) = 0.$$

This is a degenerate conic made up of the tangent to the cubic at the inflection point and the harmonic polar of the inflection point with respect to the cubic. Their equations are, respectively,

$$X + Y + Z(K - 2) = 0 \quad \text{and} \quad X - Y = 0.$$

Analogous results may be produced for the inflection points $(0, 1, -1)$ and $(1, 0, -1)$.

If the entire configuration is subjected to the isogonal transformation, the cubic and the harmonic polars remain invariant. However, the tangent lines transform into circumscribed conics each of which meets the cubic four times at a vertex. This follows from the fact that the tangents to the cubic at the inflection points each meet the cubic three times at an inflection point which

lies on the sides of triangle ABC. If $K=0$ in (1), then the centers of the degenerate conics mentioned above are located at the incenter of triangle ABC, and each of the circumscribed conics which meet (1) four times at a vertex passes through the incenter.

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WHAT ARE SET FUNCTIONS?

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1. **Point functions and set functions.** In speaking of a function, one usually thinks of a *point function* $f(x)$: A rule is given by which, in a certain domain, to each number x (or, more generally, to each point x of a certain space) there correspond numbers $f(x)$. If in this statement we replace x by a *set* X of numbers or points x, we come immediately to the more general notion of a *set function* $f(X)$; that is: A rule is given by which to each set X, belonging to a certain family of sets of a space, there correspond numbers $f(X)$. In the most important case of a single-valued set function, to each set X only one number $f(X)$ is attached.

Particular instances of set functions have been well known to mathematicians for a long time, even since antiquity. We have only to remember the length of an interval or of an arc, the area of a plane region or of a surface, the volume of a solid, the content or the measure of a set. Of course, the definite integral is also a very important example of a set function. In fact, the integral $\int_a^b f(x)\,dx$ is an interval function; to be specific, it is a function of the interval $[a, b]$. In the case of multiple integrals one integrates over regions. The idea of more general integration over (measurable) sets is due to H. Lebesgue [1]. In fact, it is Lebesgue [2] who may be considered as the founder of the theory of set functions; since his original studies the theory has been further developed by many other mathematicians. We now know that such a general theory is interesting and useful, furnishing a solid, unifying basis for more special theories, and is a valuable source for many applications. A systematic presentation and

discussion of set functions will be found in the forthcoming book of H. Hahn and A. Rosenthal [3].

Further generalizations of basic notions are possible, as for the range of values of set functions. Instead of numbers one can take points of a space (for instance, of Banach spaces) or, even more generally, sets as values of the function. In the case of this last generalization, where sets correspond to sets, one arrives at the notion of mapping one space into another space. We shall not go so far here, but shall consider only single-valued set functions whose range of values consists of real numbers (including $+\infty$ and $-\infty$).

The notion of set functions becomes fruitful by the introduction of some specific properties, such as additivity.

2. **Additive set functions.** Let $\mathfrak{M}$ be a system of sets in a quite general space, and let the set function $\phi(X)$ be defined for each set X that is a member of $\mathfrak{M}$, denoted symbolically by $X\epsilon\mathfrak{M}$. From the first we shall exclude the case that $\phi(X)=+\infty$ for *all* $X\epsilon\mathfrak{M}$ or that $\phi(X)=-\infty$ for *all* $X\epsilon\mathfrak{M}$. Then ϕ is called *additive* in $\mathfrak{M}$ if for any two *disjoint* sets $A\epsilon\mathfrak{M}$ and $B\epsilon\mathfrak{M}$, for which also their sum $(A+B)\epsilon\mathfrak{M}$, we have

$$\phi(A+B)=\phi(A)+\phi(B). \tag{1}$$

Here $\phi(A)$ and $\phi(B)$ cannot have infinite values of different signs, since otherwise the right side of (1) would be meaningless.

EXAMPLES: Let $\mathfrak{M}$ be the system of all subsets of a given infinite set E.

(*a*) For every $A\epsilon\mathfrak{M}$ let $\phi(A)$ be the number of elements of A. (In particular, let $\phi(A)=+\infty$ if A is an infinite set.)

(*b*) For every $A\epsilon\mathfrak{M}$ let $\phi(A)$ be one plus the number of elements of A. (In particular, let $\phi(A)=+\infty$ again if A is an infinite set.)

(*c*) Let $\phi(A)=0$ if A is finite, and $\phi(A)=+\infty$ if A is infinite.

Then ϕ is additive in the Examples (*a*) and (*c*), while ϕ is non-additive in the Example (*b*).

Let ϕ be additive in $\mathfrak{M}$, and let all sets discussed belong to $\mathfrak{M}$. In studying ϕ it will be suitable to assume that $\mathfrak{M}$ is a *field;* that is, with any two sets $A\epsilon\mathfrak{M}$ and $B\epsilon\mathfrak{M}$, their sum $A+B$ and their difference $A-B$ also belong to $\mathfrak{M}$. It can easily be seen also that their intersection $A\cdot B$ belongs to $\mathfrak{M}$.

Of course, by induction, we obtain immediately from (1):

$$\phi(A_1+A_2+\cdots+A_n)=\phi(A_1)+\phi(A_2)+\cdots+\phi(A_n),$$

provided that the sets A_i, $(i=1, 2, \cdots, n)$, are disjoint.

For two sets which are not necessarily disjoint one obtains:

$$\phi(A)+\phi(B)=\phi(A+B)+\phi(A\cdot B). \tag{2}$$

The demonstration of this follows from the fact that $A=A\cdot B+(A-B)$ and $B=A\cdot B+(B-A)$; thus $\phi(A)=\phi(A\cdot B)+\phi(A-B)$ and $\phi(B)=\phi(A\cdot B)+\phi(B-A)$. Hence, by addition, $\phi(A)+\phi(B)=\phi(A\cdot B)+[\phi(A\cdot B)+\phi(A-B)+\phi(B-A)]=\phi(A\cdot B)+\phi(A+B)$.

3. **Totally additive set functions.** The set function ϕ, additive in $\mathfrak{M}$, is called *totally additive* (or *completely additive*) in $\mathfrak{M}$ if for every sequence of *disjoint* sets $A_\nu \epsilon \mathfrak{M} (\nu = 1, 2, \cdots)$, for which also their sum $\underset{\nu}{S} A_\nu \epsilon \mathfrak{M}$, we have

$$\phi(\underset{\nu}{S} A_\nu) = \sum_\nu \phi(A_\nu). \tag{3}$$

In Example (a) of §2, ϕ is a totally additive set function. But, in Example (c), ϕ is an additive set function which is *not* totally additive; for if $B = \underset{\nu}{S} b_\nu$ is any denumerable subset of E, we have $\phi(b_\nu) = 0$, but $\phi(B) = +\infty$.

More significant examples are the Lebesgue measure, which is totally additive in the system of measurable sets, and the Jordan content which is additive, but not totally additive, in the system of the sets measurable in the sense of Jordan.

For the study of totally additive set functions, it is natural to assume $\mathfrak{M}$ to be a *σ-field*, that is, a field which along with each sequence of sets $\{A_\nu\}$ also contains their sum $\underset{\nu}{S} A_\nu$. Now let ϕ be totally additive in the σ-field $\mathfrak{M}$, and let all sets discussed belong to $\mathfrak{M}$. Then the following theorems can easily be proved:

THEOREM (a). Let $\{A_\nu\}$ be a monotone increasing sequence of sets (that is $A_{\nu_1} \subseteq A_{\nu_2}$ for $\nu_1 \leqq \nu_2$), then $\phi(\underset{\nu}{S} A_\nu) = \lim_\nu \phi(A_\nu)$.

THEOREM (b). Let $\{A_\nu\}$ be a monotone decreasing sequence of sets (that is, $A_{\nu_1} \supseteq A_{\nu_2}$ for $\nu_1 \leqq \nu_2$). If not all $\phi(A_\nu)$ are infinite, then $\phi(\underset{\nu}{D} A_\nu) = \lim_\nu \phi(A_\nu)$, where $\underset{\nu}{D} A_\nu$ designates the intersection of the sequence $\{A_\nu\}$.

THEOREM (c). If ϕ is *monotone increasing* (that is, $A \subseteq B$ implies $\phi(A) \leqq \phi(B)$), then we have for every sequence $\{A_\nu\}$ of sets

$$\phi(\underset{\nu}{S} A_\nu) \leqq \sum_\nu \phi(A_\nu). \tag{4}$$

The following important result is due to H. Hahn [**4**]:

THEOREM (d). The set function ϕ attains the absolute maximum and the absolute minimum of its values.

From (d) it follows immediately that ϕ is bounded either above or below (since ϕ cannot attain both the values $+\infty$ and $-\infty$); and, moreover, if ϕ is finite, then ϕ is also bounded.

We can reduce ϕ to monotone increasing functions by defining the *positive-function* ϕ^+ and the *negative-function* ϕ^- of ϕ in the following way.

For $A \epsilon \mathfrak{M}$, we set

$$\phi^+(A) = \sup \phi(X) \quad \text{and} \quad \phi^-(A) = -\inf \phi(X), \tag{5}$$

where X designates those subsets of A which belong to $\mathfrak{M}$. Then we have

$$\phi = \phi^+ - \phi^-. \tag{6}$$

Besides, the *absolute-function* $\bar{\Phi}$ is defined by

$$\bar{\Phi} = \phi^+ + \phi^-. \tag{7}$$

The functions ϕ^+, ϕ^-, $\bar{\phi}$ are monotone increasing and, with ϕ, they are also totally additive. The reduction to monotone increasing functions, given by (6), is very useful for the proofs of many theorems.

4. **Zero-sets for ϕ.** A set $A \in \mathfrak{M}$ is called a *zero-set* for ϕ if $\phi(X) = 0$ for all subsets X of A which belong to $\mathfrak{M}$. If ϕ is totally additive in the σ-field $\mathfrak{M}$, then the system of all zero-sets for ϕ is also a σ-field.

$\mathfrak{M}$ is called *complete* for ϕ if every subset of each zero-set for ϕ is contained in $\mathfrak{M}$.

A given field (or σ-field) can be extended into a complete field (or σ-field). More specifically, if ϕ is an additive set function in the field $\mathfrak{M}$, then there is a field $\mathfrak{M}^0 \supseteq \mathfrak{M}$ and an extension of ϕ into a set function ϕ^0 which is additive in $\mathfrak{M}^0$ such that $\mathfrak{M}^0$ is complete for ϕ^0. In this statement the words "field" and "additive" can be replaced by the words "σ-field" and "totally additive."

5. **ψ-continuous set functions.** Let ϕ and ψ be two totally additive set functions in the σ-field $\mathfrak{M}$. Then ϕ is called *ψ-continuous* (or *totally continuous with respect to* ψ) if for every zero-set X for ψ we have also $\phi(X) = 0$; or, what amounts to the same thing, if every zero-set X for ψ is also a zero-set for ϕ. On the other hand, ϕ is called *purely ψ-discontinuous* if every set $M \in \mathfrak{M}$, with $\phi(M) \neq 0$, contains a zero-set X for ψ with $\phi(X) \neq 0$.

Example of a ψ-continuous function: Let μ be the one-dimensional Lebesgue measure. Then the Lebesgue integral $(M)\int f(x)dx$ (defined for the one-dimensional measurable sets M) is μ-continuous.

Example of a purely ψ-discontinuous function: The function ϕ defined in Example (a) of §2 is purely μ-discontinuous, if in the example E designates a linear infinite set and μ is again the one-dimensional Lebesgue measure.

It has been proved that a totally additive set function ϕ can be represented as the sum of a ψ-continuous and a purely ψ-discontinuous set function (both of which are also totally additive in $\mathfrak{M}$). If ϕ is finite, then there is only one such representation.

For the notion of the ψ-continuous set functions, the zero-sets for ψ play a distinguished role. A more general theory, also containing other interesting particular cases, can be developed if the sets S of any σ-field $\mathfrak{S} \subseteq \mathfrak{M}$ are distinguished as "singular sets," provided that every $X \in \mathfrak{M}$ which is a subset of S belongs also to $\mathfrak{S}$.

6. **Measure.** The general theory of measure, which contains the Lebesgue measure as a particular case, was founded by C. Carathéodory [5]. Let E be a given set, and consider the system $\mathfrak{E}$ of all subsets A of E. A set function $\phi(A)$, defined in $\mathfrak{E}$, is called a *measure function* if the following conditions are satisfied:

(a) For the empty set Λ we have: $\phi(\Lambda) = 0$.

(b) $\phi(A)$ is monotone increasing (*cf.* §3, (c)).

(c) $\phi(\underset{\nu}{S} A_\nu) \leq \sum_\nu \phi(A_\nu)$.

Because of (*a*) and (*b*), we always have: $\phi(A) \geqq 0$. The essential inequality (*c*) is the same as used above in (4).

A set $M \epsilon \mathfrak{S}$ is called *ϕ-measurable* if for M, together with *every* set $A \epsilon \mathfrak{S}$, we have

$$\phi(A) = \phi(A \cdot M) + \phi(A - M). \tag{8}$$

The ϕ-measurable sets M form a σ-field $\mathfrak{M}$, and ϕ is totally additive in $\mathfrak{M}$. This gives the relation between the measure functions (which, in general, are not additive) and the totally additive set functions.

7. **Integration.** The integral of a point function $f(x)$ can be defined with respect to any totally additive set function ϕ. Of course, the model for a general theory of integration is the Lebesgue integral. H. Lebesgue [1] performed the integration with respect to the Lebesgue measure in an n-dimensional Euclidean space R_n. Generalizing the Lebesgue integral and, simultaneously, also the Stieltjes integral, J. Radon [6] replaced the Lebesgue measure by an arbitrary totally additive set function. He still took the space R_n as the basis; but then M. Fréchet [7], referring to J. Radon, easily generalized his definition for abstract spaces.

Subsequently, such definitions and theories of integration have been developed in many ways by quite a few mathematicians. Perhaps the most simple and most fascinating definition of general integration is due to H. Hahn [8], and it has been extensively discussed in the above-mentioned book [3].

Let $\mathfrak{M}$ be a σ-field and let ϕ be a totally additive set function in $\mathfrak{M}$. According to §4 it is no restriction to assume $\mathfrak{M}$ to be complete for ϕ, since $\mathfrak{M}$ may be replaced by the extended σ-field $\mathfrak{M}^0$, if it is necessary to do so. Let $A \epsilon \mathfrak{M}$; then the subsets $M \epsilon \mathfrak{M}$ of A form also a σ-field, say $\mathfrak{A}$. Moreover, let $f(x)$ be a point function defined on A.

The measurability of $f(x)$ on A can be defined in the same way as in Lebesgue's theory; that is, for every number y, the set of those points $x \epsilon A$ at which $f(x) > y$ has to belong to $\mathfrak{M}$.

Now let $f(x)$ be measurable on A and let $\phi(A)$ be finite. The function f is called *ϕ-integrable* on A if there is a set function $\lambda(M)$, which afterwards is said to be the *ϕ-integral* of f, defined in $\mathfrak{A}$ and satisfying the following two conditions:

(1) $\lambda(M)$ is totally additive in $\mathfrak{A}$.

(2) $\lambda(M)$ has the following mean-value property:

Let $M \epsilon \mathfrak{A}$ and $c' \leqq f(x) \leqq c''$ for all $x \epsilon M$; then $c'\phi(M) \leqq \lambda(M) \leqq c''\phi(M)$ if $\phi^-(M) = 0$, (that is, if ϕ is monotone increasing on M). $c''\phi(M) \leqq \lambda(M) \leqq c'\phi(M)$ if $\phi^+(M) = 0$, (that is, if ϕ is monotone decreasing on M). [If herein $c = \pm \infty$ and $\phi(M) = 0$, one has to set $c\phi(M) = 0$.]

If f is ϕ-integrable on A, then it can be proved that the value $\lambda(A)$ is uniquely determined by (1) and (2), and hence it can be designated by

$$\lambda(A) = (A) \int f d\phi. \tag{9}$$

On the basis of this definition the whole theory of integration can be developed, including the representation of the integral by means of Lebesgue sums (which H. Lebesgue [1] had used as the definition of his integral in the particular case that ϕ designates the Lebesgue measure in R_n). Moreover, the Radon-Nikodym theorem [6, 9] can be obtained, which gives the following characterization of the set functions that are ϕ-integrals: In order that a set function $\lambda(M)$ be a ϕ-integral, it is necessary and sufficient that $\lambda(M)$ be totally additive and ϕ-continuous.

8. Differentiation. The theory of differentiation of set functions, founded also by H. Lebesgue [2], encounters more difficulties.

Let E be a metric space, and let a be any point of E; we designate by $S_{a\rho}$ the sphere with center a and radius ρ. Let ϕ and ψ be totally additive set functions in a σ-field $\mathfrak{M}$ of subsets of E. Moreover, let $\mathfrak{M}$ satisfy the following condition: to every $\rho > 0$ there is a (non-empty) set of $\mathfrak{M}$, contained in the sphere $S_{a\rho}$. We say the sequence $\{M_\nu\}$ of sets of $\mathfrak{M}$ converges to a if there is a sequence of positive numbers $\{\rho_\nu\}$ with $\rho_\nu \to 0$, such that $M_\nu \subseteq S_{a\rho_\nu}$.

If there is a sequence $\{M_\nu\}$ converging to a, such that the sequence

$$\left\{\frac{\phi(M_\nu)}{\psi(M_\nu)}\right\}$$

converges to the value d, then d is called a *derivate* of ϕ with respect to ψ at the point a on $\mathfrak{M}$. The greatest and smallest of such values give the *upper* and *lower derivate*, respectively, and, if both coincide, we have the *derivative* of ϕ with respect to ψ at the point a on $\mathfrak{M}$.

But these definitions are still too general and would not furnish a satisfactory theory of differentiation without further restrictions on the systems of sets applied. This was first observed by H. Lebesgue [2] who, therefore, operating in the Euclidean space R_n and using the n-dimensional Lebesgue measure μ for ψ, introduced the notion of *regularity* of sets. A family $\mathfrak{M}$ of measurable sets M was called *regular* by him if for every point a there exists a number $\zeta(a) > 0$ with the following property: Let $M_\nu \in \mathfrak{M}$, $(\nu = 1, 2, \cdots)$, be any sequence of sets converging to a and let S_ν be the smallest n-dimensional sphere containing M_ν. Then the following condition must be satisfied:

$$\frac{\mu(M_\nu)}{\mu(S_\nu)} \geqq \zeta(a).$$

In the theory of differentiation, developed by H. Lebesgue with the aid of the notion of regularity, the Vitali covering theorem played an essential role. Now in different ways, conditions formed after the pattern of the Vitali covering theorem can be stated, such that a general and satisfactory theory of differentiation of ϕ with respect to ψ is possible. It includes theorems of the following type. The derivative of ϕ with respect to ψ exists almost everywhere. The ψ-integral of this derivative, or of any derivate, furnishes the ψ-continuous part of

the primitive function ϕ. Since the theory of differentiation of set functions is rather complicated, we do not go into further details here. Those interested in this subject may be referred to the discussion in Chapter V of the above-mentioned forthcoming book [3].

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MATHEMATICAL NOTES

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Material for this department should be sent directly to E. F. Beckenbach, University of California, Los Angeles 24, California.

SOME INTERESTING SQUARES

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I was about to send to the MONTHLY as a problem the following innocent looking proposal:

PROBLEM: Find squares of four or more digits (of which the last one is not zero), so that their last three digits, when reversed, form a cube.

The solution of this problem has led me to find a family of squares which have some remarkable properties that, in my opinion, deserve communication in this note.

Let $A^2 = \cdots abc$ be the required squares and let $cba = d^3$. As $c \neq 0$, d^3 can only take the five values

$$125, \quad 216, \quad 343, \quad 512, \quad 729.$$

Then A^2 is one of

$$\cdots 521, \cdots 612, \cdots 343, \cdots 215, \cdots 927,$$

where the dots stand for other digits.

As squares cannot end in the digits 2, 3 or 7, the second, third and fifth possibilities are immediately eliminated.

Every square in 5 is of the type

$$(10x + 5)^2 = 100x^2 + 100x + 25 = 100x(x + 1) + 25.$$

Consequently every square which ends in 5 must also end in 25, and this eliminates the fourth possibility.

Therefore the squares that we are seeking must be of the type

$$A^2 = 1000y + 521,$$

and these are the squares which have the interesting properties that motivate this note. They could be considered independently of the above problem, but the problem is inserted to show how I was led to treat them.

There is an infinite number of these squares, the first few being:

$39^2 = 1\ 521,$	$461^2 = 212\ 521,$	$789^2 = 622\ 521,$
$211^2 = 44\ 521,$	$539^2 = 290\ 521,$	$961^2 = 923\ 521,$
$289^2 = 83\ 521,$	$711^2 = 505\ 521,$	$1\ 039^2 = 1\ 079\ 521.$

Observe that A is the sum at any point of the series of summands

$$A = 39 + 172 + 78 + 172 + 78 + 172 + 78 + \cdots,$$

where, after 39, the numbers $172 = 4 \cdot 43$ and $78 = 2 \cdot 39$ alternate as summands. As a matter of fact this series may also be extended towards the left into negative numbers

$$A = 39 - 78 - 172 - 78 - 172 - \cdots,$$

whose squares are positive and all end in the prime number 521.

Note also that in $A^2 = 1000y + 521$, y is again the sum at any point of the series of summands

$$y = 1 + 43 + 39 + 3 \cdot 43 + 2 \cdot 39 + 5 \cdot 43 + 3 \cdot 39 \\ + 7 \cdot 43 + 4 \cdot 39 + 9 \cdot 43 + 5 \cdot 39 + \cdots,$$

where each successive summand after 1 is alternately 43 multiplied by the consecutive odd numbers $1, 3, 5, \cdots$, and 39 multiplied by the consecutive numbers $1, 2, 3, \cdots$.

Hence as

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

and

$$1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}$$

are respectively the successive squares and triangular numbers, we can express

y in the form

$$y = 1 + 43n^2 + 39\,\frac{n(n \pm 1)}{2},$$

where n can be any non-negative integer.

Therefore A^2 is of the general type

$$\begin{aligned} A^2 &= 500(125n^2 \pm 39n + 2) + 521 \\ &= 62500n^2 \pm 19500n + 1521 \\ &= (250n \pm 39)^2. \end{aligned}$$

CLASSROOM NOTES

Edited by C. B. Allendoerfer, Haverford College

All material for this department should be sent directly to C. B. Allendoerfer, Haverford College, Haverford, Pennsylvania.

Editorial Note: The following papers were among those presented before the twenty-ninth Summer Meeting of the Association, New Haven, Conn., September 1, 1947, under the general title: "How to Solve It," a Symposium on Mathematical Problems at the College Level.

PROBLEMS IN MECHANICS

J. L. Synge, Carnegie Institute of Technology

Problem I: *Practical—Find a number by a standard procedure.* A pendulum consists of two equal uniform bars AB, BC smoothly jointed at B and suspended from A. The mass of each bar is m, and its length is $2a$. Find the normal periods for small oscillations in a vertical plane under gravity, in the form $2\pi k(a/g)^{1/2}$, where k is a numerical constant.

Problem II: *Aesthetic—Existence or non-existence of something simple.* For a rigid body, in general motion, show that there is no point at rest. Show also that, in general, there is one point, and only one, with no acceleration.

Problem III: *Puzzle—We won't let it get away unsolved.* For a system of particles, prove that the kinetic energy of motion relative to the mass center may be expressed in the form

$$T = \frac{1}{2m}\,\Sigma m_i m_j V_{ij}^2$$

where m is the total mass of the system, m_i the mass of a typical particle, V_{ij} the velocity of m_i relative to m_j, and the summation contains one term for each pair of particles.

In considering, attacking, and solving a problem, we might note the following things:

1. Motivation—why do it?
2. Classification—what is it?
3. Plan
4. Details
5. Mistakes

Let us get *classification* out of the way first, because it is the easiest to deal with. All our three problems belong to *mechanics*. The student will recognize at once that they all belong to *dynamics* and not to *statics*, but he should draw finer distinctions. He should recognize that the first problem belongs to dynamics proper (kinetics), or, more precisely, to the theory of vibrations. The other two belong to *kinematics*—the idea of force plays no part in them.

The question of motivation (to use the jargon of the psychologists) is very important. Why does anyone feel an inclination to solve problems such as these? (I leave out such sordid motives as solving problems for the sake of obtaining a good grade—we are thinking on a higher plane.) Some people are moved to solve problems in mechanics because of their practical implications, and some are moved by the pleasure of employing a technique over which they have mastery. But this is not the whole story, as I shall indicate later.

If I had to work up interest in the first problem, I would point out that this is a simple case of an eigen-value problem, and that eigen-value problems belong to an interesting and important domain of mathematics which goes over into quantum mechanics. Further, on the more practical side, the general question of natural frequencies is important to engineers, and, although the given problem is not perhaps of any immediate practical importance, it is typical of a class of problems which are.

However, I feel that if we go only so far we omit something very essential. Students of mathematics are human beings, and as such are susceptible to aesthetic feelings. If you know where to look for it, there is something aesthetic about this first problem. The double pendulum is not, in a sense, a complicated thing, but when you describe it and its motion in mathematical terms it does become rather complicated. Out of this complication, there emerges something very simple—a pair of numbers, representing the frequencies. Anyone will, I believe, experience some aesthetic emotion in obtaining something simple from something complicated. Without any claim to originality in this matter (although I do not recall previously seeing the statement so put), let me enunciate an "aesthetic principle" as follows:

$$\text{COMPLEXITY} \xrightarrow[\text{understanding}]{\text{Through intellectual}} \text{SIMPLICITY}$$

I believe that this aesthetic principle, properly interpreted, is valid not only for mathematics and science generally, but for the arts as well.

The aesthetic aspect is more pronounced in the second problem. Here we are indeed faced with something complicated and hard to visualize—the three-dimensional motions of a rigid body. But we are asked something simple. First,

is there any particle instantaneously at rest? Secondly, is there any particle which instantaneously has no acceleration? A further aesthetic charm of variety arises from the fact that we can answer the first question in a flash by considering the infinitesimal screw displacement of the body in infinitesimal time, but the second question forces us into analytical methods.

So much for motivation. As regards a plan of attack, it is easy enough for a mature and experienced person to lay one out. But it is a very hard thing for a student to do, and here he needs all encouragement. After he has solved a problem, he should discuss his solution (either with himself or someone else), and describe what steps he has carried out. Our first problem lends itself very well to planning, because it is of a standard type. Assuming that the student knows the Lagrangian method, the steps might be planned as follows:

1. Assign generalized coördinates.
2. Calculate the kinetic and potential energies approximately, each as a quadratic form.
3. Write down the equations of motion.
4. By a trigonometric or exponential substitution, obtain a determinantal equation for the frequencies.
5. Solve this equation. This will be easy, because the equation will be quadratic, there being two degrees of freedom.

The second problem hardly lends itself so well to detailed planning. In fact, overemphasis on planning would lead us at once into an analytic attack, and so hide from us the simple descriptive way of answering the first part of the problem. Thus planning is something that can be overdone—we must not become slaves to a routine, or we shall miss many interesting things which lie off the beaten track.

As for details, I pass them over because of lack of time. But mistakes are interesting. I plead guilty personally to an incurable tendency to make mistakes —trivial, annoying mistakes. In fact, on working over the first problem (as I thought I should before talking about it here), I first got imaginary frequencies! It was one of those ridiculous slips, omitting a factor 2 through carelessness. I wish I knew how to avoid doing this sort of thing. There are those who hold that a mistake in principle is a very serious sin, whereas a mistake in detail is easily forgiven. Frankly, I do not know the proper philosophy here, and would welcome advice.

I have not had time to say much about the third problem. I put it in as a foil for the other two, regarding it as a problem which lacked the practical appeal of the first and the aesthetic appeal of the second. Possibly I have not done justice to it in so treating it.

THE NEGLECTED SYNTHETIC APPROACH

L. M. KELLY, University of Missouri

The writer would like to enter a mild protest against the existing order of things in the field of mathematical education. This is, of course, not an unusual

desire. These remarks are directed wholly toward the training offered a mathematics major. Our principal thesis is that the synthetic approach, which was once so popular, has been relegated to a role of unwarranted obscurity. The classical conflict of the 1800's between the proponents of the analytic and synthetic methods has happily long since been dead and buried and it is not our intention of making any effort to revive it. However the fact that the analytic method is admittedly the most potent tool at the disposal of a mathematician does not warrant the complete disregard of all the other equipment. Long usage allows us to employ the terms analytic and synthetic without very precise definition. It is not altogether certain however that even rough agreement as to their meaning is generally attained. For purposes of this discussion no sharp delineations need be made but we will point out that our usage does not confine the reference to geometry nor does it indicate the absence or presence of coördinates as is sometimes done. Possibly a rough synonym for our use of the term synthetic could be "unorthodox." To the writer the absence or presence of "machinery" is the criterion but this probably does little to clarify the issue.

An example which illustrates the points we have in mind is the well known problem "*to find a point the sum of whose distances from three fixed points is a minimum.*" There are any number of published solutions to this problem ranging from methods of high school geometry (Johnson's *Modern Geometry*) to relatively sophisticated methods employing Lagrange multipliers. The analytic attack is obvious. A coördinate system is introduced, an expression for the distance sum is obtained, this expression is then differentiated partially, first with respect to x and then y, the two results are set equal to zero and the resulting equations solved simultaneously for the coördinates of the desired point. This sounds simple but, as a matter of fact, the algebra is quite involved. (See Goursat-Hedrick, vol. 1, page 130.) Even if the equations could be conveniently solved the result would be the coördinates of the point, which are not of particular interest.

The interesting property of the point is revealed by the following simple and quite natural argument which is essentially that contained in *What is Mathematics* by Courant and Robbins. Let A, B, and C be the three fixed points and P the minimizing point. (We assume its existence.) Consider the ellipse defined by the two foci A and B and passing through the point P. We maintain that the segment CP is normal to this ellipse, otherwise the foot P' of the normal to the ellipse from C would provide us with a point the sum of whose distances from the three fixed points was less than the alleged minimum. It follows that the angles CPA and CPB are equal and similar considerations show CPB equals APB. Thus, in general, the three segments PA, PB, and PC make angles of 120 degrees with one another. Certain special cases require examination but it is not our purpose to carry out an exhaustive analysis. Certainly it will be conceded that this is a synthetic approach and that it is simple and quite natural as well as elementary. If we ask where in the course of a normal training, undergraduate or graduate, a student would be encouraged to pursue such a line of attack the answer would seem to be "nowhere." Such proofs would hardly be tolerated, let

alone encouraged, in the ordinary calculus course. Similarly in all the other traditional courses with the possible exception of one in College Geometry. It seems regrettable that more institutions do not offer a course or two of the *pot pourri* variety built around a book like *What is Mathematics* where the synthetic method could be exploited along with the analytic.

A further advantage in addition to those already mentioned might be pointed out. Very often the synthetic approach makes the proof of an interesting proposition accessible to a larger group than would be the case if more sophisticated methods were employed. Such is the case in the following problem of obvious interest to cartographers and navigators. Problem: *Is it possible to map a spherical cap congruently (i.e., isometrically) onto the plane.* If a navigator tried to find out from a mathematician he would probably be confronted with some of the technical jargon of differential geometry such as Gauss curvature and differential forms. A very simple synthetic argument however is available. We first appeal to a lemma already proved in this MONTHLY (May, 1947, p. 283) and requiring nothing more advanced for its proof than some spherical trigonometry. This states that if a spherical triangle be congruently reproduced in the plane (that is with length of sides preserved) each angle of the plane triangle is less than the corresponding angle of the spherical triangle. This granted, we imagine, if possible, a one to one distance preserving map of a spherical cap onto the plane and ask what this mapping would do to the points A, B, and C forming an equilateral spherical triangle and its circumcenter D. A, B, C would correspond to the vertices A', B', and C' of a plane equilateral triangle and D would thus have to correspond to D' its circumcenter. But this is impossible since then the triple ABD would be congruent to the planar triple $A'B'D'$ and both the angle D and D' would be 120 degrees contrary to the lemma. This proof is due to L. M. Blumenthal. It is sometimes claimed that such methods are only for the ingenious. There will continue to be a dwindling group of ingenious people if ingenuity is not encouraged and cultivated.

A PROBLEM OF COLLINEAR POINTS

H. S. M. COXETER, University of Toronto

I would like to begin by expressing my wholehearted agreement with Mr. Kelly's plea for the synthetic approach. But my definition of "synthetic" would differ slightly from his. Synthetic geometry is ultimately based on certain primitive concepts and axioms, appropriate to the particular kind of geometry under consideration (e.g., projective or affine, real, complex or finite). Each problem belongs to one kind (or to a few kinds), and I would call a solution synthetic if it remains in that kind, analytic if it goes outside. The use of coördinates is one way of going outside; the use of trigonometry is another. These remarks are illustrated by the following problem, which was proposed by Sylvester in the *Educational Times*, Mathematical Question 11851, vol. 59 (1893), p. 98:

Let n given points have the property that the line joining any two of them passes

through a third point of the set. Must the n points all lie on one line?

Sylvester himself was doubtless aware of the negative answer in the case of the complex projective plane, where the nine points given by

$$x^3 + y^3 + z^3 - xyz = 0$$

(which are the points of inflexion of any one of the cubic curves $x^3+y^3+z^3 = cxyz$) lie by threes on twelve lines in such a way that every two of the points belong to such a set of three. Some years later, Veblen and Bussey described the finite affine geometries, one of which, called $EG(2, 3)$, consists of the nine points (x, y) whose coördinates are residues modulo 3, and the twelve lines

$$Xx + Yy + Z \equiv 0 \pmod 3$$

which join them. Since any two points determine a line, and every line contains three points, all the points in this geometry form a set of the desired kind (with $n=9$).

But the question as applied to the ordinary real plane remained unanswered for forty years, and began to seem as intractable as the four-color map problem. Finally, Grünwald established the affirmative answer: such points must all be collinear. Robert Steinberg transformed Grünwald's affine proof into a projective one (this MONTHLY, vol. 51 (1944), p. 169), and L. M. Kelly discovered a still shorter (and quite unrelated) Euclidean proof (see below).

It seems to me that parallelism and distance are essentially foreign to this problem, which is concerned only with incidence and order. Thus I would not regard the proofs by Grünwald and Kelly as strictly synthetic. Steinberg's projective proof becomes even more elementary when we transform it into a "descriptive" proof, using the single primitive entity *point* and the single primitive relation *between*, in terms of which lines and serial order can be defined (see Veblen, *Transactions of the American Mathematical Society*, vol. 5 (1904), pp. 353–371).

If the n points are not all collinear, some three of them must form a triangle ABC. Lines joining A to all other points of the set meet the line BC in B and C and possibly other points. Let P' be a point of BC not among these. Then the line AP' contains no point of the set except A. Joins of pairs of points belonging to the set meet AP' in A, P', and possibly other points. Let P be the *first* one of these, going from A towards P'. (P may possibly coincide with P'.) Then no join of two points of the set can pass through a point between A and P. We shall obtain a contradiction by showing that this "empty" segment AP is not really empty after all.

By its definition, the point P lies on a line containing at least three points of the set, say Q, R, S, named so that the segment PR contains Q but not S. (We do not care whether S lies beyond R or beyond P.) Since A and R belong to the set, the line AR contains a third point of the set, say O. Two cases now arise. If O lies between A and R, the line SO must intersect the segment AP (by Pasch's Theorem). Similarly, if O lies outside the segment AR, the line QO must inter-

sect the segment AP. In either case we have the desired contradiction. Hence the triangle ABC cannot exist, and the n points must in fact be collinear.

Alternative proof of collinearity, by *L. M. Kelly*. Assume the contrary. If the points be labeled $p_1, p_2, \cdots, p_n$, then there exists a point p_i and a line p_jp_k such that the perpendicular distance from p_i to the line p_jp_k is the shortest non-zero perpendicular distance from any of the points to any of the lines defined by pairs of the points. Let q be the foot of the perpendicular from p_i to the line p_jp_k. On the line p_jp_k there is at least one more point of the set, p_l. At least two of p_j, p_k, p_l must lie on one side of q. Suppose for definiteness that p_jp_kq is the order (p_k may coincide with q). Then the distance from p_k to the line p_jp_i provides us with a contradiction since it is shorter than the assumed minimum p_iq.

CAN WE TEACH GOOD MATHEMATICS TO UNDERGRADUATES?

R. G. HELSEL and T. RADÓ, Ohio State University

We decided upon our topic shortly after Professor Birkhoff invited us to contribute to this symposium. Later Professor Pólya sent us detailed information concerning the nature of the symposium and we realized then that our topic was perhaps too general; however, we felt the question is of such import to teachers of college mathematics that it should not be put aside. One of the goals of our instruction certainly should be to show the student mathematics of the highest caliber, if we can.

Let us consider then the ingredients of good mathematics. First, it must be *relevant*. Fortunately, most of the subjects we teach are filled with pertinent topics. Calculus, for example, is overflowing with relevancy. Second, good mathematics must be *rigorous*. In other words, all of the reasons must be given. Some persons feel that to teach calculus rigorously the student must learn his ϵ's and δ's and be acquainted with the real number system as defined by Cantor or Dedekind. This is certainly not the case. In fact the sophomore regards any discussion of the real number system as being irrelevant, and it is, at that level. Finally, good mathematics must be *elegant*. If we fail to show our students elegant mathematics, we deprive them of the very thing which affords us our greatest pleasure.

Having listed the ingredients of good mathematics, there remains the question of whether it is possible to serve such a dish to the undergraduate. To answer this, consider the three parts of a theorem: *hypotheses*, *proof*, and *conclusion*. The conclusion, which we assume to be interesting and relevant, is fixed. There may be leeway in the method of proof, but the real freedom is in the choice of assumptions. Thus we propose to *pack the hypotheses* with carefully chosen assumptions that have the power to push through the proof in a rigorous and striking manner and yet are either obvious to the student or acceptable on the basis of past experience. Existential and qualitative assumptions are more likely to meet these requirements than quantitative assumptions.

We shall illustrate this general principle of *packing the hypotheses* by an ex-

ample from calculus. Perhaps our example is not well chosen, but it has the advantage of being brief. Anyone who is interested can find better examples for himself. Let us establish then the formula $d(\log_a x)/dx = c/x$. This certainly is a relevant and interesting relation for the student, since he already knows that $d(x^{n+1})/dx = (n+1)x^n$ if $n \neq -1$. The student also knows, or believes, that the graph of $\log_a x$ is continuous, monotone increasing, and has a tangent at each point. We shall use as our first hypothesis: H_1*.d(*$\log_a x$*)/dx exists for* $x>0$*, call the derivative function* $g(x)$. As our second hypothesis we shall use the one functional relation involving logarithms which the sophomore does know: H_2.$\log_a x + \log_a \alpha = \log_a \alpha x$ *for* $x > 0$ *and* $\alpha > 0$. Differentiating the left side of this equation, regarding α as a constant, we obtain, through H_1, $g(x)$. Differentiating the right side, with the help of the chain rule for the derivative of a function of a function, we obtain $\alpha g(\alpha x)$. Now let x have the value 1 to obtain the equation $g(1) = \alpha g(\alpha)$, or $g(\alpha) = g(1)/\alpha$, and the proof is complete. Note that the value of the proportionality constant is the slope of the graph of $\log_a x$ at the point where it crosses the x axis, an interesting fact which is often overlooked. Similar proofs can be made for the derivative formulas of the trigonometric and exponential functions.

Thus it is our conviction that by the device of *packing the hypotheses* it is possible to show our students mathematics of the highest caliber. Of course the teacher must use great care in selecting relevant topics and creating interest in them. Also he must choose existential and qualitative hypotheses which are completely acceptable to the student. If the assumptions are properly chosen, the teacher can follow through with a rigorous and striking proof which will show the student what mathematics is really like. In addition the student will soon become aware of the power of existential assumptions and, it is to be hoped that, those who go on to advanced courses will have some interest in existence proofs.

Finally, we might ask whether it is following a high or low path to *pack the hypotheses* as suggested? In reply, consider how the working mathematician grabs everything that is plausible and relevant when he sets about to verify a surmise. Afterwards he may check through the details of theorems that he uses, but initially he merely assumes that they are valid. For example, most mathematicians have used the Jordan Curve Theorem at some time or other, but few indeed have ever gone through a rigorous proof. So it appears that the principle of *packing the hypotheses* is not only a means to show our students good mathematics but it is also in keeping with the manner in which mathematicians actually do their work.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 798. *Proposed by J. M. Elkin, Chicago, Ill.*

Prove that $\Pi(n_i!)$, where Σn_i is constant, is a minimum when $\Sigma|n_i-n_j|$ is a minimum, and that, consequently, the most likely distribution of the four suits in a bridge hand is four cards of one suit and three cards each of the other three suits.

E 799. *Proposed by Leo Moser, University of Manitoba*

Prove that for all n

$$\sum_{r=0}^{n}\left[2^{n-r}\binom{n}{r}^2-\binom{2n-r}{n}\binom{n}{r}\right]=0.$$

E 800. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The polar planes, with respect to a tetrahedron, of the isotomic conjugates of a set of collinear points are coaxal.

Note. If the cevians AP, BP, CP, DP, for a tetrahedron $ABCD$, of a point P, cut the planes of the faces BCD, CDA, DAB, ABC, in A', B', C', D', then the isotomic conjugate P' of P is the point of concurrency of the lines joining A, B, C, D to the isotomic conjugates A'', B'', C'', D'' of A', B', C', D' in the faces of the tetrahedron.

SOLUTIONS

Professor Umbugio's Simple Scheme

E 766 [1947, 223]. *Proposed by H.E.G.P.*

Professor Umbugio, who was introduced to our readers in our foregoing April number, invented a remarkable scheme for reviewing books. He divides the time he allows himself for reviewing into three fractions, α, β, and γ. He devotes the fraction α of his time to a deep study of the title page and the jacket. He devotes the fraction β to a spirited search for his name and for quotations from his works. Finally, he spends the fraction γ of his alloted time in a proportionally penetrating perusal of the remaining text. Knowing his characteristic taste for simple and direct methods, we cannot fail to be duly impressed by the differential equations on which he bases his scheme:

$$(1) \qquad dx/dt = y - z, \qquad dy/dt = z - x, \qquad dz/dt = x - y.$$

He considers a system of solutions x, y, z which is determined by initial conditions depending on a (small) parameter ϵ, independent of t. Therefore x, y, and z depend on both t and ϵ, and we appropriately use the notation:

$$(2) \qquad x = f(t, \epsilon), \qquad y = g(t, \epsilon), \qquad z = h(t, \epsilon).$$

The functions (2) satisfy the equations (1) and the initial conditions

$$(3) \qquad f(0, \epsilon) = 1/3 - \epsilon, \qquad g(0, \epsilon) = 1/3, \qquad h(0, \epsilon) = 1/3 + \epsilon.$$

Professor Umbugio defines his important fractions α, β, and γ by

$$\lim_{\epsilon\to 0} f(2, \epsilon) = \alpha, \qquad \lim_{\epsilon\to 0} g(5, \epsilon) = \beta, \qquad \lim_{\epsilon\to 0} h(279, \epsilon) = \gamma.$$

Deflate the Professor! Find α, β, γ without much numerical computation.

I. *Solution by R. E. Greenwood, University of Texas.* From Professor Umbugio's equations (1) we readily find that

$$dx/dt + dy/dt + dz/dt = 0,$$
$$x(dx/dt) + y(dy/dt) + z(dz/dt) = 0.$$

Integrating and taking into account the initial conditions (2), we obtain

$$x + y + z = 1,$$
$$x^2 + y^2 + z^2 = 1/3 + 2\epsilon^2,$$

whence we deduce that

$$(x - 1/3)^2 + (y - 1/3)^2 + (z - 1/3)^2 = 2\epsilon^2.$$

Thus, as $\epsilon\to 0$, we see that x, y, z each approaches $1/3$, no matter what t may be, and hence

$$\alpha = \beta = \gamma = 1/3.$$

Geometrically, the solutions of (1) are concentric circles on the plane $x+y+z=1$, with center $(1/3, 1/3, 1/3)$ and radii $\epsilon\sqrt{2}$. When $\epsilon\to 0$ we obtain the point $(1/3, 1/3, 1/3)$.

II. *Solution by C. F. Pinzka, North Plainfield, N. J.* Solution of (1) by the conventional methods yields

$$x = a_1 + a_2 \cos \sqrt{3}\, t + a_3 \sin \sqrt{3}\, t,$$
$$y = b_1 + b_2 \cos \sqrt{3}\, t + b_3 \sin \sqrt{3}\, t,$$
$$z = c_1 + c_2 \cos \sqrt{3}\, t + c_3 \sin \sqrt{3}\, t,$$

where the a's, b's and c's are constants which satisfy (1) and (3). Substituting in (1) and (3) and solving for the constants gives

$$x = 1/3 - \epsilon \cos \sqrt{3}\, t - (\epsilon \sin \sqrt{3}\, t)/\sqrt{3},$$

$$y = 1/3 + (2\epsilon \sin \sqrt{3}\, t)/\sqrt{3},$$
$$z = 1/3 + \epsilon \cos \sqrt{3}\, t - (\epsilon \sin \sqrt{3}\, t)/\sqrt{3}.$$

Taking the required limits, we find that $\alpha = \beta = \gamma = 1/3$.

Also solved by Paul Brock, G. Y. Cherlin, E. S. Keeping, and W. D. Lambert.

An Application of Pappus' Theorem

E 767 [1947, 223]. *Proposed by Milton Schwartz, Temple University*

From any point P on BC of parallelogram $ABCD$ line segments are drawn to A and D. From any point Q on AD line segments are drawn to B and C. Through the intersections of these four segments (PA, PD, QB, QC) a line is drawn meeting AB in R and CD in S. Prove that BR equals DS.

Solution by Paul Brock, Hunter College. Let PA, QB intersect in T and PD, QC in V. Then, by Pappus' theorem, T, V, and the intersection, L, of the diagonals are collinear. It is obvious that triangles LRB and LSD are congruent. Therefore BR equals SD.

Also solved by W. G. Brady, D. H. Browne, G. Y. Cherlin, H. S. M. Coxeter, W. P. DeWitt, William Douglas, H. E. Fettis, L. M. Kelly, B. R. Leeds, D. K. Pease, Kaidy Tan, Maud Willey, and the proposer.

Editorial Note. This problem has been proposed in the problem department of *School Science and Mathematics* and solutions have appeared in the Nov. 1931 and the Oct. 1932 issues. Also, Problem E 129 of this MONTHLY (May, 1935) is the same problem. In these references one can find solutions using only high school methods.

As a slight generalization we may take $ABCD$ to be a trapezoid, with AB and DC as the parallel sides. Then we may show that $AR/RB = CS/SD$.

Sums of Consecutive Integers

E 768 [1947, 224]. *Proposed by Irving Kaplansky, University of Chicago*

A number n has the property that for any $p < q < n$,

$$S = p + (p+1) + \cdots + q$$

is never divisible by n. Show that this is true if and only if n is a power of 2.

Solutions by S. H. Gould, University of Wisconsin. Set $n = m \cdot 2^k$, with m odd. If $m = 1$, then $2S = (q+p)(q-p+1)$, which has one factor even and the other odd, cannot be divisible by $2n = 2^{k+1}$, since its even factor is less than $2n$. But if $m > 1$, then S is divisible by n, with $0 < p < q < n$, if we take

$$q = (m + 2^{k+1} - 1)/2,$$

and

$$p = (1 + q - 2^{k+1}) \qquad \text{for } m > 2^{k+1},$$

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4277. *Proposed by C. D. Olds, San Jose State College, California*

In a non-associative algebra, it is necessary to distinguish the possible interpretations of x^n. Thus, for example, in a non-commutative non-associative algebra x^3 can mean $x\cdot x^2$ or $x^2\cdot x$. In a general non-commutative non-associative algebra the number of interpretations of x^n is $2(2n-3)!/n!(n-2)!$. Is there a formula for the number of interpretations of x^n in a general commutative non-associative algebra?

4278. *Proposed by Peter Ungár, Budapest, Hungary*

Construct two divergent series, $\sum a_k$ and $\sum b_k$ with $a_1 \geqq a_2 \geqq \cdots \geqq 0$, $b_1 \geqq b_2 \geqq \cdots \geqq 0$, but such that if $c_k \leqq \min(a_k, b_k)$, $c_k > 0$, then $\sum c_k$ is convergent.

4279. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Prove that a tetrahedron $ABCD$, whose vertices A, B, C are the inverse, with respect to a sphere with center D, of a right-angled triangle, and the tangential tetrahedron $A_1B_1C_1D_1$ of that tetrahedron are each inscribed in the other and that the midpoints of AD_1, BC_1, CB_1, DA_1 are coplanar.

SOLUTIONS

Hermite Polynomials

4215 [1946, 470]. *Proposed by Hüseyin Demir, Columbia University*

Prove that the Hermite polynomials defined as follows

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$$

have the property

$$n!\sum_{p=0}^{n} \frac{H_p^2(x)}{p!} = H_{n+1}^2(x) - H_n(x)H_{n+2}(x).$$

Solution by Hsien-yü Hsü Yenching University, Peiping, China. In Polya-Szegö, *Aufgaben und Lehrsätze* II, pp. 294–295, Hermite polynomials are de-

fined as follows

$$h_n(x) = \frac{1}{n!} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2},$$

and satisfy the difference equation

$$nh_n(x) = - xh_{n-1}(x) - h_{n-2}(x), \qquad n = 2, 3, \cdots .$$

We notice that

$$H_n \equiv H_n(x) = (-1)^n n! h_n(x),$$

whence the difference equation is

$$H_n = xH_{n-1} - (n-1)H_{n-2}, \qquad n = 2, 3, \cdots . \tag{1}$$

Upon eliminating x from this equation and the analogous equations for H_{n+1} and H_{n+2} we obtain immediately

$$\begin{aligned} H_{n+1}^2 - H_nH_{n+2} &= n(H_n^2 - H_{n+1}H_{n-1}) + H_n^2 \\ &= n(n-1)(H_{n-1}^2 - H_nH_{n-2}) + nH_{n-1}^2 + H_n^2 \\ &= \cdots \\ &= n!\left\{(H_1^2 - H_2H_0) + \frac{1}{1!} H_1^2 + \frac{1}{2!} H_2^2 + \cdots + \frac{1}{n!} H_n^2\right\} \\ &= n! \sum_{p=0}^{n} H_p^2/p!, \end{aligned}$$

since $H_1^2 - H_2H_0 = 1$.

Solved also by F. E. Cothran, A. B. Farnell, L. M. Kelly, Norman Miller, S. T. Parker, W. A. Pierce, W. H. Spragens, M. S. Webster, M. Wyman, Professor Otto Szász's class, and the Proposer.

Editorial Note. Several solvers mentioned that equation (1) of the above solution is found in Dunham Jackson, *Fourier Series and Orthogonal Polynomials*, p. 176, ff. The solution by members of Professor Szász's class in Orthogonal Developments proceeds from the (so-called) Christoffel's formula

$$\sum_{p=0}^{n} \frac{H_p(x)H_p(y)}{p!} = \frac{H_{n+1}(x)H_n(y) - H_n(x)H_{n+1}(y)}{n!(x-y)}.$$

See Szegö, *Orthogonal Polynomials*, p. 102. Webster's solution employs the relation

$$H_m(x)H_n(x) = n!\sum_{r=0}^{n} \binom{m}{n-r} \frac{H_{m-n+2r}}{r!} \qquad (m \geqq n),$$

established by E. Feldheim, *Journal of the London Mathematical Society*, 1938, pp. 22–29.

Cesàro Continuity

4216 [1946, 470]. *Proposed by Herbert Robbins, Annapolis, Md.*

Write $\{x_n\}\underset{c}{\rightarrow}\alpha$ if $\lim_{n\to\infty}(x_1+\cdots+x_n)/n=\alpha$. A function $f(x)$ is said to be C.c. (Cesàro continuous) at $x=\alpha$ if $\{x_n\}\underset{c}{\rightarrow}\alpha$ implies $f(x_n)\underset{c}{\rightarrow}f(\alpha)$. Show that if $f(x)$ is of the form $Ax+B$ then it is C.c. at every value of x, and that if $f(x)$ is C.c. at even a single value $x=\alpha$, then $f(x)$ is of the form $Ax+B$.

Solution by R. C. Buck, Brown University. The first part of the problem is trivial. For the second part suppose that $f(x)$ is C.c. at a single value α. By changing to parallel coördinate axes, we can take $\alpha=0$ and $f(\alpha)=0$. Then the sequence $\{a, b, c, a, b, c, \cdots\}$ is Cesàro convergent to zero if $a+b+c=0$. From the Cesàro continuity of $f(x)$ at zero we conclude that $f(a)+f(b)+f(c)=0$, or $f(a)+f(b)=-f(-a-b)$. Thus also $f(a)=-f(-a)$, whence for any x and y

$$f(x+y)=f(x)+f(y),\qquad f(nx)=nf(x).$$

If a sequence $\{x_n\}$ is Cesàro convergent to zero, so that $\lim\sigma_n=0$ where $n\sigma_n=x_1+\cdots+x_n$, then

$$\lim\frac{1}{n}\sum_{k=1}^{n}f(x_k)=\lim f(\sigma_n)=0.$$

Hence $f(x)$ is continuous in the usual sense at zero, and by the additivity of $f(x)$, at all other points x. It is well known that the only continuous function obeying $f(x+y)=f(x)+f(y)$ is $f(x)=Ax$.

Solved also by P. T. Bateman, Colin Blyth, Jr., Harley Flanders, William Gustin, and J. B. Kelly.

Integers

4217 [1946, 471]. *Proposed by P. A. Piza, San Juan, P. R.*

Let a and b be positive integers whose sum is the square c^2 and whose difference is the cube d^3. Given that each of c^2, d^3 and a is a 4-digit integer and that the sum of the 4 digits included in each is equal to b, find a and b.

Solution by Monte Dernham, San Francisco. If

$$a=10^3m+10^2h+10t+u,\qquad b=m+h+t+u,$$

then $d^3=999m+99h+9t$. Now the only four-digit cubes divisible by 9 are $12^3=1728$, $15^3=3375$, $18^3=5832$, $21^3=9261$. In any case $b=18$ so that $c^2=a+18=d^3+36$. Now d^3+36 is a square for only the first of the cubes listed. Hence the unique answer is given by

$$b=18,\qquad a=1746,\qquad c^2=1764,\qquad d^3=1728.$$

Solved also by M. Aissen, Murray Barbour, R. G. Blake, Paul Brock, D. H. Browne, R. B. Herrera, N. Kaufman and R. J. Koch, Helen Nickerson, Clay Perry, C. F. Pinzka, W. A. Pierce, E. D. Schell, Maud Willey, and the Proposer.

Integers with k Factors

4220. [1946, 537]. *Proposed by P. Erdös, Syracuse University.*

Let m be a positive integer with no prime factors greater than n and $m \leqq n^{(k+1)/2}$. Then m can be written as the product of k integers $\leqq n$; the exponent $(k+1)/2$ is the best possible.

Solution by N. J. Fine, Washington, D. C. Let m satisfy the given conditions' and suppose that m cannot be written as the product of k integers $\leqq n$. Write m as a product of primes:

$$m = \prod_{i=1}^{r} p_i, \qquad p_1 \leqq p_2 \leqq \cdots \leqq p_r.$$

We can define $A_1, A_2, \cdots, A_k$, and $r_1, r_2 \cdots, r_k$ by

$$A_1 = \prod_{i=1}^{r_1-1} p_i \leqq n, \qquad p_{r_1}A_1 > n,$$

$$A_2 = \prod_{i=r_1}^{r_2-1} p_i \leqq n, \qquad p_{r_2}A_2 > n,$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$A_k = \prod_{i=r_{k-1}}^{r_k-1} p_i \leqq n, \qquad p_{r_k}A_k > n.$$

The possibility of continuing this construction at least as far as A_k is guaranteed by our assumptions. Clearly $A_{j+1} \geqq P_{r_j}$, $j=1, 2, \cdots, k-1$, and $m \geqq A_1A_2 \cdots A_kp_{r_k}$. Hence

$$m^2 \geqq (A_1A_2 \cdots A_kp_{r_k})^2 = (A_1A_2)(A_2A_3) \cdots (A_{k-1}A_k)(A_1p_{r_k})(A_kp_{r_k})$$
$$\geqq (A_1p_{r_1})^2(A_2p_{r_2}) \cdots (A_{k-1}p_{r_{k-1}})(A_kp_{r_k}) > n^{k+1}.$$

But this is a contradiction and proves the first part of the theorem.

Given a fixed positive integer k and an arbitrary $\epsilon > 0$, let

$$\delta = \min\left(\frac{1}{2}, \frac{\epsilon}{k+1}\right).$$

Then there is an integer $n_0(\delta)$ so large that for all $n > n_0(\delta)$, there exists a prime p satisfying

$$n^{1/2} < p < n^{1/2+\delta}$$

The integer $m = p^{k+1}$ has no prime factors exceeding n, cannot be written as the product of k integers $\leqq n$, and is less than $n^{(k+1)/2+\epsilon}$. Thus the proposed exponent is the best possible.

Solved also by P. T. Bateman.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, 27, N. Y. and not to any of the other editors or officers of the Association.

Introduction to the Theory of Equations. By Lois W. Griffiths. New York, John Wiley and Sons, Inc., 1947. 9+278 pages. $3.50.

The problem of content in a course in the Theory of Equations is one that is open to discussion. Many of the topics considered are, of course, standard with all instructors. However, the majority of the books in this subject offer more material, albeit briefly, than can possibly be presented in the usual half year allotted to this course. Therefore, it is not only possible, but necessary, that each instructor use some selection in outlining his course. On the other hand, Miss Griffiths has chosen to present in her book somewhat less than the usual number of topics, but to present the material selected in greater detail. While the additional detail is commendable, the merit of this book in the eyes of the individual instructor will probably be determined to a large degree by his or her agreement or lack of agreement in the choice of subject matter.

As the author has stated in the preface, "Development from the particular to the general is an outstanding feature of this book." In each case, she has gone through the details of a proof with a particular example before attempting to prove the general case. For instance, she isolates the real roots of a particular quartic equation in complete detail before embarking upon a general discussion of Sturm's Theorem. Using this approach, the major proofs are presented at least twice. This feature makes this book particularly useful for the student who proposes to master the material with no aid other than the text.

Before listing in more detail the topics discussed, it is well to note that only the algebraic equation is considered. This is particularly worthy of note since Newton's method for the approximation of the real roots of an equation is not mentioned, nor is the method of Budan for the isolation of the real roots of an equation mentioned at any point in the book. The first chapter is devoted to the binomial equation, roots of unity, and a very elementary treatise on complex numbers. The second chapter includes the solution of the cubic and quartic equation, with a discussion of the discriminant of each. The third and fourth chapters give some general theorems on the roots of an equation (including the factor theorem, remainder theorem, and upper and lower bounds on the real roots), Sturm's theorem, and Horner's method for the computation of real roots. Chapters five and six are used to develop the theory of determinants, with an introduction of the concept of a matrix. Systems of linear equations are discussed in chapter seven. Chapter eight gives a more rigorous, though brief, treatment of complex numbers followed by a statement of the fundamental theorem of algebra. The last of the nine chapters is devoted to a short discussion of symmetric functions. There are many good problems throughout the

book, and the answers are given for all of the odd-numbered problems.

This book presents a somewhat different menu than many texts in this field. In general the selection presented here would emphasize the initial word in the title, and the presentation is sufficiently elementary that no student who would ordinarily be permitted to enroll for this course should have any difficulty in comprehending the material set forth. To this extent, this book is to be recommended. There will be some, however, who will question whether or not this book may have too little material beyond a good text in college algebra. As stated before, the answer will lie in the agreement or disagreement of the individual instructor with the material presented.

R. L. WILSON

Introduction to Mathematical Statistics. By P. G. Hoel. New York, John Wiley and Sons, Inc., 1947. 10+258 pages, $3.50.

The reviewer's opinion that the minimum of mathematical preparation for a satisfactory introductory course in mathematical statistics is a year of the calculus is widely shared but so far no really suitable textbook for such a course has been available. It is with considerable anticipation, therefore, that he noted that Hoel's new book is intended to fill this need, particularly in view of the author's known competence in this field. But a rapid inspection of the contents reminds one of the difficulties one faces in planning a first course in mathematical statistics. (1) Is this first course also to be in most cases the terminal course? If it can be generally admitted that any real statistical competence is not likely to be gained in a one-year course, then an author can more easily avoid spreading himself too thin. (2) Is one to concern himself chiefly with the mathematics of statistics which can be so formal as to be almost meaningless or is the science of statistics also to be a matter of principal interest? The reviewer has always found the formal mathematics in elementary statistics the easier half of what is to be taught; to give the student an orientation in a new discipline is more difficult. (3) The students electing such a course vary not only in their mathematical preparation and maturity but also even more widely in their previous acquaintance with the uses to which statistical methods may be put. The most enthusiastic students are commonly those who have already acquired a considerable practical acquaintance with the methods taught but who are now satisfying a most laudable desire to gain some real understanding of them. These and not average students who have just completed a calculus course are the ones to whom the more purely mathematical type of course should be taught. It appears to the reviewer that Hoel's book is best adapted to students with mathematical maturity beyond that usually conferred by a calculus course and with some previous acquaintance with statistical methods.

The first seven chapters cover what has heretofore commonly been the material for an introductory course but the mode of treatment is a distinct departure. Not many words are wasted, for the job is done in 127 pages. Little is actually omitted that ought to be included and there are frequent numerical

illustrations which indicate the importance and nature of the applications. What is said is almost all accurate and one can take little exception to the mathematics. But the discussion is always brief and the mathematics neat. Brevity with clarity is a merit of an elementary textbook provided it is taught by an instructor who can amplify the discussion and provide additional examples. But pedagogically speaking, mathematical elegance ought sometimes to be eschewed in favor of the direct attack. The obvious case in point here is the early introduction of the moment generating function and its free use. This gets in a concise and pleasing way many results, and the student of mathematical statistics should not go for long without making its acquaintance; but had not beginners better first calculate the variance of a normally distributed variable or of one obeying a binomial distribution law by direct integration or summation? Finally in these seven chapters actual computing methods are treated very briefly indeed in spite of the fact that there is much to be taught about computation and that no one becomes a competent statistician without having a great deal of computation behind him.

The remaining five chapters include an introduction to the small sample significance tests and to χ^2 tests which have now become standard. The exposition is still brief but it is still a long step beyond what has heretofore been available in an elementary text in accuracy and mathematical soundness. The reviewer does not see why at least the geometrical proof of the independence of the mean and variance in samples from normal and the development of the approximation which leads to the χ^2 test of goodness of fit could not have been attempted in view of other derivations that are included. The reviewer does have differences with the author with respect to the Student-Fisher t test. On page 143 it ought to be stated that this test does not remedy the defect of the large sample test. In fact for small samples the defect cannot be remedied; instead in strict accuracy we are forced to use a different test. Again in both numerical illustrations, on this page and on page 146, the regions of rejection are both taken as one-sided, which is only correct if it were decided in advance of performing the experiment that the alternate hypotheses admissible were also one-sided, which seems very debatable.

In addition some rather recent material is included. Linear discriminant functions and the theory of runs are introduced with developments that seem really substantial for college juniors. Non-parametric tests, interval estimation (confidence intervals), sampling inspection methods including sequential analysis, stratified sampling, some of the Neyman-Pearson theory of significance tests, and a little about maximum likelihood methods are all included briefly, some of them so very briefly in comparison with what would be required to give any real comprehension of them, that it makes their value to the student doubtful.

Very few typographical errors were noted. The book suffers from the very common inconvenience of requiring the reader to search for the page on which, say, (3), Chapter V referred to is to be found. At the end of each chapter refer-

ences are given to other, fuller, or more mathematical treatments of material in that chapter. Good lists of problems follow each chapter, and the instructor will find often included as exercises nontrivial derivations of theoretical results which supplement the text.

I think it is evident that the reviewer's chief concern is that Hoel's book may prove a little more mature and to treat too much material a little more concisely than is best for beginners fresh from a calculus course. A really experienced instructor could undoubtedly amplify and supplement the text and have in his students' hands a basic reference book that is much more sound, more complete, and more modern than anything that has been available at anywhere near the proper level. The book does represent a very long advance in the right direction.

C. C. CRAIG

A First Course in Mathematical Statistics. By C. E. Weatherburn. Cambridge, at the University Press; New York, The Macmillan Company, 1946. 15+271 pages. $3.50.

This book has a good many points of similarity to the even more recent text by Hoel, reviewed above. It is written at about the same mathematical level and for the most part, as is natural, deals with the topics found in the other book. It is, however, even more concerned with the mathematics of statistics, so far as its mathematical level permits, and by not including an account of some of the more recent developments dealt with in Hoel, space is found for a fuller treatment of some of its subjects. Greater mathematical clarity seems to be obtained at points, especially in the first half, but the discussion of the meaning of the results from the viewpoint of application to experimental work is even more brief. This is primarily a book for a comparatively mature student who already has a working knowledge of methods of statistical investigation, who wishes to gain a sounder understanding of their validity, and who has a good command of the calculus or a little more as his mathematical equipment. For students who are quite new to statistics, a good deal would need to be added by the instructor to keep a course based on this book from being quite formal.

The first five chapters deal with moments, probability, the binomial, the univariate and bivariate normal and the Poisson frequency functions and finally bivariate regression theory. Confined to 105 pages the discussion is necessarily often brief. Here also the moment generating function is introduced and freely used and Weatherburn goes beyond Hoel in deriving Sheppard's adjustments (as corrections good on the average) and in defining and using Thiele's seminvariants (or cumulants). The introduction to probability is not so very compressed as in Hoel's book and the student is likely to feel better prepared to go on to the study of frequency laws. As the author says, he has deliberately avoided the more modern mathematical formulations of the theory of probability, but the reviewer would be better pleased if it were at least mentioned that there are logical difficulties in the concept of the limit of an observed frequency ratio as

the number of observations increases indefinitely.

The remaining seven chapters are concerned with sampling theory, sampling distributions, and tests of significance. A rather clear division is made between results only valid for large samples and those good for samples of any size. The distributions of the Student-Fisher t and of the variance ratio in samples from normal and the χ^2 distribution are derived and the usual applications are made. Interesting features are the introduction of the Γ- and β-type distribution functions as the basic functions for the distributions used for the exact tests in samples from normal universes, and the derivation of the χ^2 distribution by the method used by Fisher in 1922. There is a chapter on the analysis of variance and covariance in which the exposition is clear for the simple two, three and four component (Latin square) analyses of variance but is less clear for the one and two component analyses of covariance dealt with. The final chapter on multivariate regression theory also seems to suffer from an effort to achieve brevity.

A great deal of what the reviewer said in a general way concerning Hoel's book could be repeated here. Again it is very pleasing to have another book on mathematical statistics at this level that one can recommend to students as mathematically sound and adequate. There are good lists of problems and some of them embody additional theoretical developments that could have been included in the text. Computing methods are not given a very prominent place.

There are a good many places where the reviewer would like to see fuller discussion and illustration; fewer where he would want to revise what is said. In the discussion of the standard errors of sample parameters there should be some mention of the efficiency of a statistic as well as its bias and the two concepts should be distinguished from each other. The development of the large sample standard error of a sample variance and the account of the approximation obtained is really not satisfactory. Presumably the author doesn't mean to say on page 179 that the sum of two stochastic variates could be independent of either summand. In the comparison of two sample means by the Student-Fisher t test, the assumed equality of the variances in the universes sampled needs emphasis and the fact that this assumption can be tested by the use of the variance ratio needs to be included and illustrated. In fact the student is not shown at all how to find the probability that the variance ratio be less than a given proper fraction. The relation between the variance ratio with one degree of freedom and the Student-Fisher t ought to be pointed out.

But these are minor criticisms. We really have here another very useful reference book and an excellent text for a properly prepared class. The lack of books where statistical mathematics could be found at this level has been a serious handicap in statistical instruction and this book and Hoel's will be of much aid to both students and teachers.

C. C. CRAIG

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

EDITORIAL NOTE: The request for *Club Reports* for 1946–47 has been responded to most satisfactorily by the advisers of mathematics clubs. The reports will appear on these pages in the order in which they have been received. If there are clubs who wish to report on the year's activities please send the material to the Editor of this department.

CLUB REPORTS, 1946–47

Mathematics Club, Purdue University

The Purdue *Mathematics Club* has completed a most successful first year. Following the organizational meeting, papers have been presented at each meeting. Some of the topics that were discussed include:

A trip through the universe, by Professor Carl Holton

Mathematics and bridge, by Professor I. W. Burr

Closed curves having a pencil of constant chords, by Professor C. K. Robbins.

One activity of the club consisted of a series of problems presented in the school paper. Solutions were submitted by interested persons on the campus. The correct solution was then published in a subsequent edition of the paper.

Officers for 1946–47: President, Arthur F. Sterling; Secretary, Nancy F. Van Ness; Treasurer, Nelson J. Herbert; Program Chairman, William F. Haldeman; Publicity Chairman, Charles O. Davis.

Kappa Mu Epsilon, Montclair State Teachers College

The papers presented at the regular monthly meetings were:

Navigation, by Lawrence Campbell, Walter Rissler and Robert Poppke.

The number e, by Professor Howard F. Fehr

Map making, by June Van Hoeven

Ruler and compass constructions, by Max Sobel.

Twenty new members were initiated during the year.

In April, a banquet was held for the double purpose of celebrating the twentieth anniversary at Montclair of faculty advisor Professor V. S. Mallory, and renewing friendships between alumni and undergraduate members.

The semester closed with the traditional party at Dr. Mallory's home. At this time awards were given to the three seniors who had attained the highest marks in mathematics throughout their four years at college. Mrs. Florence Elder, Max Sobel, and Bessie Kaplan each received a copy of Gaylord Merriman's *To Discover Mathematics*.

The officers who served during the year 1946–47 were: President, Dorothy McGrory; Vice-President, Muriel Roversi; Recording Secretary, Bessie Kaplan; Treasurer, Helga Schrank; Corresponding Secretary, Professor H. F. Fehr.

The newly elected officers for the year 1947–48 are: President, Walter Rissler;

Vice-President, Ann McCumsey; Recording Secretary, Ruth Muller; Treasurer, Jean McRae; Corresponding Secretary, Professor H. F. Fehr.

Mathematics Club, Illinois Institute of Technology

The policy of the club during the year was to present a number of lectures which would be of interest to the engineering students and beginning mathematics majors as well as presenting several lectures of an advanced nature for those with more mathematical background. This policy proved quite successful.

The lecturers, both student and guest speakers, presented the following papers:

Some remarkable theorems concerning moving figures, by Dr. L. R. Ford, who gave a very popular lecture dealing with the curve traced by a fixed point on a line segment of fixed length whose end points follow any two curves.

The probability function and the gamma function, by Robert Teasdale

Prime numbers, by Milton Searl

Motivation interaction, by Dr. Anatol Rapaport, whose lecture evoked considerable interest and discussion. The paper dealt with the mathematical evaluation of the amount of work a person of given psychological characteristics would do under varying circumstances.

Non-euclidean systems of geometry, by Malcolm Smith

Bessel functions, by Chandler Sammons

What is a curve? by Dr. Karl Menger, in which the main historical concepts as well as the present concepts of a curve were presented and proved quite fascinating to many of the club members. Menger's Universal Curve was also discussed.

Differential equations, by Don Mandel Friedlen

Vectors and fields, by H. Flanders.

In addition to their regular meetings the club participated in Junior Week, an open house activity. Featured in the clubs display was an exhibit of calculating machines and a brachristochrome. First prize was won by the club in one phase of the competition.

Officers for the September 1946 term were: President, Rober Teasdale; Publicity and Program Director, Milton Searl; Secretary, Chandler Sammons; Faculty Advisor, Dr. Anatol Rapaport.

Officers for the February 1947 term were: President, Malcolm Smith; Publicity and Program Director, Louis Joseph; Secretary, Milton Searl; Faculty Advisor, Dr. Anatol Rapaport.

Pi Mu Epsilon, University of Arizona

On April 7, 1941, the local mathematical group became the *Arizona Alpha* Chapter of the national organization, *Pi Mu Epsilon*. Since that time, annual elections have been held and a cup awarded for achievement in mathematics.

Two years ago the proposal was made and passed which put the local chapter under student management with a member of the faculty popularly elected to

act as advisor. Since that time it has been the purpose to hold at least four dinner meetings during each scholastic year, the banquets being followed by an hour lecture on some interesting aspect of mathematics and given by any student sufficiently advanced in mathematics. The initiation banquet scheduled as speaker a faculty member, either local or from another institution. The public was invited to all of these talks.

The papers presented this year were:

Application of mathematics to radio, by Robert Masching, a junior student in electrical engineering

The construction of magic squares, by F. E. Haupt, Instructor in the department of mathematics

Mathematics as related to other great fields of learning, by R. J. Hannelly, Dean of Phoenix Junior College.

Robert O. Wenzel received the 1946 *Pi Mu Epsilon* cup.

Excluding various business meetings, nomination committee meetings and so on, the season's activities were concluded with a student-faculty picnic.

This year's officers were: President, Donald B. Marsh, Jr.; Treasurer, Paul R. Vandiver; Secretary, Louisa F. Simons; Faculty Advisor, Dr. D. L. Webb.

The officers elected for next year are: President, John F. Pfeiffer; Treasurer, John Williams; Secretary, Harriet W. Darley; Faculty Advisor, J. F. Foster.

Mathematics Club, Camp Dodge Annex of Iowa State College

The *Mathematics Club* was organized in November, 1946, to stimulate interest in mathematics among the freshman students of the college Annex. The Club was sponsored by the *Iowa Alpha* Chapter of *Pi Mu Epsilon*. The programs at monthly meetings emphasized those phases of mathematics which were usually not found in textbooks. The program topics included:

Generalizations of trigonometry, by Professor H. P. Thielman

Games of chance, by Professor F. A. Brandner.

For the final meeting of the school year a motor trip was made to the Drake University observatory at Des Moines, where Professor B. E. Gillam of Drake University gave an excellent lecture on popular astronomy.

Refreshments were served at each meeting at the Recreation Hall of the Camp Dodge Annex. Quarterly dues were assessed to each of the seventy-five members to help defray this cost.

Much able assistance in organizing and planning of programs was given by Professor Fred Robertson, head of the mathematics department at the Camp Dodge Annex, and Ralph Robinson, treasurer of Pi Mu Epsilon at Iowa State.

Officers of the club were: President, Robert Triska; Vice-President, John Hart; Secretary, Kenneth Brown; Treasurer, James O. Loy; Faculty Advisor, Mr. R. L. Lambert.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

ACKNOWLEDGMENT

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SIXTH EDITION OF ARCHIBALD'S OUTLINE

A sixth edition of the well-known *Outline of the history of mathematics* by Raymond Clare Archibald is now in process of preparation. Professor Archibald would be glad to receive criticisms and suggestions for improvement of the fifth edition.

Although the primary purpose of the Slaught Memorial Papers is to make available *new* expositions of worth-while subjects, there is such a widespread interest in the *Outline* that it has been decided to make an exception in this case. Accordingly, the sixth edition will appear as a Slaught Memorial Paper, and all subscribers to the MONTHLY will receive a copy free of charge.

CORRECTION TO A RECENT REPORT

There are three errors in my recent report of the 1947 William Lowell Putnam Competition. They may be corrected by adding the name of Gerard Washnitzer of Brooklyn College to the list of contestants receiving honorable mention,

by adding the University of British Columbia to the list of institutions entering teams, and by removing Boston College from the latter list.—G. W. MACKEY.

INSTITUTE FOR ADVANCED STUDY

The School of Mathematics of the Institute for Advanced Study will allocate a small number of stipends to gifted young mathematicians and mathematical physicists to enable them to study and to do research work at Princeton during the academic year, 1948–1949. Candidates must have given evidence of ability in research comparable at least with that expected for the degree of Doctor of Philosophy. Blanks for application may be obtained from the School of Mathematics, Institute for Advanced Study, Princeton, New Jersey, and are returnable by February 1, 1948.

THE EIGHTH ANNUAL WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The eighth annual William Lowell Putnam Mathematical Competition, under the sponsorship of the Mathematical Association of America, will be held on Saturday, March 20, 1948. This Competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is open to undergraduates in the United States and Canada who have not received a degree.

The examination consists of two parts of three hours each. The questions will be taken from the fields of calculus (elementary and advanced) with applications to geometry and mechanics not involving techniques beyond the usual applications, higher algebra (determinants and theory of equations), elementary differential equations, and geometry (advanced plane and solid analytic geometry). Any college or university wishing to enter a team or individual contestants may secure an application blank from Professor George Mackey, Hunt Hall 12, Harvard University, Cambridge 38, Massachusetts, by a postcard request. All applications must be filed with Professor Mackey not later than March 1, 1948. If three candidates are presented from a college or university, they are to constitute a team; if more than three are presented from any one college or university, the team of three must be named on the application.

The examination may be given at any place where a team, or at least three candidates, can be assembled. Exceptions to the rule may be made in cases of unusual necessity. Sealed copies of the examinations will be sent to the supervisor of the examination in time for the examination day and are not to be opened before the hour set. At the supervisor's first opportunity after the afternoon examination, the books are to be sent by registered mail or by express to Professor Mackey, who will forward them to a qualified reader chosen by the Association.

The prizes to be awarded to the departments of mathematics of the institutions with the winning teams are \$400, \$300, \$200 and \$100 in the order of their rank. In addition, there will be prizes of \$40, \$30, \$20 and \$10 awarded to the members of these teams according to the rank of the team; a prize of \$40 to

each of the five highest contestants and a prize of $20 to each of the succeeding five highest contestants. Each of the winners will receive a suitable medal. Honorable mention will be given to several teams next in order after the four winning teams and to the fifteen individuals next in order after the ten individual winners. For further encouragement of the Competition, there will be awarded at Harvard University (or at Radcliffe College in the case of a woman) an annual $1200 William Lowell Putnam Prize Scholarship to one of the first five contestants, this to be available either immediately or on the completion of the student's undergraduate work.

Reports on the seven previous competitions and examination questions will be found in this MONTHLY for May, 1938, 1939, 1940, 1941, 1942, October, 1946, and for August–September, 1947.

PERSONAL ITEMS

Assistant Professor C. F. Adler of New York University has been promoted to an associate professorship.

Dr. G. E. Albert of the Naval Ordnance Plant, Indianapolis, Indiana, has been appointed to an associate professorship at the University of Tennessee.

Adjunct Professor W. E. F. Appuhn of the Polytechnic Institute of Brooklyn has been appointed head of the department and professor of mathematics at St. Francis College, Brooklyn.

Assistant Professor B. H. Arnold of Montana State College has been appointed to an assistant professorship at Oregon State College.

W. H. Bradford, John McNeese Junior College of Louisiana State University, Lake Charles, Louisiana, has been promoted to an associate professorship.

Assistant Professor B. W. Brewer of Texas A. & M. has been appointed to an assistant professorship at Oregon State College.

Assistant Professor H. K. Brown of Lehigh University has been appointed to an assistant professorship of Mechanical Engineering at Northeastern University.

Ethel B. Callahan of Hartwick College, Oneonta, New York, has been promoted to a professorship.

Randolph Church of the United States Naval Postgraduate School has been promoted to a professorship.

Assistant Professor Mary Dean Clement of Wells College has been appointed to an assistant professorship at the University of Miami.

Professor A. B. Coble of the University of Illinois has been appointed visiting professor at Haverford College.

Dr. J. B. Coleman has been appointed to a professorship at Presbyterian College, Clinton, South Carolina.

Associate Professor H. L. Dorwart of Washington and Jefferson College has been promoted to a professorship.

Associate Professor E. D. Eaves of the University of Tennessee has been promoted to a professorship.

Professor L. R. Ford of the Illinois Institute of Technology was visiting professor at the University of Utah during June and July of the 1947 Centennial.

Professor Mario O. Gonzalez of Havana University has been appointed visiting professor at the University of Alabama.

Associate Professor Cornelius Gouwens of Iowa State College has been promoted to a professorship.

Assistant Professor Albert Grau of the University of Kentucky has been appointed to an associate professorship at the University of Alabama.

Professor L. M. Graves of the University of Chicago has been appointed to a visiting professorship at Indiana University for the current academic year.

William Gustin of the University of California at Los Angeles has been appointed to an assistant professorship at Indiana University.

D. W. Hall of the University of Maryland has been promoted to a professorship.

Dr. N. A. Hall of United Aircraft Corporation has been appointed to a professorship of thermodynamics at the University of Minnesota.

Dr. O. G. Harrold of Princeton University has been appointed to a professorship at the University of Tennessee.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

APRIL MEETING OF THE METROPOLITAN NEW YORK SECTION

The sixth annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at Pratt Institute, Brooklyn, N. Y., on Saturday, April 19, 1947.

There were one hundred and seventy-two present, including the following sixty-eight members of the Association: R. G. Archibald, L. A. Aroian, Aaron Bakst, Brother Bernard Alfred, Samuel Borofsky, C. B. Boyer, Benjamin Braverman, Paul Brock, A. B. Brown, Hobart Bushey, Jewell Hughes Bushey, Mannis Charosh, Ruth T. Coleman, T. F. Cope, W. H. H. Cowles, Constance H. Davis, D. R. Davis, J. N. Eastham, W. H. Fagerstrom, J. M. Feld, Edward Fleisher, R. M. Foster K. G. Fuller, Irving Gerst, B. P. Gill, Harriet M. Griffin, George Grossman, Frank Hawthorne, G. C. Helme, Morris Hertzig, E. Marie Hove, L. C. Hutchinson, Joseph Jablonower, H. S. Kieval, Helen L. Kutman, Nathan Lazar, C. H. Lehmann, A. A. LePori, M. E. Levenson, S. B. Littauer, H. F. MacNeish, May H. Maria, F. H. Miller, A. J. Mortola, M. A. Nordgaard, P. B. Norman, L. F. Ollmann, Max Peters, Walter Prenowitz, J. K. Reckzeh, Rose Roll, G. J. Ross, S. G. Roth, John Salerno, Charles Salkind, A. A. Schwartz, Aaron Shapiro, James Singer, J. E. Thompson, H. E. Wahlert, Etta A. Waite, Israel Wallach, Alan Wayne, Margaret C. Weeber, D. E. Whitford, John Williamson, J. M. Wolfe, H. J. Zimmerberg.

Professor W. H. H. Cowles and Mr. Morris Hertzig, Vice-Chairmen of the Section, presided at the morning and afternoon sessions respectively. At the opening of the morning session Dean Nelson S. Hibshman of the School of Engineering welcomed the Section to Pratt Institute. At the opening of the afternoon session a brief business meeting was held, with Professor H. E. Wahlert, Chairman of the Section, presiding. The following officers were elected for the coming year: Chairman, W. H. H. Cowles, Pratt Institute; Vice-Chairmen, Brother Bernard Alfred, Manhattan College, and George J. Ross, Erasmus Hall High School; Secretary, James Singer, Brooklyn College; Treasurer, Aaron Shapiro, Midwood High School. A report was presented by the Committee on Awards and Prizes. It was moved, seconded and carried without opposition that the Committee be continued for another year with a view to presenting its recommendations in writing, subject to the approval of the Executive Committee, with the call of the next annual meeting of the Section. At the close of the afternoon session all those present were invited to remain for tea at the Women's Club as guests of The Faculty Wives' Club of Pratt Institute.

The following papers were presented at the morning and afternoon sessions:

1. *The mathematics of magic squares*, by Harry Sitomer, New Utrecht High School, introduced by the Secretary.

The first n^2 positive integers can be written uniquely in the form $N=na+b$ where $a=0, 1, 2, \cdots, n-1$, and $b=1, 2, 3, \cdots, n$. Thus a magic square can be decomposed into two auxiliary squares, the first (called an A square) containing the a's, the second (called a B square) containing the b's; and corresponding a's and b's occupy those similarly-placed cells occupied by N. Any line in A (row, column, or diagonal) has a sum L_A, and a line in B has a sum L_B. Then $nL_A+L_B=n(n^2+1)/2$, the sum of each line in a magic square. Thus the problem of constructing magic squares is resolved through the construction of A squares and conjugate B squares. The simplest A square is regular, that is, $L_A=n(n-1)/2$. In regular B squares, $L_B=n(n+1)/2$. Often an A square can be transformed into a conjugate B square by adding one to each a and rotating the square 90°. This method of constructing magic squares simplifies the construction, permits a census of a set of squares, and leads naturally to methods for constructing magic cubes, hypercubes, and so forth.

2. *Elementary geometry as an algebraic system*, by Professor Walter Prenowitz, Brooklyn College.

An ordered linear geometry (for example, euclidean, ordered affine, or hyperbolic geometry) can be converted into an algebraic system by defining (1) $a+b$ for distinct points a, b to be the set of points between a and b (segment ab), and (2) $a+a$ to be a. The resulting system is a special type of multigroup or generalized group with many-valued composition. The difference $a-b$ of points a, b, defined as the set of points x for which $b+x$ contains a, is the prolongation of segment ab beyond a, provided $a\neq b$. A formalism is developed which enables one to derive geometrical properties algebraically. The fundamental classes of geometrical figures in non-metrical geometry, (1) convex sets, (2) linear spaces (points, lines, planes, $\cdots$), (3) half-spaces (rays, half-planes, $\cdots$) can be formulated respectively as semi-groups (subsets closed under $+$), subgroups (subsets closed under $+$, $-$) and residue classes of congruence relations.

3. *Mathematics in psychology*, by Dr. Lloyd Henry Beck, Department of Psychology, Yale University, introduced by the Chairman.

The design, execution, and analysis of psychological experiments involve the use of mathematics. The design requires hypothesizing correspondences between psychological elements and mathematical elements, and between psychological operations and mathematical operations. The execution obtains data relative to the hypothesis and requires measurement of the psychological elements. This measurement ranges from a statement of presence or absence at one extreme to quantification at the other. The execution frequently requires apparatus designed on the basis of mathematics in other sciences. The analysis of the data requires at a minimum description of association between variables, that may be described graphically or by an arbitrary function. In addition the design correspondences may enable one to deduce the experimentally-observed association at three levels: (1) the form of the function is given theoretically but the constants have to be determined by the data; (2) the form and some of the constants are given theoretically, the others being determined from the data; (3) both the form and the constants are given theoretically.

4. *What statistics, if any, in a required general mathematics course? Report of an attack on this problem at Queens College,* by Professor T. F. Cope, Queens College.

As a result of about eight years experience with a required mathematics course at Queens College, the speaker reported that the subject of statistics was of intrinsic interest to students in the arts and the social sciences, and it could, in his opinion, be used to great advantage by departments of mathematics in teaching mathematics to these students. He discussed in some detail the topics in statistics that are included in the work of the second term of a required one-year course at Queens College, and the methods used in presenting these topics.

5. *The tentative secondary school syllabus in mathematics for grades 7 through 12,* by Joseph Orleans, George Washington High School, introduced by the Secretary.

The tentative course of study in mathematics for grades 7 to 12, recently prepared by a committee for the New York State Department of Education under the chairmanship of the State Supervisor of Mathematics, must be considered in three parts: (1) the work of the seventh, eighth, and ninth years, (2) the geometry of the tenth year, and (3) the content of the eleventh and twelfth years. The work listed for the seventh, eighth, and ninth years resembles in general the content of what has come to be regarded as the mathematics of the Junior High School. It may involve a change in sequence of topics and shifting of emphasis; but it arouses no serious differences of opinion. The work of the eleventh and twelfth years likewise consists of the present half-year courses in intermediate algebra, trigonometry, advanced algebra, and solid geometry, with topics rearranged into two comprehensive one-year courses, with some deletions and some additions. It is the tenth year that may be the basis for serious differences of opinion among teachers of mathematics. In presenting the year of plane geometry, the committee proposes a new emphasis on the importance of definitions and of assumptions in mathematics and in non-geometric situations. The new course also calls for keeping alive throughout the tenth year the arithmetic and algebraic skills and concepts learned in previous years, and it includes the introduction of a short unit of coördinate geometry and a new emphasis on types of thinking in geometric and non-geometric situations.

6. *A report on high school mathematical preparation,* by Professor F. H. Miller, The Cooper Union School of Engineering.

The speaker reported on the results of a questionnaire, prepared by Professor S. G. Roth and himself, and circulated among fifty-seven high schools in the metropolitan New York area. Questions of varying degrees of difficulty, arranged under twelve headings on topics in algebra and

trigonometry, were listed. Chairmen of high school mathematics departments were asked to indicate whether these items were considered in elementary, intermediate, or advanced algebra courses, in trigonometry courses, or not at all. Results of the questionnaire, obtained from twenty-eight high schools, were presented and discussed by the speaker. The full questionnaire, distribution of replies received, and conclusions drawn from the results will appear in an early issue of the *Journal of Engineering Education*. The hope was expressed that similar projects will be undertaken in other regions so that college mathematics teachers may better determine the degree of preparation of their entering students.

C. B. BOYER, *Secretary*

APRIL MEETING OF THE KANSAS SECTION

The thirty-second annual meeting of the Kansas Section of theMathematical Association of America was held at the University of Wichita, in Wichita, on Saturday, April 19, 1947. Morning and afternoon sessions were held. Professor C. A. Reagan presided at these sessions. The morning session was a joint meeting with the Kansas Association of Teachers of Mathematics.

The attendance was one hundred fifty-seven including the following thirty-two members of the Association: Frances M. Breneman, Virginia L. Chatelain, L. E. Curfman, Paul Eberhart, H. P. Fawcett, W. H. Garrett, Edison Greer, J. R. Hanna, A. J. Hoare, W. C. Janes, L. E. Laird, C. F. Lewis, H. W. Linscheid, Anna Marm, Thirza Mossman, O. J. Peterson, P. S. Pretz, O. M. Rasmussen, C. B. Read, C. A. Reagan, L. M. Reagan, E. S. Robbins, R. G. Sanger, G. W. Smith, R. G. Smith, W. T. Stratton, Sister M. Helen Sullivan, C. B. Tucker, Gilbert Ulmer, E. B. Wedel, A. E. White, Ferna E. Wrestler.

At the business meeting the following officers were elected for next year: Chairman, Sister M. Helen Sullivan, Mt. St. Scholastica College; Vice-Chairman, R. G. Sanger, Kansas State College; Secretary-Treasurer, Anna Marm, Bethany College.

The following papers were presented:

1. *The development of teachers of mathematics*, by Professor Harold P. Fawcett, Ohio State University.

2. *Development of schools of mathematics*, by Professor R. G. Sanger, Kansas State College.

The development of certain schools of mathematics, such as the Pythagorean School, the one at the University of Paris, and the one at Göttingen, was considered. In addition, an attempt was made to ascertain the cultural, political, and social conditions amidst which mathematics might flourish.

3. *On non-euclidean planes*, by S. G. Kneale, University of Kansas.

In the euclidean plane, statements involving the notion of length follow from the three axioms to which the length is subjected. Thus, we arrive at the notion of a non-euclidean plane with a non-euclidean distance, where the last is defined by any numerically valued function of couples of points which satisfies these axioms. Many elementary configurations will change their shape in a geometry with a different definition for distance. Some radical and interesting changes will occur if the distance between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is defined as the greater of two numbers $|x_2-x_1|$ and $|y_2-y_1|$. The perpendicular bisector (that is the locus of points equidistant from two fixed points) is considered in this connection. In the plane with the distance defined as

above, the perpendicular bisector is in general a broken line whose shape depends on the relative position of the two points with respect to the coördinate axes. In two special cases, when the two points have the same x or y coördinate, the "perpendicular bisector" covers an entire area of infinite measure.

4. *Trigonometry as developed by use of vectors*, by F. B. Sleat, Kansas State College.

Elementary principles of vector analysis were employed to derive a great many of the formulae that are to be found in plane and spherical trigonometry.

5. *Pythagorean numbers*, by Professor G. W. Smith, University of Kansas.

Professor Smith exhibited a method which would yield all the primitive Pythagorean number triples, and then proved that for every right triangle whose sides form a Pythagorean number triple the radii of the inscribed and of the three escribed circles are integers.

ANNA MARM, *Secretary*

APRIL MEETING OF THE PACIFIC NORTHWEST SECTION

The first meeting of the newly organized Pacific Northwest Section of the Mathematical Association of America was held at the University of British Columbia, Vancouver, British Columbia, on April 10–11, 1947.

Forty-seven persons attended, including twenty-seven Association members, as follows: J. P. Ballantine, R. F. Bell, Daniel Buchanan, L. G. Butler, D. G. Chapman, C. L. Clark, C. M. Cramlet, Howard Eves, N. S. Free, W. H. Gage, F. L. Griffin, P. C. Hammer, H. H. Irwin, R. D. James, S. A. Jennings, J. R. F. Kent, M. S. Knebelman, A. S. Merrill, W. E. Milne, D. C. Murdoch, F. S. Nowlan, T. S. Peterson, A. R. Poole, J. J. Rowland, W. H. Simons, J. R. Vatnsdal, F. E. Wood.

The officers elected for next year were: Chairman, M. S. Knebelman, State College of Washington; Vice-Chairman, W. E. Milne, Oregon State College; Secretary-Treasurer, F. S. Nowlan, University of British Columbia. The Section voted to accept the invitation of the University of Oregon to hold the next meeting there in March, 1948.

Fifteen papers were presented at the sessions on Thursday afternoon and Friday morning, and a third meeting, devoted to an informal discussion of problems bearing on mathematical instruction in the Pacific Northwest was held on Thursday evening following a complimentary dinner given by the University of British Columbia for the visiting mathematicians and their guests. In addition, on Thursday afternoon, Professor S. A. Jennings, University of British Columbia, gave an interesting and instructive paper entitled *Topographical Methods in Algebra.*

The papers presented at the Thursday and Friday sessions are listed below:

1. *A formula of Liouville*, by Professor H. Gage, University of British Columbia.

The only published proof for formula (Q), stated by Liouville in the sixth of his famous eighteen articles, was given by E. T. Bell, who paraphrased an identity between elliptic functions.

Professor Gage presented an elementary proof. A general formula, which includes several of Liouville's results as particular cases, was also developed.

2. *Tertiary combination*, by Professor A. R. Poole, Oregon State College.

A set of postulates for a system with a single tertiary law of combination was given, and the fact that the tertiary combination cannot be obtained by suitably combining binary combinations was demonstrated.

3. *Representations of the Sylow subgroups of the symmetric group of degree* p^n, by Professor S. A. Jennings and D. G. Duncan, the latter introduced by Professor Jennings, University of British Columbia.

L. Kaloujnine has shown that the Sylow subgroups of the symmetric groups of degree p^n when p is prime may be represented as permutations of n dimensional vectors with coefficients in $GF(p)$. These representations were discussed and new results on the structure of these groups were obtained.

4. *On subgroup chains in a finite group analogous to the upper and lower central series*, by Professor D. C. Murdoch, University of British Columbia.

Various series of normal subgroups of a group G, analogous to the upper and lower central series, were discussed, in particular those series obtained by the use of such "commutators" as $a^s \cdot a^{-1}$ where s is an outer automorphism of G, and $(a^{s^{-1}}b^{s^{-1}})^s \cdot b^{-1}a^{-1}$ where s is an arbitrary permutation of the elements of G.

5. *Interference with a controlled process*, by Professor P. C. Hammer, Oregon State College.

In the terminology of statistical quality control a process is in control with respect to the production of a certain quality characteristic measured by a variable x if a sequence $x_0, x_1, x_2, \cdots, x_n$ of realizations of x might reasonably have been drawn at random from a fixed frequency distribution of x-values. If the occurrence of x_{k-1} results in a designed change of the distribution from which x_k is drawn, then the process is no longer in control. Letting a be a standard value desired for x, and m_k the mean of the distribution of x_k, Professor Hammer assumes interference in such a fashion that $m_k = m_{k-1} - c(x_{k-1} - a)$ where c is a positive number. The distribution of $\xi_k = x_k - m_k$, $k = 0, 1, 2, \cdots$, is assumed fixed, and the ξ_k are assumed to be mutually independent. This form of interference results in an increased standard deviation of x_k and a non-zero correlation between x_k and its predecessor x_{k-1}. The existence of an asymptotic distribution of x_k for $0 < c < 2$ is demonstrated.

6. *Successive approximations adapted to computation of complex roots*, by Professor W. E. Milne, Oregon State College.

A computational scheme is devised for practical calculation of irreducible quadratic factors of the form $z^2 + pz + q$ of a polynominal with real coefficients. The procedure is to approximate the real quantities p and q by a process of successive approximations. The procedure is devised with special reference to convenient computation with a calculating machine.

7. *On the numerical solution of linear integral equations*, by Professor G. Hacker, State College of Washington.

Linear integral equations of the first and second kinds are solved numerically by Gaussian quadrature in the form

$$\varphi(x) = f(x) + \lambda \sum_{i=1}^{n} A_i \cdot K(x, \mu_i)\, \varphi(\mu_i)$$

for $x=\mu_i$ where μ_i are the Legendre polynomial real zeros, and A_i the Gauss weight factors. Interpolation then provides the required numerical description of $\phi(x)$. Comparison of the method with those of Prasad and Bateman is made.

8. *Graphical solution of first order linear differential equations*, by Professor J. P. Ballantine, University of Washington.

The general equation discussed is $y'=[f(x)-y]/T(x)$, and y' is seen to be the slope of the line from any point (x, y) of the integral curve to $(x+T, f(x))$. The locus of the latter point can be drawn in advance, so that the tangent can be quickly drawn at any point of the integral curve. The solution is then obtained in the form of a graph which is really a series of straight line segments. Good results can be obtained, since each straight line segment is tangent to the curve at its midpoint and not at one of its ends. Special cases when $f(x)=0$, and $T(x)$ is a constant or a multiple of x are important. The method is very rapid for these cases.

9. *The general problem of stability of economic equilibria*, by Professor D. G. Chapman, University of British Columbia.

If a dynamic economic situation is represented mathematically by a set of equations involving time, then to determine the stability of an economic equilibrium so defined, it is necessary to study the behaviour of the solutions of this set of equations for large t. There is also a further question, namely, will the process of adjustment used by the individuals within the situation result in the same or any equilibrium point being attained? By the use of difference equations these two problems are studied, and it is shown for various economic situations that the problem of stability can be fully solved by answering both questions.

10. *Oscillating satellites with the force varying inversely as the nth power of the distance*, by Professor Daniel Buchanan, University of British Columbia.

The restricted problem of three bodies was discussed for the case when the force varies inversely as the nth power of the distance. The straight line and equilateral triangle equilibrium points were found, and the oscillating satellites near these points were obtained.

11. *Some related space curves*, by Professor Howard Eves, Oregon State College.

Let $x=x(s)$ be the parametric representation, in terms of the arc length s as parameter, of a given skew curve, and let α, β, γ be the unit vectors in the directions of the tangent, binormal, and principal normal respectively. The given curve may then be represented as $x=\int\alpha ds$. Associated with the curve are the two curves $x_2=\int\beta ds$, and $x_3=\int\gamma ds$. This speaker discussed some of the geometrical relations existing between the given curve and these related curves. In particular, certain linear combinations of the given curve and its related curves were considered. In addition to a number of new theorems, some known results were obtained in a novel way, for example, the general equations of all Bertrand curves and of all Mannheim curves were found.

12. *New methods in teaching polar coordinates in analytic geometry*, by Professor F. S. Nowlan, University of British Columbia.

Two essentially different systems of polar coördinates were considered. In the first, ρ is never negative, whereas in the second ρ is never positive. The replacements of ρ by $-\rho$ and θ by $\pi+\theta$ carry one from either system to the other. The methods developed enable one to avoid certain difficulties associated with the usual treatment of polar coördinates. Furthermore, it was shown that these methods have important applications to the calculus.

13. *The teaching of elementary calculus*, by Professor R. D. James, University of British Columbia.

Most of the calculus texts in use at present begin with a more or less systematic discussion of variables, functions, and limits, and then proceed to the topic of differentiation. It is Professor James' opinion that such a traditional beginning is unwise and that the most natural approach to the calculus is by way of the definite integral. He set forth reasons for this point of view and outlined the necessary steps in the alternative approach.

14. *A graphical treatment of the covariants of a binary cubic*, by Professor F. E. Wood, University of Oregon.

With three distinct points A, B, C of the complex plane, with complex coördinates a, b, c respectively are associated three points A', B', C' by making the cross ratios of (AB, CC'), (BC, AA'), (CA, BB') each equal to a given complex number α. Using all such α there is established a set Γ of point triples, associated with the given triple. In this paper are obtained the geometric relations between the triples of Γ and their relations to some of the covariants of the binary cubic form obtained from the cubic equation which has a, b, c as roots.

15. *New formulas for the numerical integration of $y'=f(x, y)$*, by R. J. Brown introduced by Professor F. L. Griffin.

It is assumed that the solution $y=y(x)$ of the differential equation $y'=f(x, y)$ has a valid Taylor series near the given point (x_0, y_0), from which three other points (x_1, y_1), (x_2, y_2), (x_3, y_3) at equal intervals, $\Delta x=h$, have been calculated initially to the desired accuracy. To predict a fourth such point, y_4 is taken as a linear function of y_0, y_1, y_2, y_3, and derivatives y_0', y_1', y_2', y_3'. The eight coefficients are determined so as to make the linear formula agree with the Taylor series through h^7, but the resulting formula is unsatisfactory because its large coefficients magnify the errors in the initially computed y_1, y_2, y_3. By omitting the terms in y_2 and y_0', however, a single infinitude of formulas is obtained, each of which is correct through h^4, some of which give results above the true y_4 and some below. Applications are made to some standard textbook examples. In one familiar case one formula gives eight decimal places correct when the initial computed values are obtained with sufficient accuracy.

F. S. NOWLAN, *Secretary*

CALENDAR OF FUTURE MEETINGS

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pennsylvania State College, Spring, 1948
ILLINOIS
INDIANA
IOWA, Fairfield, April 16–17, 1948
KANSAS, Atchison, April 10, 1948
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, January 24, 1948
OHIO
OKLAHOMA
PACIFIC NORTHWEST, Eugene, Oregon, March, 1948
PHILADELPHIA, Philadelphia, Pa., Nov. 27, 1948
ROCKY MOUNTAIN
SOUTHEASTERN, Charleston, S.C., March 19-20, 1948
SOUTHERN CALIFORNIA, Redlands, March 13, 1948
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, Schenectady, N. Y., May 1, 1948
WISCONSIN, Beloit, May 8, 1948

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THE TEACHING OF MATHEMATICS IN COLLEGES AND UNIVERSITIES*

E. P. VANCE, Oberlin College

1. Introduction. In the early part of 1947 the President of the United States requested a report on science teaching in the colleges and universities of this country; he desired information on the kinds of research being done both in and out of government, where research was being done, how it was being financed and administered, and so on. In connection with this request the President's Scientific Research Board asked the American Association for the Advancement of Science to study the effectiveness of science teaching in colleges and universities. This request was turned over to the Cooperative Committee on Science Teaching sponsored by the A.A.A.S.† The findings of this Committee have formed an important part of the report made by the Scientific Research Board on our scientific personnel resources.‡ The Board also gathered data on total enrollment in science classes, the number of scientists now employed in various types of institutions, and other matters related to the development of scientific talent.

In order to obtain necessary information the Cooperative Committee decided to solicit data and opinions from chairmen of science departments in colleges and universities. Thus the Scientific Research Board sent out a request for information in the form of a comprehensive questionnaire to two hundred and fifty such chairmen, five at each of fifty colleges and universities. Since these fifty institutions had granted more than ninety percent of the Ph.D.'s in science during the past ten years, the composite replies constitute a significant contribution to the study of the effectiveness of science teaching and a general view of the over-all picture as it occurs today. Forty-one of the fifty institutions which received requests for information in mathematics replied, some in more detail than others. The purpose of this article is to summarize those findings which seem to be of interest to mathematicians.

The questions on the questionnaire were general in character and were placed in seven different categories, each of which was believed to represent an area of problems pertinent to the effectiveness of science teaching. Since the same questionnaire was used for all the sciences, some of the topics mentioned were, of course, not particularly apposite to the field of mathematics.

2. The size of enrollments. In the undergraduate schools during the year 1946–47, it was generally true that there were between two and three times the number of students enrolled in mathematics in 1940. Only one school re-

* The author wishes to thank Dr. M. H. Trytten, Director of the Office of Scientific Personnel, National Research Council, for making available to the author the material on mathematics collected by the President's Scientific Research Board.

† Professor Raleigh Schorling of the University of Michigan is the representative of the M.A.A. on this Committee.

‡ See Manpower for Research, Vol. IV of Science and Public Policy, Oct. 11, 1947, U. S. Government Printing Office, Washington 25, D. C., Price 35 cents.

ported no increase, while the largest increase was five hundred percent at one of the large western universities. In many cases the enormous influx of students made it necessary to appoint as regular instructors in mathematics a large number of individuals who would not have been appointed under ordinary conditions. Such teaching personnel would be unable to teach the more advanced courses; thus a serious problem is indicated if beginning students continue in their mathematical training. Also many of the large universities have employed many graduate assistants, and have been forced to have freshman and sophomore mathematics taught to a large extent by these graduate assistants. Some are excellent teachers, while others are of questionable ability, and under ordinary circumstances would not be given teaching assignments. Almost all institutions report that classes are much larger. In fact, the shortage of classrooms, along with the inadequacy of essential staff, has forced several schools into handling students in classes of one hundred or more. It is common opinion that the general quality of undergraduate instruction has deteriorated in the past few years.

Several methods are being used to remedy this situation and to raise the effectiveness of the teaching of mathematics. Some schools are planning the courses ahead in detail, and along with this, employing standardized examinations. Others are sectioning their students according to the background and previous records of the students, then distributing the experienced and proven teachers to best advantage. In many of the larger schools graduate fellows and inexperienced instructors have met regularly with a supervising professor to map out methods of presentation; this has led frequently to a plan for close supervision of course-content, methods of presentation, and departmental testing. Other aids such as visiting of classes and conferences with individual teachers have been employed.

There were no reports of undergraduate students being denied admission to mathematics courses. This is indeed a fine record. As soon as more competent teachers are available, it is generally believed that undergraduate teaching will measure up again with that in 1940.

New teachers must come from those students now in our graduate schools. Here the picture is very favorable, although it must be remembered that many graduate students enrolled at the present time will obtain their degrees, and find their way into research with the government, or in industry, rather than make teaching their life profession. Also the question naturally arises whether the graduate schools are able, with the huge increase in the size of enrollment, and the shortage of capable personnel, to give competent training to their graduate students.

At present the registration in mathematics in the graduate schools is at least double any enrollment prior to the war, and all indications are that it will continue to increase rapidly in the years just ahead. This is partly a result of the accumulation of many men who had planned to go directly from college to graduate school, but unfortunately had their training interrupted, after only

the undergraduate course was completed. This group, plus many who finished their undergraduate courses this past spring and were offered attractive assistantships, have filled the graduate schools to overflowing. Although some schools have increased their graduate registration only slightly, others have felt a tremendous increase. For example, one midwestern university in 1940–41 had a graduate enrollment in mathematics of 243 course-registrants while in 1946–47 there were 1344 course-registrants.

No definite generalization can be made with regard to any significant change in the quality of graduate teaching as reported by representatives of the universities. Typical replies were the following from three different institutions:

On the graduate level, instruction is perhaps superior to what we had before the war, but if enrollments should increase, the manpower situation will become hopeless.

Graduate instruction has been adversely affected by the loss during the war period of full time members of the staff, which have not, up to the present time, been repaired by satisfactory replacements (not to mention additions).

At the graduate level the regular staff members of prewar status are seriously overloaded with the probability that graduate teaching will ultimately suffer if relief does not become available. This problem will become increasingly acute as graduate students reach the stage of thesis research in increasing numbers.

3. Space and equipment. Existing needs for space and equipment are more pertinent in the teaching of the experimental sciences, although to some extent they are also factors in the effectiveness of mathematics teaching. Office space seems to be definitely inadequate, with the result that informal conferences between faculty members and among faculty members and students are difficult. Also the present lack of sufficient classrooms adequately equipped with blackboards has an important bearing on teaching; because of this shortage in several universities, classes are being held continuously from 8:00 A.M. until 10:00 P.M. Nor does there seem to be much hope for any relief in the future. One university, for example, states that in a new building for its College of Liberal Arts the space assigned to mathematics will be fifty percent less than the department occupied twenty years ago, and this in spite of expansion of both the student body and the teaching staff.

The availability of research journals and reference treatises is essential to work in mathematics. Most institutions which reported have been able to maintain their libraries except for some books and periodicals from foreign countries; in many cases these latter items are now being obtained. The need for funds, of course, is ever present.

Other than library facilities, the lack of equipment affects a mathematics department mainly in its courses in statistics. Due to the increase in importance and emphasis on statistics, and the resulting large enrollments, problems in teaching have arisen. Computing machines, although again available, are expensive. Many colleges obtained some computing machines as surplus property, but a majority of these machines lacked the multiplication and division features and hence were not well adapted for use in statistics classes. The situation in one midwestern university, although much more desperate than at most institutions,

might be presented. In the second term, 1942–43, there were approximately 100 registrations in statistics, while in the fall term, 1946–47, the registration for the same course had grown to over 1200. Twenty-two computing machines were available for student use in 1943, and there had been no increase when the report was made. The former statistics laboratory has been taken over by the psychology department, and thus the mathematicians are now limited to a computing room which will hold at most sixteen students. To offset this, it should be stated that the large majority of replies were negative to the question, "Is the quantity and state of repair of your instructional equipment a major limiting factor in determining the effectiveness of teaching in your department at present?"

4. Availability of teachers. In the majority of universities, the undergraduate staff of teachers is much larger and less competent than in 1940. Moreover, the size of the staff has not increased in proportion to the increase in the student body. The problem of adequate staff is made more difficult because of the unpredictability of future needs. Many of the new teachers in colleges and universities are former high school teachers. Also, in certain localities, men from research laboratories have given excellent assistance by teaching of evening classes.

A common response pertaining to the size of the permanent and graduate staff seemed to be, "Our present staff is essentially the same in number and competence as before the war." The situation is not entirely a happy one, however. One department chairman of a midwestern university commented,

> Our present staff numbers about the same as the pre-war staff, but it does not compare quite so well in competence. War work took a good portion of a carefully built-up pre-war staff. Even in normal times such a staff can not be rebuilt over night. Competition of other institutions has considerably affected our staff. Top men are being "pirated" . . . .

From an Eastern university came the comment:

> In 1940 we had thirteen full-time staff members of our department, all with the Ph.D. degree in mathematics, and about eight or nine part-time teaching assistants. This year (1947) we have nine full-time staff members and thirty part-time teaching assistants. We are trying to improve this situation for next year; but the competition for well trained men is keen, and we are finding it difficult to enlarge our full time staff.

These indicate that competition from other universities is felt. Also the competition with industry and government is important. For example, another head of one of the midwestern institutions stated,

> Government competition has taken away the head of our statistics department, and to date we have been unable to replace him. . . . Two of our good mathematics teachers who went to work with the Bell Aircraft Company during the war have not returned because we are not able to offer them a salary commensurable with what they were receiving in industry.

The economic law of supply and demand is fundamental throughout the picture. During the war young students in mathematics were unable to pursue their graduate studies under normal conditions. A large portion of the graduate students were drafted or drawn off into war jobs which offered them no oppor-

tunity for completing their graduate training. Thus in the war years, the number of Ph.D.'s granted in mathematics decreased very rapidly until in 1944 it was only about thirty percent of the normal figure. A corresponding decrease also occurred in the number of Master's degrees granted. Thus, mathematics is now faced with a tremendous shortage of men who have completed their graduate training. When this shortage is considered along with the fact that many colleges and universities are expanding, while some institutions that never required the Ph.D. degree for staff appointments are now trying to build graduate departments, the seriousness of the situation becomes evident. Also the competition for scientific personnel with government and industry, a competition which will be felt for some time, will affect the quality of available staff members at all universities. This is particularly true in the field of mathematical statistics where men who have just obtained the Ph.D. degree are being offered salaries over twice those normally available for men at that stage of accomplishment.

It is generally felt that present salary schedules for science teachers are having and will have in the future an important bearing on the quality of science teaching. If universities can not compete with government and industry for the service of a large portion of the best young scientists available, and consequently these men are lost to the academic world, then the teaching profession will be forced to fill their places with men of inferior quality. This will most certainly have an adverse influence on the training which young scientists receive in the future.

The question was asked: "How effective is science teaching at the graduate level for the training of teachers of science and how might this effectiveness be improved?" Most of the heads of mathematics departments who replied believed that the training given was entirely satisfactory. In fact, only six had any additional comments to make. In some cases the effectiveness of teachers who do research and who must provide leadership in guiding graduate students of science is impaired by too high teaching loads. Also the effectiveness in training future teachers might be enhanced according to some, if more general courses were offered and more use were made of the applications of theory in mathematical and physical problems. Some implied that more training in philosophy and history of mathematics, together with its applications to other sciences, would improve the quality of output. Some insisted that too little attention is paid to the development of teaching skills; in this connection it is suggested the method of practice teaching used for training secondary school teachers might be of some help. In fact, as one man commented (and many would agree, I feel confident),

> There has been practically no training of graduate students for the teaching of mathematics on the college level. I think those departments of mathematics, who turn out the most Ph.D.'s and consequently the largest proportion of college teachers in this subject, should make definite provision for training of graduate students in good teaching habits and methods for their subject.

As to the desirability of having more women teachers of science at the col-

lege and university level, there seems to be a definite hesitancy. Possibly this is caused by the belief stated on several questionnaires that very few women are good enough mathematicians to make a substantial contribution to the training of graduate students. Many institutions, however, would be quite willing to have more women teachers at the college level if they met the scientific qualifications for the position.

5. The influence of the Federal government. At the present time the presence of Federal contracts for research has had little obvious effect on the quality of science teaching in departments of mathematics. In four or five cases the granting of such contracts was reported as beneficial to the effectiveness of teaching, while it was considered harmful by only two.

With regard to policies that educational institutions should follow in the acceptance of Federal contracts for research, including a consideration of the effects of such contracts on teaching, there were several distinctly different replies. Some stated that contracts should be accepted covering fundamental research only, and only when the proposed contracts fit into existing research programs. Others felt that all contracts should be accepted when at all possible, since in the near future there would not be enough stipends available to subsidize graduate students, and research contracts would supply this deficiency. Still other schools felt that such contracts might hamper the universities unduly in the execution of their primary function of training students for teaching and fundamental research. Since this is such a fundamental matter, and one to which most mathematicians have given much thought, I quote the comments of three influential mathematicians, and chairmen of large departments, in this country:

1. I believe that educational institutions in the acceptance of Federal contracts for research should stipulate that in most cases the research men involved should be appointed on a split basis on which about half time should be spent in university teaching and only half time on the contracts. It should also be specified that so far as national safety is not involved, individuals should retain the right of publication. Under these conditions the teaching profession will be enriched by the research activity of the teachers without the customary cost for research purposes of the staff. If scientists devote their entire time to research, the total progress of the research itself may be very great but the benefits to teaching and to the students are likely to be negligible.

2. I certainly do not believe an educational institution should secure a Federal grant, then rob some other institution for men to do the work. This takes men out of teaching where they are so much needed at present. The teaching will certainly be poorer and the rate of training men will be reduced. I believe too much emphasis is being put on research *at the present time.* For a few years the emphasis should be put on teaching and training young men. Unless this is done, the situation will get worse instead of better. Taking men from the teaching staff of our universities is "killing the goose that laid the golden egg." Why doesn't the Federal government do something to help the universities obtain trained staffs in the immediate emergency of increased enrollments? These large enrollments are due largely to the action of the government in sending the veterans to college.

3. In the acceptance of Federal contracts for research, consideration should be given to the following:

(a) Is the research significant in the judgment of men who are competent in the field?

(b) Would the work be done if a Federal contract were not forthcoming?

(c) Is the university in question avoiding its proper obligations? If a university does not encourage and support research, it is no longer a university.
(d) Would the granting of such a contract accentuate the unfortunate separation of education and research?
(e) Under Federal contracts, to what extent would research be directed into specific channels, which may or may not be desirable?

It is interesting to note that several universities, requested by the Federal government to provide assistance in training scientific employees for them, did so willingly, and reported that in no case did it affect the quality of teaching. For example, graduate instruction was provided for qualified personnel at Fort Monmouth; instruction was given to members of the Ames Aeronautical Laboratory, Moffett Field; courses were set up for employees of the United States Navy at the Underwater Sound Laboratory in New London; and, of course, several schools are participating in the NROTC program, thereby offering a place for the training of "junior mathematicians."

6. The role of research. Only a small number of institutions stated that there were any significant changes in the type or amount of research now being carried on. If a change was reported, it was either an increase in applied research, or an increase in both applied and fundamental. The reasons given in case of an increased amount of research were either (1) an increase of staff, (2) an increase of university funds for research, or (3) a contract or project with some branch of the Government.

The comments immediately above were the result of the question, "Has the role of research in your department changed significantly since before the war? Are you, for example, doing more or less applied research as distinguished from fundamental research?" The idea was expressed frequently in this connection that one must not place too much emphasis on the distinction between pure and applied mathematics; mathematics, is indivisible, and one must not emphasize the diversity but rather its unity. Research must always remain an excursion into the unknown, with truth alone as its objective.

With the present overcrowded enrollments in all universities, it is gratifying to know that over three-fifths of the reports indicated that staff members had adequate time and opportunity for research and study.

7. General. The last section of the request for information concerning the effectiveness of science teaching should be considered in two parts. In the first, the following questions were asked: "Is sufficient attention given, particularly in the undergraduate curriculum, to

(a) the meaning and significance of the scientific method?
(b) the recent advances in the field of science being studied?
(c) the fields of science related to the major field of study and the overlapping fields as, for example, biophysics and geochemistry?
(d) the "general education" of the future scientist—training in other fields—training for citizenship?"

Because of the generality of the questions, it seems impossible to summarize the replies. Many who replied answered parts (a), (c) and (d) in the affirmative, though not in emphatic fashion. With respect to point (b), it seemed generally agreed, especially in any undergraduate curriculum, that keeping abreast with the recent advances in mathematics would be an impossibility. It should be stated in this connection, that there appears to exist a definite dissatisfaction with present freshman and sophomore courses. Complete and thorough studies of this undergraduate curriculum are taking place in many institutions.

What is somewhat alarming to the writer is the fact that many mathematicians in this country stated that the above questions were not pertinent to mathematics. It is my personal feeling that insufficient attention has been given to the meaning and significance of the scientific method, that the tendency has been to train students well in particularly narrow fields, but not to give them a broad background in the whole field and in related fields.

The second part of this general section asked for a discussion of the somewhat controversial statement: "This country has, in the past, produced about one-seventh of the Nobel prize winners in science. Do you consider this fact to be a reflection on the quality of science teaching in this country in the past?" Since Nobel prizes are not awarded in mathematics, many declined to comment. Some admitted that this was a reflection on the quality of science teaching, since our educational system was too "practical"; that is, if a future use of knowledge is not in prospect, the knowledge is considered useless. This attitude, they maintained, was destructive of real scientific mentality. The majority contended, however, that the situation with respect to Nobel prize winners was no reflection on science teaching in this country. Several reasons were stated. American educational philosophy is to extend educational opportunity to the many, and not to the intensive training of a few. Also, it is asserted, there seems to be in this country a smaller proportion of time and energy which can be given to research, in comparison with that of scientists of other countries. In this connection also the belief was expressed that there has not been sufficiently early recognition of research ability in this country, and opportunity has not been given scientists to carry on research, unhampered by activities necessary to the individual's economic situation.

DENUMERABILITY OF THE RATIONAL NUMBER SYSTEM

L. S. JOHNSTON, University of Detroit

It has long been known that the set of rational numbers is denumerable, and several counting systems have been devised. It is proposed in this note to exhibit a counting system which furnishes the solution for each of the following problems:

(i) To find the ordinal number for a given rational; for example, to find the ordinal number of the rational 37/11.

(ii) To find the rational number corresponding to a given ordinal; for example, to find the 593rd rational number.

If

$$0 \leqq \frac{x}{y} < \frac{z}{w}, \quad \text{then} \quad \frac{x}{y} < \frac{x+z}{y+w} < \frac{z}{w},$$

whence also

$$\frac{x}{y} < \frac{2x+z}{2y+w} < \frac{x+z}{y+w} < \frac{x+2z}{y+2w} < \frac{z}{w}.$$

This is of course the density property of the rational number system, and between any two fractions we may insert a fraction of intermediate value as shown in the last display. We shall use $[m, n]$ for the fraction $(mx+nz)/(my+nw)$. Then we have the array

$$\begin{array}{lllllllll}
[1,0] & [0,1] \\
[1,0] & [1,1] & [0,1] \\
[1,0] & [2,1] & [1,1] & [1,2] & [0,1] \\
[1,0] & [3,1] & [2,1] & [3,2] & [1,1] & [2,3] & [1,2] & [1,3] & [0,1] \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

where we have omitted for typographical convenience the $<$ sign between two successive terms. This array, which we shall call S, has an infinite number of rows, but a finite number of elements in each row. For $x=0$, $y=z=w=1$, $m=0, 1, 2, 3, \cdots$, $n=1, 2, 3, \cdots$, $[m, n]$ is the proper fraction $n/(m+n)$; if m and n are taken always relatively prime, as they are in this note, then $[m, n]$ is a proper fraction in its lowest terms.

We shall denote by $s(i, j)$ the term in the ith row and the jth column of S, and shall mean by $s(i, j)+s(i, j+1)$ the vector addition $[m, n]+[p, q] = [m+p, n+q]$. Inspection of S, in the light of its construction, discloses the following properties

(a) $$s(i, j) = s(i+1, 2j-1),$$

(b) $$s(i, j) + s(i, j+1) = s(i+1, 2j).$$

From (a) it is evident that an element appearing in S at all appears an infinite number of times thereafter, but always in an odd-numbered column after its first appearance. If, therefore, an element $[m, n]$ appears in an even-numbered column of S, it can not have previously appeared in any column of S, nor can it ever appear again in any even-numbered column.

Now delete from S every term of the first column except the first element, $s(1, 1)$, every term in odd numbered columns after the first, and also the first element, $s(1, 2)$, in the second column. Then the residual array includes zero and every proper fraction (note that it has already been stated that every $[m, n]$ will be shown to appear in S) but does not include unity. Denoting by $R(0)$ the set of all proper fractions together with zero, every number of $R(0)$ appears in S.

We shall call the elements of the residual array described above counted elements of S. A little inspection shows that the number of counted elements in the first i rows of S is 2^{i-1}, and therefore that the element $s(i, 2j)$ is the $(2^{i-2}+j)$th counted element of S. We shall define $Q(m, n)=2^{i-2}+j$, where i and j depend for their values upon some method of computing them from m and n. This number $Q(m, n)$ is the ordinal number of the proper fraction $n(m+n)$ in the set $R(0)$.

To compute i and j for a given m and n, we note first that

$$\begin{aligned}[m, n] &= m \times [1, 0] + n \times [0, 1] \\ &= m \times s(1, 1) + n \times s(1, 2).\end{aligned}$$

To proceed beyond this point it is expedient to exhibit a specific case, the general method being obvious thereafter. Consider the fraction 4/11, or

$$\begin{aligned}[7, 4] &= 7s(1, 1) + 4s(1, 2) \\ &= 3s(1, 1) + 4s(1, 1) + 4s(1, 2).\end{aligned}$$

But from property (b) of S, we have $4s(1, 1)+4s(1, 2)=4s(2, 2)$; from property (a) we have $3s(1, 1)=3s(2, 1)$. Then we have

$$\begin{aligned}[7, 4] &= 7s(1, 1) + 4s(1, 2) \\ &= 3s(2, 1) + 4s(2, 2) \\ &= 3s(3, 2) + s(3, 3) \\ &= 2s(4, 3) + s(4, 4) \\ &= s(5, 5) + s(5, 6) \\ &= s(6, 10) + 0\end{aligned}$$

showing that $[7, 4]=s(6, 10)$, whence $Q(7, 4)=2^{6-2}+5=21$. Hence 4/11 is, in our numbering system, the 21st rational fraction, counting zero as the first.

It will be noted that we have developed above an algorithm quite suggestive of the Euclidean Algorithm on Highest Common Factor. It might also be called a "Method of Finite Descent."

The algorithm can be constructed more simply. Write the columns

(A)	(B)
7,4	1
3,4	1
3,1	2
2,1	3
1,1	5
1,0	10

Column (A) is the set of coefficients of the terms $s(i, j)$ in the first column on the right hand side of the algorithm, and column (B) is the set of j's in the first s of each pair in the algorithm. The sequence of elements in (A) is quite apparent on inspection, and needs no elaboration.

Now let $A(k)$ be the kth element in (A), and let $B(k)$ be the kth element in (B). If the second number in $A(k+1)$ is the same as the second number in $A(k)$, we say that $A(k+1)$ shows a permanence; if the second number in $A(k+1)$ is different from the second number in $A(k)$, we say that $A(k+1)$ shows a variation. Thus $A(1)=7, 4$; $A(2)=3, 4$; and $A(3)=3, 1$. $A(2)$ then shows a permanence, while $A(3)$ shows a variation. (It is quite evident that we are borrowing an idea from Descartes here). To construct column (B), we note that the first element of (B) is always unity. If $A(k+1)$ shows a permanence, then $B(k+1) = 2\times B(k)-1$, while if $A(k+1)$ shows a variation, then $B(k+1)=2\times B(k)$. Thus it will be noted that $A(3)$ shows a variation, and $B(3)=2\times B(2)$. But $A(4)$ shows a permanence, whence $B(4)=2\times B(3)-1$. This gives column (B) quite readily. It will be noted that the last two elements of (A) are always 1, 1 and 1, 0, which shows a variation in the last element. This guarantees that the last element of (B) is always even, as has been predicted. It is also quite evident from the nature of the algorithm that any given number pair $[m, n]$ reduces, by a finite number of steps, to the final forms in (A). This bears out a prediction already mentioned twice, namely, every proper rational fraction does appear in S. The number of elements in (A), and of course in (B), is the required value of i, and the last element in (B) is the required value of $2j$. Some emphasis upon the algorithm is worth while. Consider the number pairs [23, 7] and [7, 23].

For the pair [23, 7]:

(A)	(B)
23,7	1
16,7	1
9,7	1
2,7	1
2,5	2
2,3	4
2,1	8
1,1	15
1,0	30

For the pair [7, 23]:

(A)	(B)
7,23	1
7,16	2
7, 9	4
7, 2	8
5, 2	15
3, 2	29
1, 2	57
1, 1	114
1, 0	228

whence $[23, 7]=s(9, 30)$ whence $[7, 23]=s(9, 228)$.
This is an interesting property of S. It is evident that S must exhibit $[m, n]$ and $[n, m]$ in the same row. Furthermore, since there are, in the ith row of S, $2^{i-1}+1$ elements, it follows that if $[m, n]=s(i, 2j)$, then $[n, m]=s(i, 2^{i-1}-2j+2)$. In our problem above we have $30+228=2^{9-1}+2$. This is often a convenient check on the computations in (B).

The algorithm can be condensed still further. Inspection shows that if $A(k)$ is followed by a sequence of r variations, then $B(k+r)=2^r\times B(k)$, while if $A(k)$ is followed by a sequence of r permanences, then $B(k+r)=2^r\times B(k)-(2^r-1)$. To apply this, consider again the pair $[23, 7]$. Here $[23, 7]$ is followed by three permanences, reducing to $[2, 7]$. Then $B(4)=1$. $[2, 7]$ is reduced by three variations to $[2, 1]$, giving $B(7)=2^3\times B(4)=8$. Then $[2, 1]$ is reduced by one permanence to $[1, 1]$, giving $B(8)=15$. Finally, $[1, 1]$ is reduced to $[1, 0]$ by one variation, giving $B(9)=30$.

We may set up a tabular arrangement for the procedure above.

(A)		(B)
23, 7		$B(1)=1$
	$3P$	
2, 7		$B(4)=1$
	$3V$	
2, 1		$B(7)=8$
	$1P$	
1, 1		$B(8)=15$
	$1V$	
1, 0		$B(9)=30$

Similarly for $[7, 23]$

(A)		(B)
7, 23		$B(1)=1$
	$3V$	
7, 2		$B(4)=8$
	$3P$	
1, 2		$B(7)=2^3\times 8-(2^3-1)=57$
	$2V$	
1, 0		$B(9)=2^2\times 57=228.$

The similarity to the Euclidean Algorithm is even more conspicuous here.

The algorithm is reversible. Consider, for example, the problem of finding $[m, n]$ for $s(6, 10)$. Here write (B) first:

10
5
3
2
1
1

continuing the set of unit elements at the end until six elements appear, since $i=6$. Then write alongside (A) the column (B), remembering that if an element of (B) is even, then the corresponding element of (A) must have shown a variation, while if an element of (B) is odd, then the corresponding element of (A) must have shown a permanence. Then we have

(B)	(A)
10	1, 0
5	1, 1
3	2, 1
2	3, 1
1	3, 4
1	7, 4

whence $s(6, 10)=[7, 4]=4/11$. Similarly, for $s(9, 228)$ we have

(B)	(A)
228	1, 0
114	1, 1
57	1, 2
29	3, 2
15	5, 2
8	7, 2
4	7, 9
2	7, 16
1	7, 23

whence $s(9, 228)=[7, 23]=23/30$.

We have shown how to compute $Q(m, n)$ for a given i and j. The procedure here is reversible. For example, let us find $[m, n]$ from $Q(m, n)=21$. Here we have $2^{i-2}+j=21$. Since j is never zero, we must find the largest integer for i consistent with $2^{i-2}<21$. Then $i=6$, whence $j=5$, and $2j=10$. Then we must find $[m, n]$ such that $[m, n]=s(6, 10)$. This gives, as we have seen, $[m, n]=[7, 4]$. Similarly, to find $[m, n]$ if $Q(m, n)=242$, we have $2^{i-2}<242$, whence $i=9$. Since $2^{i-2}+j=242$, we have $j=114$, $2j=228$. Then we must find $[m, n]$ from $s(9, 228)=[m, n]$, and we have seen that the solution is $[7, 23]$.

We are now prepared to order the whole field of rational numbers. Let $x=p$, $z=p+1$, with the same hypotheses on y, w, m, and n as before. Let $R(p)$, $(p=0, 1, 2, 3, \cdots)$, be the set of positive numbers of the form $p+n/(m+n)$. Let $E(p, Q)$ be the number in $R(p)$ whose ordinal number in $R(0)$ is determined by $Q(m, n)$. It is easy to see from our foregoing remarks that if a proper fraction bears the ordinal number $Q(m, n)$ in the set $R(0)$, then $p+[m, n]$ bears the ordinal number $Q(m, n)$ in the set $R(p)$. Thus $4/11$ is the 21st number in the set $R(0)$, $15/11$ is the 21st number in $R(1)$, and so on. Then we have $E(0, 21)=4/11$, $E(1, 21)=15/11$, and so on. Now form the array:

$$\begin{array}{cccc}
E(0,1) & E(0,2) & E(0,3) & E(0,4)\cdots \\
E(1,1) & E(1,2) & E(1,3) & E(1,4)\cdots \\
E(2,1) & E(2,2) & E(2,3) & E(2,4)\cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & E(p,Q) & \cdots
\end{array}$$

Counting the elements of this array in the familiar Cantor order, it is easy to see that the element $E(p, Q)$ is the $[(p+Q)(p+Q-1)/2+Q]$th term in the array. If we were interested only in positive rationals, this last expression would be the ordinal number of $E(p, Q)$ in the set of all positive rationals, including 0. To extend the ordering to negative rationals, we enlarge our counting system and give to every positive rational, including zero, an odd ordinal number and to each negative rational an even ordinal. This can be done easily by defining our ordinal numbers in the following fashion: For a positive rational of the form $p+[m, n]$, the ordinal number is given by

$$N = (p + Q)(p + Q - 1) + 2Q - 1,$$

and for a negative rational of the form $-(p+[m, n])$ the ordinal number is

$$N = (p + Q)(p + Q - 1) + 2Q - 2.$$

We shall carry through a complete solution of the two problems stated at the beginning of the note.

(i) To find the ordinal number of 37/11.

Here $p+[m, n]=3+[7, 4]$. We have already found that $Q(7, 4)=21$. Hence we have $N=24\times 23+42-1=593$; from our extension to negative rationals, we have, for $-37/11$, the ordinal number 592.

(ii) To find the 593rd rational number.

Since the ordinal number is odd, our result will be a positive rational. We must have $(p+Q)(p+Q-1)+2Q-1=593$. Since Q is never zero, we must have also $(p+Q)(p+Q-1)<594$, where $p+Q$ and $p+Q-1$ are the two largest consecutive integers satisfying the inequality. Inspection gives $p+Q=24$, whence $Q=21$, $p=3$. But we have already shown that if $Q(m, n)=21$, then $[m, n] = [7, 4]=4/11$, whence the required number is $3+4/11=37/11$.

We exhibit one more example. Let us find the 6630th rational number. Here we have $(p+Q)(p+Q-1)+2Q-2=6630$, or $(p+Q)(p+Q-1)<6632$. This gives $p+Q=81$, whence $p=5$, $Q=76$. Retracing the algorithm for m and n from $Q(m, n)$, we find that the required element is $s(8, 24)=[18, 7]$. The required number is then $-(5+7/25)=-132/25$.

ON THE TWO-ANGLE POLE OF A LINE TO A TRIANGLE

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1. Introduction. In a recent paper [1], I mentioned a theorem generalizing those ideas that lead to the notion of the orthopole and the isopole; it suggests the consideration of a point which is related to a triangle $A_1A_2A_3$, a straight line l, and two angles θ and ϕ, and which therefore may be called the *two-angle pole* of the line to the triangle.

In this paper [1], I gave several properties of the two-angle pole, generalizing well known theorems on the orthopole and the isopole. We will now consider the two-angle pole in connection with subjects which have been discussed previously in the MONTHLY, namely, theorems related to pairs of triangles, and to the orthopolar line of a straight line as to a quadrilateral.

2. The two-angle pole. The theorem which leads to the notion of the two-angle pole is the following one.

THEOREM. *Parallels through the vertices of a triangle $A_1A_2A_3$ cut the line l at P_1, P_2, P_3, forming with l an angle θ; the lines through P_1, P_2, P_3, forming with the corresponding sides of $A_1A_2A_3$ an angle ϕ, form a triangle $Q_1Q_2Q_3$ similar to $A_1A_2A_3$, the ratio ρ being*

$$\rho = \frac{\sin(\theta + \phi)}{\sin \theta}.$$

When $\theta+\phi=\pi$, we find the well known isopole theorem.

The center ω of the circle $Q_1Q_2Q_3$ will be called the pole of l as to $A_1A_2A_3$ for the angles θ, ϕ.

Consider complex coördinates, the base-circle being the circumcircle Γ of $A_1A_2A_3$, the origin being O, and the unit point Ω; let t_1, t_2, t_3 be the coördinates of A_1, A_2, A_3, and denote by s_1 and s_3 the expressions

$$s_1 = t_1 + t_2 + t_3, \qquad s_3 = t_1t_2t_3.$$

Denote by $\bar{a}$ the conjugate to a.

If

$$\lambda = e^{2i\theta}, \qquad \mu = e^{2i\phi},$$

the two-angle pole of the straight line l, having the equation

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1,$$

is easily found to have for coördinate

$$x = \frac{s_1 - \lambda a - \mu \dfrac{\bar{a}}{a} s_3}{1 - \lambda}.$$

3. Pairs of triangles. The following theorem has been considered several times [2] and also in the MONTHLY [3]:

THEOREM. *If two triangles are inscribed in the same circle, the orthopoles of the sides of each of them as to the other triangle are six points on a circle having as center the midpoint of the segment between the orthocenters of the two triangles.*

Let τ_1, τ_2, τ_3 be the coördinates of the vertices of a second triangle $\beta_1\beta_2\beta_3$, and denote by σ_1 and σ_3 the expressions

$$\sigma_1 = \tau_1 + \tau_2 + \tau_3, \qquad \sigma_3 = \tau_1\tau_2\tau_3.$$

The pole for the angles θ, ϕ of $\beta_2\beta_3$ as to $A_1A_2A_3$ is

$$\frac{s_1 - \lambda(\tau_2 + \tau_3) - \dfrac{\mu s_3}{\tau_2\tau_3}}{1 - \lambda}$$

or

$$\frac{s_1 - \lambda\sigma_1}{1 - \lambda} + \frac{\left(\lambda - \dfrac{\mu s_3}{\sigma_3}\right)\tau_1}{1 - \lambda}.$$

Hence the circle C passing through the two-angle poles of the sides of $B_1B_2B_3$ as to $A_1A_2A_3$ has for its center

$$\frac{s_1 - \lambda\sigma_1}{1 - \lambda},$$

and the square of its radius is

$$\rho^2 = \frac{(\lambda\sigma_3 - \mu s_3)^2}{\mu s_3\, \sigma_3(1 - \lambda)^2}.$$

But the pole for the angles $-\theta$, $-\phi$ of A_2A_3 as to $B_1B_2B_3$ is

$$\frac{\sigma_1 - \dfrac{t_2 + t_3}{\lambda} - \dfrac{\sigma_3}{\mu t_2 t_3}}{1 - \lambda^{-1}}$$

or

$$\frac{\sigma_1 - \dfrac{s_1}{\lambda}}{1 - \lambda^{-1}} + \frac{\left(\dfrac{1}{\lambda} - \dfrac{\sigma_3}{\mu s_3}\right)t_1}{1 - \lambda^{-1}};$$

this shows that the pole under consideration is also on the circle C.

If two triangles $A_1A_2A_3$ and $B_1B_2B_3$ are inscribed in the same circle, the poles

for the angles θ, ϕ of the sides of $B_1B_2B_3$ as to $A_1A_2A_3$ and the poles for the angles $-\theta$, $-\phi$ of the sides of $A_1A_2A_3$ as to $B_1B_2B_3$ are on the same circle.

The center of the circle is the point of the axis of the segment between the orthocenters of the triangles from which that segment subtends an angle 2θ.

It may be recalled that if ψ is the *angle associated** *with* the two triangles $A_1A_2A_3$, $B_1B_2B_3$ [4], we have

$$e^{i\psi} = \frac{\sigma_3}{s_3}.$$

Hence

$$\rho = \frac{\sin(\theta - \phi - \psi/2)}{\sin\theta}.$$

When $\lambda/\mu = s_3/\sigma_3$, that is, when $\phi = \theta - \psi/2$, it follows that $\rho = 0$.

If ψ is the angle associated with two triangles $A_1A_2A_3$ and $B_1B_2B_3$, which are inscribed in the same circle, the poles for the angles θ, $\theta - \psi/2$ of the sides of $B_1B_2B_3$ as to $A_1A_2A_3$ and the poles for the angles $-\theta$, $-\theta + \psi/2$ of the sides of $A_1A_2A_3$ as to $B_1B_2B_3$ all coincide with the point on the axis of the segment joining the orthocenters of $A_1A_2A_3$ and $B_1B_2B_3$ from which that segment subtends an angle 2θ.

When the two given triangles have a common mean, that is, when $s_3 = \sigma_3$,

$$\rho = \frac{\sin(\theta - \phi)}{\sin\theta}.$$

4. The two-angle polar line in the cyclic quadrilateral. It is well known that the orthopoles of a straight line as to the four triangles having for vertices three out of four given points are on a straight line [5], [6]. The same property may be extended to the isopoles of the line for a given angle [7].

We will now obtain a similar property about two-angle poles, but in the particular case when the four given points are on the same circle.†

Let A_4 be the point on Γ having as coördinate t_4, and denote by S_1, S_3, S_4 the expressions

$$S_1 = t_1 + t_2 + t_3 + t_4,$$
$$S_3 = t_2t_3t_4 + t_3t_4t_1 + t_4t_1t_2 + t_1t_2t_3,$$
$$S_4 = t_1t_2t_3t_4.$$

Then the coördinate to the two-angle pole of l as to $A_1A_2A_3$ is

$$x = \frac{S_1 - \lambda a - t_4 - \dfrac{\mu\bar{a}S_4}{at_4}}{1 - \lambda},$$

* The angle ψ is twice the constant angle formed by the Simson lines of any point on the circumcircle as to the two triangles.

† The property does *not* exist for two-angle poles in the case of *any* four given points.

and, inasmuch as

$$\bar{x} = \frac{-\lambda\bar{S}_1 + \bar{a} + \dfrac{\lambda}{t_4} + \dfrac{\lambda a t_4}{\mu\bar{a}S_4}}{1-\lambda},$$

elimination of t_4 shows that the point under consideration is on the line

$$x + \frac{\mu\bar{a}S_4\bar{x}}{\lambda a} = \frac{S_1 - \lambda a - \dfrac{\mu\bar{a}S_3}{a} + \dfrac{\mu\bar{a}^2S_4}{\lambda a}}{1-\lambda}.$$

The two-angle poles of a straight line l as to the four triangles formed by three of the vertices of a cyclic quadrilateral are on a straight line L.

We call it the two-angle polar line of l as to the quadrilateral.

If $O\Omega$ is parallel to one of the bisectors of the angles formed by the pairs of opposite sides or diagonals of the quadrilateral, $S_4=1$ and the last equation becomes

$$x + \frac{\mu\bar{a}\bar{x}}{\lambda a} = \frac{S_1 - \lambda a - \dfrac{\mu\bar{a}S_3}{a} + \dfrac{\mu\bar{a}^2}{\lambda a}}{1-\lambda}.$$

The two-angle polar line of l as to a cyclic quadrilateral forms an angle $\phi-\theta$ with the symmetric of l as to the bisectors of the angles formed by the pairs of opposite sides or diagonals of the quadrilateral.

Furthermore, L passes through the point

$$\frac{S_1 - \lambda a}{1-\lambda}.$$

But S_1 is the so-called *orthocenter* of the quadrilateral (point obtained by producing three times its length the segment between O and the center of gravity of the vertices), and a is the image of O in l.

The two-angle polar line passes through the point T of the axis of the segment joining the orthocenter to the image of O in l, and such that the segment subtends at T an angle 2θ.

When the angle θ is a constant and ϕ variable, L passes through the fixed point T.

5. Some envelopes. If θ and ϕ being constant, l passes through a fixed point having as coördinate c, then a is $c-\tau\bar{c}$, where τ is a turn, and the equation to L becomes, $O\Omega$ having still the same position,

$$x - \frac{\mu\bar{x}}{\lambda\tau} = \frac{S_1 - \lambda c - \lambda\tau\bar{c} + \dfrac{\mu(S_3 - \bar{c}/\lambda)}{\tau} + \dfrac{\mu c}{\lambda\tau^2}}{1-\lambda}.$$

Hence the coördinate of the contact point of L with its envelope is

$$x = \frac{S_1 - \lambda c + 2\bar{c}\lambda\tau + \dfrac{\mu c}{\lambda\tau^2}}{1 - \lambda}.$$

When l passes through a fixed point, L envelopes a deltoid.

This is an extension of a theorem given by McBrien [6] for the orthopolar line.

If, θ and ϕ being constant, l envelopes a circle concentric to Γ, it follows that a is $k\tau$, where k is a constant and real and τ is a turn; moreover, the equation to L becomes

$$x + \frac{\mu\bar{x}}{\lambda\tau^2} = \frac{S_1 - \lambda k\tau - \dfrac{\mu S_3}{\tau^2} + \dfrac{\mu k}{\lambda\tau^3}}{1 - \lambda}.$$

The point where L touches its envelope is

$$\frac{2S_1 - \dfrac{\mu k}{\lambda\tau^3} - 3\lambda k\tau}{2(1 - \lambda)}.$$

When l envelopes a circle concentric to Γ, L envelopes an astroid.

This is also a generalization of a theorem on the orthopolar line given by McBrien [6].

When $A_1A_2A_3A_4$ remains inscribed in Γ and circumscribed to a conic, then we have the following relations of the form:

$$S_1 = p + qS_4, \qquad S_3 = \bar{q} + \bar{p}S_4,$$

S_4 being now variable and p and q being constants [8].

So we derive from the equation to L the following property:

When a variable quadrilateral remains inscribed in a circle and circumscribed to a conic, the two-angle polar line of a fixed line as to the quadrilateral passes through a fixed point.

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THE BEGINNINGS OF ANALYTIC GEOMETRY IN THREE DIMENSIONS

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1. Statement of the problem. In Section 528 of Jowett's translation of Plato's *Republic* we find the writer complaining of "the ludicrous state of solid geometry." His objection is to the fact that in his time, as in our own, geometry was far more developed in the plane than in a space of three or more dimensions. The reasons for this are fundamental. The fact that the data are more simple in the plane than in three or more dimensions, or that fewer independent variables are involved, not only makes the subject easier at the outset, but curiously enough, when we come to the higher reaches, we can introduce powerful techniques in two dimensions which are not available in spaces of greater dimensionality. Naturally Plato knew nothing of all this. His own efforts to improve the situation by studying regular and semi-regular solids produced no great result. What is really curious is that there was a similar lag in the development of analytic geometry. We shall see that a surprisingly long time passed from the publication of Descartes *Géométrie* until something really effective was accomplished for the algebraic geometry of space.

But first we must come to agreement about definitions, for there are different views as to the meaning of the principal terms. It is my own view that what we mean by analytic geometry is essentially the study of loci by means of equations connecting determining coördinates, regardless of the notation employed. Consequently, I hold that, strictly speaking, plane analytic geometry was introduced by the Greeks in their study of the conic sections. In Apollonius, for example, the study of the ellipse is based on the fact that the square of an ordinate is a constant multiple of the distance from its foot to the ends of the diameter in the conjugate direction. Logically then, I should maintain that the beginnings of analytic geometry in three dimensions are to be found in Archimedes' study of conoids and spheroids, where the ellipsoid of revolution, for instance, is the surface whose equation we should write

$$y^2 + z^2 = \frac{b^2}{a^2}\,x(2a - x).$$

However, if we look into the matter further, we find that Archimedes did not write this equation or anything like it. He used the properties of the circles which were in planes perpendicular to the axis, and of the ellipses in planes through that line.

The delay in extending algebraic methods to three dimensions appears the more surprising when we reflect that, after all, we live in a three-dimensional physical world, and the advantage of having a frame of reference to which we can refer physical objects, *saute aux yeux*. But probably this advantage was not fully understood until Newtonian mechanics made familiar the ideas of instantaneous velocity and acceleration.

2. Descartes and Fermat. In looking for the beginnings of three-dimensional analytic geometry it is natural to turn to the two great Frenchmen who introduced it in the plane. In fact, it is sometimes asserted that Descartes was the first to envisage three-dimensional analytic geometry. This seems to me to be scarcely the case. At the end of the second book of his *Géométrie*, which first appeared in 1637, we find him writing:

Au reste je n'ai parlé en tout ceci que des lignes courbes qu'on peut décrire sur une superficie plate, mais il est aisé de rapporter ce que j'en ai dit à toutes celles qu'on sauroit imaginer être formées par le mouvement régulier des points de quelque corps dans un espace à trois dimensions.

It is to be noticed first that he is interested in curves, not in surfaces. His method is to project his space curve orthogonally on two mutually perpendicular planes. The space curve is thus the total or partial intersection of two cylinders whose elements are mutually perpendicular. But at this point he makes a curious slip:

Même si on veut tirer une ligne droite qui coupe cette courbe au point donné à angles droits, il faut seulement tirer deux autres lignes droites dans les deux plans, une en chacun, qui coupent à angles droits les deux courbes qui y sont aux deux points où tombent les perpendiculaires de ce point donné, car ayant élevé deux autres plans, un sur chacune de ces lignes droites qui coupe à angles droits le plan où elle est, on aura l'intersection de ces deux plans pour la ligne droite cherchée.

This says that if we wish a normal to a space curve, we have but to construct normals to the two corresponding plane curves at the corresponding points, and pass through each of these a plane perpendicular to the plane of the curve; but it is easy to show by a particular example that we do not in this way get a normal to the space curve. I can not help wondering whether Descartes did not mean that we may get the tangent to the space curve by erecting planes tangent to the two plane curves, and orthogonal to their planes.

When we speak of Descartes in connection with analytic geometry we naturally think of the man who, excluding the Greeks, was the co-discoverer, Pierre Fermat. Here we meet a curious situation. In 1643, he sent to de Cercavi a manuscript entitled, *Ad locos ad superficiem Isogoge.* Apparently this was first published by Tannery and Henry in the first volume of their edition of Fermat's works which appeared in 1891. In a note at the bottom of page 111 the editors write:

Fermat, dont le point de départ est le livre d'Archimède *De conoidibus et spheroidibus* a bien reconnu la nécessité de généraliser la notion de la surface cylindrique, aussi que celle des conoïdes (paraboloïdes elliptiques et hyperboloïdes à deux nappes) et sphèroïdes (ellipsoïdes) d'Archimède, qui n'avait traité que les surfaces de révolution, Mais il n'a pas soupçonné l'existence du paraboioïde hyperbolique, ni l'hyperboloïde à une nappe.

This clearly suggests that he was on the look out for quartic surfaces which were not surfaces of revolution. In his third lemma in this communication he writes:

Si superficies quaepiam planis quotlibet in infinitum secetur, et communis sectio omnium secantium planorum et dictae superficiei sit quandoque circulus quandoque ellipsis et nihil praeteria, superfies illa erit spherois.

This says that if all sections of a surface are either circles or ellipses the surface is a spheroid. Now if this means that every such surface is a spheroid in the Archimedian sense it is clearly wrong, as the general ellipsoid fulfils the condition. If it is a definition of an ellipsoid it is correct. But why does he call it a "lemma"? An exactly similar difficulty occurs in the case of the conoid. A possible explanation is that the word "superficies" means "surface of revolution," but then the note of Tannery and Henry is wrong. I confess that the whole situation is puzzling to me.

The principal use which Fermat makes of these lemmas is to prove propositions such as the following: *Given a number of planes and a point p which moves in such a way that the sum of the squares of its distances to these planes, each measured in a direction which makes a given angle with the normal to the plane, is constant; the locus of p is a spheroid.*

Let us see how this locus will meet a general plane π. The distance from p along a line which makes a fixed angle with the normal is the normal distance divided by the cosine of the angle. The normal distance is the distance to the intersection of the plane with π, multiplied by the sine of the angle of the two planes. Hence the intersection with π is a curve such that the sum of constant multiples of the squares of its distances from given lines is constant, and such a curve was known to be an ellipse. Hence the surface is a spheroid.

I might say, in conclusion, that previously another mathematician might have given a definition of spheroid which justifies Fermat's lemma, but I know of no such before 1643.

3. Wallis. John Wallis in his *Tractatus de sectionibus conicis*, Oxford, 1655, reaches the quadric surfaces in the following very different fashion. He had been deeply influenced by Cavalieri's method of indivisibles. He begins by showing that triangles of equal bases and altitudes are equivalent, that is, they have the same area, because they may be conceived as being composed of the same infinite set of line segments parallel to the bases, but differently placed. The area is not changed even if the lateral sides of the triangle are curved. We might start with any triangle, move one vertex any distance we please parallel to the base, connect the new vertex with one end of the base by any simple continuous curve, then move the infinite number of segments parallel to the base, parallel to themselves to new positions where one end is on this new curve. As the triangle is conceived as the sum of all of these segments, its area has not changed. The segments are conceived as infinitely thin rectangles.

Wallis extends this same process to three dimensions. Let us take as base a plane convex figure which is symmetrical with regard to its center C. Let a line through C meet the base curve in B and S, and let V be a point outside the plane of the base. We obtain a cone by connecting V with every point of the base curve. But Wallis describes the situation differently. Through each point of the axis VC we pass a plane parallel to the base plane, and in it describe a figure similar and similarly placed to the base. Let y be the distance in such a plane between the intersections with VC and VB. The ratio of the area in this

plane to the base area is y^2/CB^2. Wallis next replaces the lines VB, VS by a conic whose tangent at V is parallel to the base. We cut this by a set of planes parallel to the base plane, the ordinate y having the same meaning as before. In each such plane we have a closed figure, similar to the base figure, symmetrical with regard to the intersection with VC, a diameter of the conic; the ratio of the area in this plane to the base area is given by the previous equation. This figure he calls a "pyramidoid" or "conoid," parabolic, elliptic, or hyperbolic, as the case may be. At the end of the article, pages 111 and 112, the words, ellipsoid, paraboloid, and hyperboloid are introduced. In a plane through the axis of the original conic we have a figure similar to this conic, all ordinates being altered in the same ratio. He also says that in a general plane cutting the axis, the section will be a conic, but this is only true when the base is a conic, which he does not explicitly assume. Kötter interpreted Wallis, proceeding much more simply, by saying the surface is generated by a series of similar and similarly placed central conics, in parallel planes, whose axes are the double ordinates of a given conic [1]. This is perfectly true when the base curve is a conic, and we can obtain the general quadric surface in this way, but it is less general than what Wallis actually says, and Kötter apologizes because in the figure for a parabolic pyramidoid the base figure has five sides when he wanted it to have six.

Wallis is particularly interested not in the surface he generates in this way, but in the volumes. The integrations he performs ingeniously. Suppose the original conic is a parabola, leading to

$$y^2 = kx; \qquad \int y^2dx = k\int xdx.$$

To find this integral, suppose we have an isosceles right triangle on which stands a right prism. The area of the section in a vertical plane at a distance x from the opposite edge can be written so that this integral gives us the volume of the prism, which, however, we know otherwise.

When the original conic is an ellipse, we know from Apollonius that

$$y^2 = \frac{b^2}{a^2}\,x(2a - x).$$

Suppose then, that we have a tetrahedron whose vertices are $(0, 0, 0)$ $(0, 0, c)(2a, d, 0)(2a, -d, 0)$. The area of a rectangular section determined by the plane, $x =$ a constant, is $cdx(2a-x)/2a^2$, so we get the volume of a pyramidoid or conoid from that of a pyramid. It is worth mentioning that the scheme of finding an integral by comparing the volume of a solid calculated by two different subdivisions is found in Pascal. Fermat, writing in 1643, certainly got no help from Wallis who wrote in 1655.

I must mention one other writer who is supposed to be the first to have introduced Cartesian geometry in three dimensions. I refer to Johann Bernoulli. Here is the opinion of Beniamino Segre:

Poco appreso Joh. Bernoulli, lettere al Leibniz di 4 dicembre 1697, del 26 agosto 1698 a del 6 Febraio 1725, utiliza le coördinate per la determinazione delle geodetiche su certi tipi di superficie [2].

This statement is not absolutely correct. In the first two places mentioned there is no discussion of coördinates. The third is more helpful:

Intelligo per superficiem datam, cujus singula puncta determinatur (sic ut lineae curva puncta) per ordinates tres x, y, z quarum relatio data aequatione ecprimeretur; sunt autem tres illae coordinatae nihil aliud, quam tres tres rectae ex qualibet superficiei curvae puncto perpendiculariter ductae in tres plana positione data, et se mutuo ad angulos rectos secantis [3].

Leibniz' reply appears on the next page:

Doctrinam de aequationibus localibus trium coordinatarum, seu de locis vere solidis, olim aggredi coepi, eorumque intersectiones sive curvas etiam non planas, sed prosequi non vacavit.

This is perfectly clear, but I must point out that the date is 1715, by which time, as we shall see, the method had been publicly explained. In 1728, Johann Bernoulli wrote to Klingsterna:

Sit $AE=x$, $EB=y$ et a puncto B erigi intelligitur recta $Bb=z$ normalis and planam AEB et superficiei curvae occurens in b et detur aequatio quaevis exprimans relationem trium coordinatarum x, y, z qua relatione natura superficiei determinatur [4].

I should also mention in this connection a word by Clairaut:

Je ne crois pas cette matière moins neuve que celle des courbes à double courbure, et je ne scai de connu sur ce sujet, que la façon d'exprimer les surfaces courbes par des equations à trois variables dont j'ai appris qu'il étoit fait mention par occasion dans un mémoire du célèbre M. Bernoulli inseré dans les Actes de Leypsic [5].

In this quotation does "célèbre M. Bernoulli" mean James or John; I find nothing of this sort in James' collected works, nor in anything of John's except what I have already given, and that is not in the Acta Eruditorum.

4. LaHire and Parent. It is time to turn from those who might have invented analytic geometry in three dimensions but didn't, to those who did. First I take Philippe de LaHire. He stated definitely in 1679:

Je considère d'abord pour déterminer un point hors d'un plan, à l'égard d'une ligne droite déterminée sur ce plan, il faut trois conditions; la première est la grandeur LA de la perpendiculaire menée du point L au plan, la seconde la perpendiculaire AB menée du point A à la ligne donnée OB, et la troisième la partie OB de cette ligne comprise entre un de ses points O et le recontre B. C'est pourquoi je fais $OB=x$, $AB=y$, $LA=v$ qui sont trois inconnus.

He takes one specific problem, to find L so that $a+OB=OL$. He sees geometrically

$$OL = \sqrt{x^2+y^2+v^2}; \qquad a+x = \sqrt{x^2+y^2+v^2}$$
$$a^2+2ax = y^2+v^2;$$

then he adds tamely enough

et comme il n'y a pas moyen de trouver d'autres équations pour faire évanouir quelques unes des inconnues, il s'ensuit que le problème est indéterminé [6].

So here we have a capable mathematician with all of three dimensional analytic geometry under his hand, but unable or unwilling to make anything of it save to calculate the distance of a point from the origin, and to note that one single condition will not locate completely a point in three-space.

Very different is the case of LaHire's disagreeable but enterprising fellow countrymen, Antoine Parent, whose paper, *Des affections des superficies*, was read before the French Académie des Sciences on August 23, 1700, and published in his *Essais et recherches de mathêmatiques et de physique*, Vol. 2, Paris, 1723, pages 181–200.

It is hardly necessary to say that "affection" means equation; Parent is interested in the equations of surfaces. His first general problem is to determine the tangent plane. In his time mathematicians did not bother with the equations of tangents, but with their construction, and in the case of a plane curve, this came from the subtangent, which is the orthogonal projection on the x-axis of as much of the tangent as lies between the point of contact and the intersection with that axis; its value is ydx/dy.

Parent begins with a sphere "pour exemple." He has a perfectly general method, but he begins with a sphere for he knows how to find its equation. Like LaHire he knows how to find the distance of a point from the origin. The radius of the sphere is r, the coördinates of the center are, b, c, a; thus

$$c^2 + y^2 - 2cy + b^2 + x^2 - 2bx + a^2 + z^2 - 2az = r^2.$$

First holding y constant, and calmly disregarding algebraic signs, he has

$$(a - z)dz = (b - x)dx; \qquad \frac{zdx}{dz} = \frac{a - z}{b - x} z.$$

Similarly,

$$\frac{zdy}{dz} = \frac{a - z}{c - y} z.$$

Clearly he has in mind the general formula, which we should write

$$f(x, y, z) = 0, \quad \text{having subtangents} \quad -\frac{f_z}{f_x} z; \quad -\frac{f_z}{f_y} z.$$

Parent next looks for those points on a surface where y takes a maximum or a minimum value. For a surface he takes a conchoid, discussed by L'Hospital, and allows one of the constants to vary; that is, he considers

$$y^2 = \frac{z - x}{x} (b + x)^2.$$

If we treat z as a constant, it follows that

$$2ydy = (b + x)\left[\frac{2(z - x)}{x} - \frac{(b + x)z}{x^2}\right] dx.$$

For an extremal we have $dy=0$; so

$$2x^2 - zx + bz = 0.$$

He does not recognize this as representing a cylinder, but sees that the projection on the xz plane is a hyperbola. By solving this for x, and substituting in the equation of the surface, we get the projection on the yz plane.

Parent's third problem is to find the locus of the points of inflection in the planes, $z=$ a constant. If we take x as the independent variable, we have at such points, in the notation of the time,

$$dx = \text{a constant}, \qquad ddy = 0.$$

He needs a surface where y appears explicitly, so he takes

$$y = \frac{z^3}{x^2 + az}.$$

Keeping z constant,

$$dy = \frac{-\,2xz^3dx}{(x^2 + az)^2},$$

$$ddy = (3x^2 - az)\frac{2z^3dx^2}{(x^2 + az)^3}.$$

For a point of inflection we shall have

$$3x^2 - az = 0.$$

Parent does not go beyond this point; it is evident, however, that he had hold of a very real piece of mathematics for which he should deserve full credit.

5. Alexis Clairaut. I have already spoken of the work of the infant prodigy, Alexis Clairaut; it is time to speak further of his real contributions to three dimensional analytic geometry, even though he was late in time, 1731. Clairaut was inspired by the work of Descartes, which I have mentioned. He was interested in curves, not in surfaces, but he set up Cartesian coördinates exactly after the manner of LaHire and Parent, though he mentioned neither of these writers. His first theorem is that an equation involving three coördinates is a surface, because the intersection with any plane parallel to a coördinate plane is a curve. An equation of the first degree represents a plane, because the intersections with all planes, $z=$ a constant, are straight lines parallel to one another, and the same is true for planes, $y=$ a constant. He deals no further with planes but develops the formula for the square of the distance between two given points. He shows how to find the equation of a cone determined by a plane curve and a point outside of its plane, and how, when we have the equations of two surfaces, we may, by elimination, find the equation of the projection of their common curve on one of the coördinate planes.

Clairaut is happier when he passes to the application of the calculus to the study of curves. But here I should say that he does not go beyond first derivatives, or first differences, as he would have called them. The first and most fundamental problem is to find the tangent. The actual length of the tangent from the point of contact to the intersection with the plane $z=0$ is $z\sqrt{dx^2+dy^2+dz^2}/dz$. From this we can easily pass to various sorts of subtangents. I note in passing that the square of dx is dx^2, but the square of x is xx.

Most of the particular problems are not sufficiently interesting to be worth reproducing. The most amusing, perhaps, is that of finding the equation of the curve cut in the base plane $z=0$ by a moving tangent to the space curve. We need the orthogonal projections on the x and y axes of so much of the tangent as runs from the point of contact to the base plane. These we get from the previous formulae for subtangents, giving

$$x' = x - \frac{zdx}{dz}; \qquad y' = y - \frac{zdy}{dz}.$$

From the finite and differential equations of the space curve we may find $x, y, dx/dz, dy/dz$ in terms of z. Eliminating, we have $\phi(x', y')=0$.

The third section of Clairaut's work deals with the applications of the integral calculus to curve theory. He recognizes the distance formula

$$ds = \sqrt{dx^2 + dy^2 + dz^2}.$$

It is naturally hard to find a curve which can be rectified in this way. Here is one:

$$y^2 - 2a^2 = \sqrt[3]{9a^4x^2}; \qquad az = y^2.$$

Thus,

$$x = \frac{(y^2 - 2a^2)^{3/2}}{3a^2}; \qquad dx = \frac{\sqrt{y^2 - 2a^2}\, ydy}{a^2}; \qquad dz = \frac{2ydy}{a};$$

$$ds = dy\,\frac{y^2 + a^2}{a^2};$$

$$s = \frac{y^3}{3a^2} + y + C.$$

We can find the area of as much of one of the cylinders as is bounded by the curve and its projection on the yz plane from the integral $\int x\sqrt{dy^2+dz^2}$. The volume bounded by the two cylinders and the base plane is given by $dv=yzdx$. He finds some other volumes of this sort.

The fourth section gives certain unconnected constructions of space curves. For instance, suppose that we have a compass of fixed opening r and one end is fixed at the point (a, b, c) of a given surface, what curve will the other point trace in the surface? Clearly

$$f(x, y, z) = 0, \qquad (x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

Here is a more difficult problem: To find the equations of a curve along which a given surface is touched by the tangents through a given point, (a, b, c). We have evidently

$$\frac{dx}{(x-a)} = \frac{dy}{(y-b)} = \frac{dz}{(z-c)}.$$

Here our modern notation gives the answer immediately. Besides the equation of the surface

$$f(x, y, z) = 0,$$

we have

$$f_x dx + f_y dy + f_z dz = 0,$$
$$f_x(x-a) + f_y(y-b) + f_z(z-c) = 0.$$

Clairaut's solution is, naturally, less compact. The lack of a suitable notation for partial derivatives was a serious disadvantage to mathematicians of, Clairaut's time, yet they were recognized by Newton. It is largely for this reason that his book of 119 pages gives the impression of being very long for the actual amount of mathematics involved, but when we remember that the author was only sixteen when his memoir was first presented to the Académie des Sciences, we can but wonder at his precocity.

6. Jakob Hermann. The year after the appearance of Clairaut's "Recherches" another article, dealing with analytic geometry in three dimensions, but clearly independent, first saw the light. The author was Jakob Hermann [7]. His interest was in surfaces rather than in curves. He believed that the subject had been neglected in the past because geometers were deterred by its apparent prolixity. He set up his axes exactly like those of Parent and Clairaut, though neither of these authors is mentioned, and then gave a list of six surfaces he meant to study. The first was

$$ax + by + cz - e^2 = 0.$$

This is shown to be a plane, by means of similar triangles. Next we have

$$z^2 - ax - by = 0.$$

This is a parabolic cylinder. The section in a plane, $z =$ a constant, is a straight line, and all such lines are parallel. In the planes $x =$ a constant or $y =$ a constant, we have a parabola. Let us find an extreme value for z. Here we should have $dz/dx = dz/dy = 0$, but that involves $a = b = 0$, which is not the case, so there is no extreme value which is fairly evident geometrically. To find a tangent plane we seek the sub-tangents, as Parent did; that is,

$$\frac{zdx}{dz} = \frac{2(ax+by)}{a}; \qquad \frac{zdy}{dz} = \frac{2(ax+by)}{b}.$$

The next surface is

$$z^2 - xy = 0.$$

This is a cone as it contains entirely a line connecting the origin with any other point of the surface. Sections perpendicular to the z-axis are rectangular hyperbolas; those in planes perpendicular to the other axes are parabolas. We find something more interesting in his fourth example,

$$z^2 - ax^2 - bxy - cy^2 - ex - fy = 0.$$

Hermann remarks:

Haec Aequatio est ad superficiem alcujus Conoidis, cujus basis exponitur ex aequatione

$$cy^2 + bxy + ax^2 + fy + ex = 0.$$

The wording here is interesting. "Conoidis" does not mean surface of revolution of Archimedes. Hermann deals separately with surfaces of revolution in a subsequent paragraph, and he refers explicitly to the Wallis wedge conoid. It seems certain that he was familiar with Wallis' work, and took his definition of a conoid. He finds the extreme values of z by equating the two partial derivatives to 0; that is,

$$2ax + by + e = 0; \qquad 2cy + bx + f = 0.$$

We solve these for x and y, and substitute in the equation of the surface. He finds the tangent plane from the sub-tangents zdy/dz, zdx/dz. This is so close to Parent's work that one wonders whether he was not familiar with the French writer's mathematics.

Hermann's fifth surface gives something more difficult, namely,

$$az^2 - bxz - cyz + ey^2 = 0.$$

This again he recognizes as giving a cone, and he undertakes the none too easy problem of seeking for the planes of circular section. He takes u and t as rectangular Cartesian coördinates in such a plane, and expresses x, y, z in terms of these and of the constants determining the aspect of the plane; then he substitutes, and writes the conditions for a circle. He arrives finally at an equation of the third degree, as we should expect. The problem was not new. It had been handled in the 75th letter of Descartes, Third series, Ed. 1683, and in No. 441 of L'Hospital's *Traité analytique des sections coniques*.

Hermann's last surface is

$$u^2 - x^2 - y^2 = 0$$

where u is a function of z.

Haec Aequatio ad omnis generis solidum spectat.

He ends with a short study of geodesic curves, but acknowledges that the general problem is beyond him.

We may say that with Hermann the study of analytic geometry in three

dimensions was on a firm basis, and surfaces of the second order familiar objects. For that reason Cajori is clearly wrong when he writes of Euler: "He was the first to discuss the equation of the second degree in three variables, and to classify the surfaces represented by it" [8]. It is however true that he was the first to reduce such equations to canonical forms, in his *Analysin Infinitorum* of 1748. What a long time this is after Descartes' *Géométrie* of 1637, where curves of the second degree were elaborately treated!

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MATHEMATICAL NOTES

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PARAMETRIC SOLUTIONS OF TWO MULTI-DEGREED EQUALITIES

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1. Introduction. The notation

$$A_1, A_2, \cdots, A_p \overset{n_1, n_2, \cdots, n_r}{=} B_1, B_2, \cdots, B_q$$

designates a so-called *multi-degreed equality* and means that the sum of the numbers on the left equals the sum of the numbers on the right for each of the $r(n_1, n_2, \cdots, n_r)$ positive integral powers of the numbers.

We shall give here parametric solutions of the two multi-degreed equalities

$$(1) \qquad A_1, A_2, A_3 \overset{2,4}{=} B_1, B_2, B_3, \qquad A_3 \neq A_1 + A_2, \qquad B_3 \neq B_1 + B_2,$$

and

$$(2) \qquad C_1, C_2, \cdots, C_7 \overset{1,2,4,6,8}{=} D_1, D_2, \cdots, D_7.$$

2. Parametric solutions of (1). To obtain solutions for (1), we may proceed as follows. Consider the multi-degreed equality

$$a,\ a \overset{2,4}{=} b,\ c,\ b+c.$$

It implies the following in which $d=b+c$; k and l are arbitrary integers.

$$ak+al,\ bk+cl,\ ck-dl,\ dk-bl \overset{2,4}{=} ak-al,\ bk-cl,\ ck+dl,\ dk+bl$$

We determine the quotient k/l in such a manner that a term on the left equals a term on the right or the opposite number of a term on the right. We shall consider only the solutions in which

$$A_3 \neq A_1 + A_2, \qquad B_3 \neq B_1 + B_2.$$

Here is a parametric solution* of (1):

$$a = m^2 + mn + n^2, \qquad b = m^2 - n^2, \qquad c = 2mn + n^2.$$

Let $(\alpha, \beta, \gamma, \delta, \epsilon)$ be the fourth-degree form $\alpha m^4+\beta m^3n+\gamma m^2n^2+\delta mn^3+\epsilon n^4$. We now get the following parametric solutions of (1):

$$
(3)\quad \begin{array}{l}
(-1,\ 3,\ 6,\ 7,\ 3),\ (-1,-2,\ 5,\ 6,\ 1),\ (0,\ 4,\ 2,-4,-2) \overset{2,4}{=} \\
(1,\ 1,\ 0,-1,-1),\ (1,-2,\ 1,\ 6,\ 3),\ (0,\ 0,\ 8,\ 8,\ 2) \\
(-2,\ 0,\ 3,\ 5,\ 3),\ (0,\ 4,\ 2,-4,-2),\ (-1,\ 2,\ 7,\ 2,-1) \overset{2,4}{=} \\
(2,\ 0,-4,\ 0,\ 2),\ (0,\ 2,\ 3,\ 3,\ 1),\ (1,-2,\ 1,\ 6,\ 3) \\
(3,\ 5,\ 3,\ 0,-2),\ (2,\ 4,-2,-4,\ 0),\ (-1,\ 2,\ 7,\ 2,-1) \overset{2,4}{=} \\
(1,\ 3,\ 3,\ 2,\ 0),\ (2,\ 0,-4,\ 0,\ 2),\ (3,\ 6,\ 1,-2,\ 1) \\
(1,\ 1,\ 0,-1,-1),\ (2,\ 8,\ 8,\ 0,\ 0),\ (3,\ 6,\ 1,-2,\ 1) \overset{2,4}{=} \\
(1,\ 6,\ 5,-2,-1),\ (2,\ 4,-2,-4,\ 0),\ (3,\ 7,\ 6,\ 3,-1) \\
(0,\ 2,\ 3,\ 3,\ 1),\ (1,\ 6,\ 13,\ 6,\ 1),\ (2,\ 8,\ 8,\ 0,\ 0) \overset{2,4}{=} \\
(0,\ 4,\ 10,\ 4,\ 0),\ (1,\ 6,\ 5,-2,-1),\ (2,\ 8,\ 9,\ 7,\ 1) \\
(0,\ 0,\ 8,\ 8,\ 2),\ (1,\ 3,\ 3,\ 2,\ 0),\ (1,\ 6,\ 13,\ 6,\ 1) \overset{2,4}{=} \\
(0,\ 4,\ 10,\ 4,\ 0),\ (1,\ 2,-5,-6,-1),\ (1,\ 7,\ 9,\ 8,\ 2).
\end{array}
$$

Each sextuplet contains for given values of the parameters m, n, 21 distinct numbers of which 15 are repeated twice.

EXAMPLE. We obtain from (3) for $m=3$, $n=1$,

$$71,\ 78,\ 112 \overset{2,4}{=} 57,\ 98,\ 104.$$

3. Parametric solutions of (2). We apply now the theorem of Birck: *The multi-degreed equality*,

* A. Gloden, Mehrgradige Gleichungen, 1944, page 90.

$$a_1, a_2, a_3 \overset{2,4}{=} b_1, b_2, b_3 \textit{ in which } a_3 \neq a_1 + a_2,\ b_3 \neq b_1 + b_2,$$

implies†

$$(4)\quad \begin{aligned} &a_1 + a_2 + a_3,\ -a_1 + a_2 + a_3,\ a_1 - a_2 + a_3,\ a_1 + a_2 - a_3,\ 2b_1,\ 2b_2,\ 2b_3 \overset{1,2,4,6,8}{=} \\ &b_1 + b_2 + b_3,\ -b_1 + b_2 + b_3,\ b_1 - b_2 + b_3,\ b_1 + b_2 - b_3,\ 2a_1,\ 2a_2,\ 2a_3. \end{aligned}$$

Each of the 6 parametric solutions of (1) confers a parametric solution of (2). Here is the solution corresponding to (3):

$$\begin{aligned} &(2, -1, 9, 13, 4),\ (0, -3, 9, 15, 6),\ (0, 3, 7, 1, -2),\ (2, -1, -7, -3, 0), \\ &(-2, 6, 12, 14, 6),\ (0, 8, 4, -8, -4),\ (2, 4, -10, -12, -2) \overset{1,2,4,6,8}{=} \\ &(-2, 5, 13, 9, 2),\ (0, 1, -1, 5, 4),\ (2, 3, -9, -17, -6),\ (0, 9, 3, -3, 0) \\ &(2, 2, 0, -2, -2),\ (2, -4, 2, 12, 6),\ (0, 0, 16, 16, 4). \end{aligned}$$

The reader may construct the five other parametric solutions of (2).

EXAMPLE. For $m=3$, $n=1$, we have

$$37,\ 105,\ 114,\ 119,\ 196,\ 208,\ 261 \overset{1,2,4,6,8}{=} 51,\ 63,\ 142,\ 145,\ 156,\ 224,\ 259.$$

ON COPRIME VALUES TAKEN BY GIVEN POLYNOMIALS

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Let r, s be integers such that $2 \leqq r \leqq s$. Let $f_1, \cdots, f_s$ be given polynomials, with integral coefficients, no r of which possess a (non-constant) common factor. An integer n will be called *admissible* if no r of the s integers $f_1(n), \cdots, f_s(n)$ possess a common factor greater than 1.

For any prime p we denote by $D(p) = D(p;\, r;\, f_1, \cdots, f_s)$ the number of numbers x among $1, 2, \cdots, p$ for which at least r of the s congruences

$$f_i(x) \equiv O \pmod{p} \qquad (1 \leqq i \leqq s)$$

are satisfied.

We shall prove the following theorem.

THEOREM. *The admissible numbers form precisely* $\prod_{p|K}(p - D(p))$ *residue classes* (mod $\prod_{p|K} p$), *where K is a number depending at most upon $r, s, f_1, \cdots, f_s$. In particular, therefore, if $D(p) = p$ for some $p \mid K$, then the set of admissible numbers is empty.*

Proof. Let $1 \leqq i_1 < \cdots < i_r \leqq s$. Since the polynomials $f_{i_1}, \cdots, f_{i_r}$ have no common factor, there exists a least positive integer $K(i_1, \cdots, i_r)$ such that, for suitable polynomials* $g_1, \cdots, g_r$ we have identically

$$f_{i_1} g_1 + \cdots + f_{i_r} g_r = K(i_1, \cdots, i_r). \tag{1}$$

† A. Gloden, loc. cit., page 44.

* The polynomials $g_1, \cdots, g_r$ will, of course, generally depend on $i_1, \cdots, i_r$.

We write

$$K = \prod_{1 \leq i_1 < \cdots < i_r \leq s} K(i_1, \cdots, i_r). \tag{2}$$

If p is any prime, n will be called p-admissible if fewer than r of the s numbers $f_1(n), \cdots, f_s(n)$ are divisible by p. Hence n is admissible if and only if it is p-admissible for every prime p.

Now if p is a divisor of at least r numbers among $f_1(n), \cdots, f_s(n)$, then, by (1) and (2), $p \mid K$. Therefore, if it is not true that $p \mid k$, then every n is p-admissible. Hence n is admissible if and only if it is p-admissible for every $p \mid K$.

The numbers which are p-admissible clearly form precisely $p - D(p)$ residue classes (mod p). But, as is well known, if $m_1, \cdots, m_h$ are coprime in pairs, then the system of congruences

$$n \equiv a_1 \pmod{m_1}, \cdots, n \equiv a_h \pmod{m_h}$$

is soluble, and the solutions form precisely one residue class (mod $m_1 \cdots m_h$). Hence the numbers which are p-admissible for every $p \mid K$, that is, the admissible numbers, form precisely $\prod_{p \mid K}(p - D(p))$ residue classes (mod $\prod_{p \mid K} p$).

REMARKS. (*i*) For $x > 0$ let $N(x)$ denote the number of positive admissible numbers not exceeding x. By the theorem just proved it follows at once that, as $x \to \infty$,

$$N(x) = x \prod_{p \mid K} \left(1 - \frac{D(p)}{p}\right) + O(1), \tag{3}$$

where the O-constant may depend upon $r, s, f_1, \cdots, f_s$.

(ii) Let $k_1, \cdots, k_s$ be given distinct integers, and consider the special case

$$f_i(n) = n + k_i \qquad (1 \leqq i \leqq s).$$

A number n is now called admissible if no r among $n + k_1, \cdots, n + k_s$ have a common factor greater than 1, and $D(p)$ is the number of numbers among $1, 2, \cdots, p$ which are congruent (mod p) to at least r among $k_1, \cdots, k_s$. The relation (3) now holds, for instance with

$$K = \prod_{1 \leqq i < j \leqq s} |k_i - k_j|.$$

(iii) We may specialize (ii) still further by putting $r = s$. Then $N(x)$ becomes the number of positive integers $n \leqq x$ such that the highest common factor of $n + k_1, \cdots, n + k_s$ is 1. It is easy to show that

$$N(x) = x \frac{\phi(k)}{k} + O(1),$$

where ϕ denotes Euler's function, and k is the highest common factor of $|k_1 - k_2|, |k_1 - k_3|, \cdots, |k_{s-1} - k_s|$.

I would like to record my indebtedness to Dr. R. Rado for his very valuable suggestions in connection with this note.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Pennsylvania.

A MODIFICATION OF NEWTON'S METHOD

H. S. WALL, University of Texas

Newton's formula for approximating the roots of an equation $f(x)=0$, namely,

$$x_{p+1} = x_p - \frac{f(x_p)}{f'(x_p)}, \qquad (p = 0, 1, 2, \cdots) \tag{1}$$

may be modified in the following way. The equation of the parabola through the point $(x_p, f(x_p))$, having the same first and second derivatives at $x=x_p$ as $y=f(x)$, is

$$y = f(x_p) + (x - x_p)f'(x_p) + \tfrac{1}{2}(x - x_p)^2 f''(x_p).$$

Let x_{p+1} be a solution of the equation which results if we put $y=0$. Then

$$x_{p+1} - x_p = \frac{-f(x_p)}{f'(x_p) + \frac{1}{2}(x_{p+1} - x_p)f''(x_p)}.$$

If we take $x_{p+1}-x_p=-f(x_p)/f'(x_p)$ in this formula, we then obtain

$$x_{p+1} = x_p - \frac{f(x_p)}{f'(x_p) - \dfrac{f(x_p)f''(x_p)}{2f'(x_p)}}, \quad (p = 0, 1, 2, \cdots). \tag{2}$$

This is the desired modification of Newton's formula.

In a number of examples to which we have applied formula (2), the convergence of the sequence $\{x_p\}$ is essentially more rapid than in the case of formula (1).

EXAMPLE 1. *To compute the n-th root of a positive number a.* If we take $f(x)=x^n-a$, then Newton's formula is

$$x_{p+1} = \frac{a/x_p^{n-1} + (n-1)x_p}{n}, \qquad (p = 0, 1, 2, \cdots), \tag{3}$$

and formula (2) is

$$x_{p+1} = \frac{(n-1)x_p^n + (n+1)a}{(n+1)x_p^n + (n-1)a}\cdot x_p, \qquad (p = 0, 1, 2, \cdots). \tag{4}$$

The following table gives the approximations to $\sqrt{2}$ obtained from these formulas, starting with $x_0=1$.

	(3)	(4)
x_1	1.500000000	1.4
x_2	1.416666667	1.41421319797
x_3	1.414215686	1.4142135623730950487956900 8
x_4	1.414213562	

The value of x_3 found by Newton's formula is correct to four decimal places, while the value of x_3 found by formula (4) is correct to nineteen decimal places.

Starting with $x_0=10$, we find that Newton's formula gives for x_4 the value 1.4442, which is correct to one decimal place, while formula (4) gives the approximation $x_4=1.414213562$, which is correct to nine decimal places.

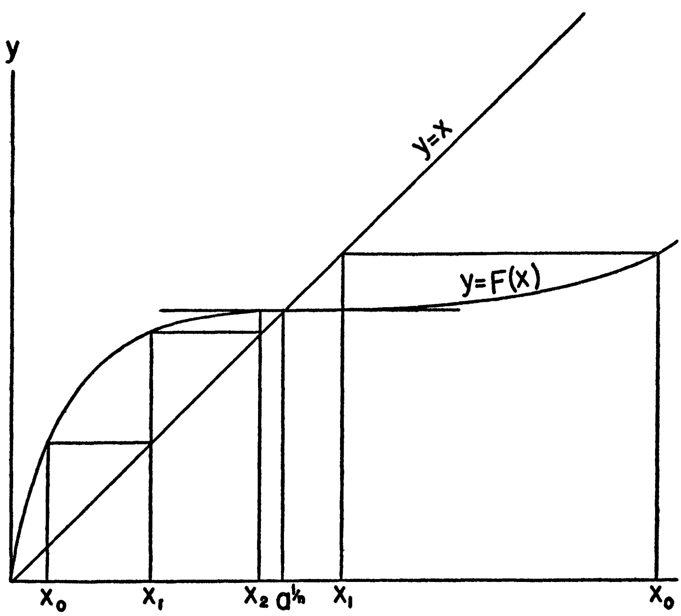

Fig. 1

If $x_0>0$, then the sequence $x_1, x_2, x_3, \cdots$, given recurrently by (4), *converges to $a^{1/n}$, and is an increasing sequence if $x_0<a^{1/n}$ and a decreasing sequence if $x_0>a^{1/n}$. The error committed in taking x_{p+1} as an approximation to $a^{1/n}$ is of the order of the cube of the error committed in taking x_p as an approximation to $a^{1/n}$.*

This can be seen in the following way. The function

$$F(x)=\frac{(n-1)x^{n+1}+(n+1)ax}{(n+1)x^n+(n-1)a}$$

has the derivative

$$F'(x) = \frac{(n^2 - 1)(x^n - a)^2}{[(n + 1)x^n + (n - 1)a]^2},$$

which vanishes without changing sign at $x = a^{1/n}$, is otherwise positive, and is greater than unity at $x = 0$. Thus, the graph of $y = F(x)$ has a point of inflection with horizontal tangent at $x = a^{1/n}$, where it crosses the line $y = x$. From the figure it is now clear that $x_p \uparrow a^{1/n}$ if $0 < x_0 < a^{1/n}$, and $x_p \downarrow a^{1/n}$ if $x_0 > a^{1/n}$.

If we put $x_p = r + \epsilon_p$, $r = a^{1/n}$, we readily find from (4) that

$$\lim_{p=\infty} \frac{\epsilon_{p+1}}{\epsilon_p^3} = \frac{n^2 - 1}{12r^2}.$$

Thus, ϵ_{p+1} is of the order of ϵ_p^3.

Figure 1 shows that the rapid convergence of the sequence $\{x_p\}$ is due to the fact that the graph of $y = F(x)$ has a point of inflection with horizontal tangent at $x = a^{1/n}$. In the general case,

$$F(x) = x - \frac{f(x)}{f'(x) - \dfrac{f(x)}{f'(x)}\dfrac{f''(x)}{2}},$$

whose derivative is

$$F'(x) = \frac{f^2(3f''^2 - 2f'f''')}{(2f'^2 - ff'')^2}.$$

Therefore, in general, the graph of $y = F(x)$ has a point of inflection with horizontal tangent at a root of the equation $f(x) = 0$.

EXAMPLE 2. *To compute the positive real root of the reduced cubic equation* $x^3 + bx - c = 0$, *b, c real*, $b \neq 0$, $c > 0$. Here, Newton's formula is

$$x_{p+1} = \frac{2x_p^3 + c}{3x_p^2 + b}, \tag{5}$$

and formula (2) is now

$$x_{p+1} = \frac{3x_p^5 - bx_p^3 + 6cx_p^2 + bc}{6x_p^4 + 3bx_p^2 + 3cx_p + b^2}. \tag{6}$$

If $b = 2$, $c = 20$, and we take $x_0 = 2$, formula (6) gives the approximations

$$x_1 = 2.46, \qquad x_2 = 2.46954551.$$

In this case, formula (5) gives, starting with $x_0 = 2$, the values

$$x_1 = 2.6, \qquad x_2 = 2.47, \qquad x_3 = 2.469546.$$

The value of the root, to nine decimal places, is 2.469545649.

If $x_0>0$, $3x_0^2+b>0$, *then the sequence* $x_1, x_2, x_3, \cdots$ *given recurrently by* (6) *converges to the positive real root* r *of the equation* $x^3+bx-c=0$. *The sequence is increasing if* $x_0<r$, *and is decreasing if* $x_0>r$. *If* ϵ_k *is the error committed in taking* x_k *as an approximation to* r, *then* ϵ_{p+1} *is of the order of* ϵ_p^3.

On writing the equation $x^3+bx-c=0$ in the form

$$x = \frac{c}{x^2} - \frac{b}{x},$$

and using the fact that $c>0$, we conclude immediately that the equation has

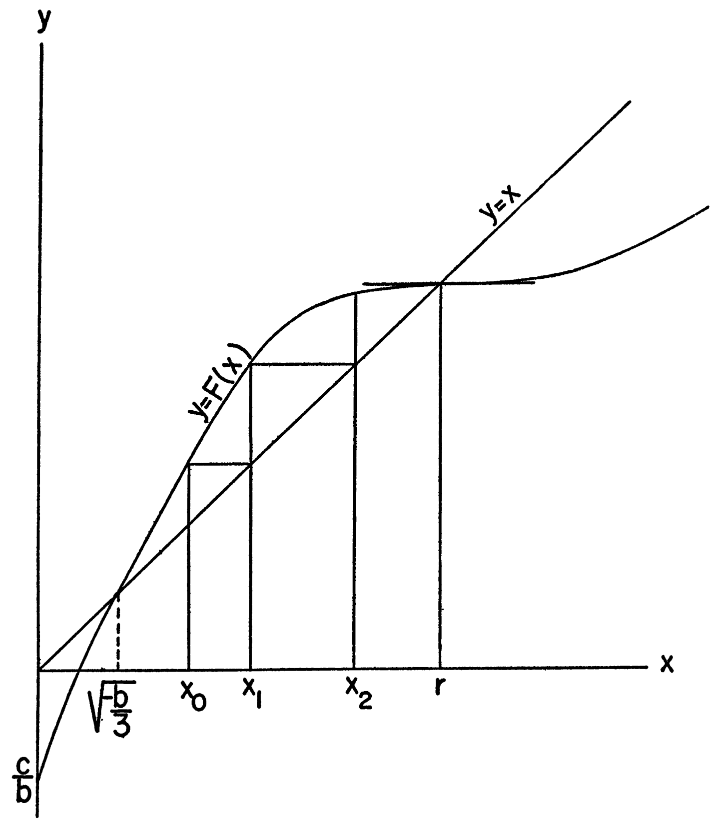

Fig. 2

just one positive real root r. This is the abscissa of the point of intersection in the first quadrant of the straight line $y=x$ and the curve $y=(c/x^2)-(b/x)$.

Consider the function

$$F(x) = \frac{3x^5 - bx^3 + 6cx^2 + bc}{6x^4 + 3bx^2 + 3cx + b^2}.$$

Since $6x^4+3bx^2+3cx+b^2=6(x^2+b/4)^2+3cx+5b^2/8>0$ for $x>0$, we see that $F(x)$ is continuous for $x>0$. Also, $F'(x)>0$ for $x>0$ except that $F'(r)=0$. It is evident from the graph that $x_p \uparrow r$ or $x_p \downarrow r$ according as $x_0<r$ or $x_0>r$, respectively. We find that $\lim_{p=\infty}[\epsilon_{p+1}/\epsilon_p^3]=(6r^2-b)/(6r^4+3br^2+3cr+b^2)$, so that ϵ_{p+1} is of the order of ϵ_p^3.

AUTHOR'S NOTE. After this article was in type, my attention was called to a note by J. S. Frame (this MONTHLY, vol. 51 (1944), No. 1, Part I, pp. 36–38) containing precisely this idea for extending Newton's formula. The above discussion brings to light other properties of Frame's formula. In addition to these, it may be easily shown that if r is a simple (real or imaginary) root of a polynomial $f(x)$, and if x_0 is taken sufficiently near to r, then the sequence (2) converges to r in such a way that $|x_{p+1}-r|$ is of the order of $|x_p-r|^3$.

EDITORIAL—THE PROOF OF EULER'S EQUATION

C. B. ALLENDOERFER, Haverford College

Judging from the volume of mail addressed to "Classroom Notes" it appears that one of the greatest mysteries of undergraduate mathematics is the equation of Euler: $e^{ix}=\cos x+i\sin x$. The objective of these correspondents is to develop this formula without the use of infinite series; and judging from the desperate devices employed by these writers in seeking to attain this end, it is highly desirable that a rigorous simple proof of this formula be available. The present note is therefore dedicated to those authors whose papers have been rejected, but whose ideas have helped me to prepare the following presentation.

The first point to be emphasized is that the expression e^{ix} has to be *defined*, and that certain properties must be ascribed to it. Otherwise any proof falls to the ground. Rigorous treatments of this appear in the classical literature; for example, see G. H. Hardy, *Pure Mathematics*, p. 409 (fifth edition), or E. T. Whittaker and G. N. Watson, *Modern Analysis*, p. 581 (fourth edition). Since these have more general objectives in view, it may be complained that they are too complicated for the purpose of defining the relatively simple expression e^{ix}. If, on the other hand, one wishes simplicity, he may straight off define e^{ix} to be the expression $\cos x+i\sin x$. This would settle the whole matter, but such a definition is unsatisfactory on intuitive grounds and appears to be drawn out of the air. It is hoped that the following definition is satisfactory on all three grounds: rigor, simplicity, and intuition.

DEFINITION. *e^{ix} is a complex valued function of the real variable x having the properties*:

$$e^{i0} = 1; \tag{1}$$

$$de^{ix}/dx = ie^{ix}. \tag{2}$$

THEOREM. $e^{ix}=\cos x+i\sin x$.

Proof: Let $e^{ix}=\rho(\cos\theta+i\sin\theta)$, $0\leqq\theta<2\pi$; $\rho\geqq 0$. From property (1) of the definition:

$$\text{when} \quad x = 0; \qquad \text{then} \quad \rho = 1 \quad \text{and} \quad \theta = 0.$$

From property (2) of the definition:

$$\frac{d\rho}{dx}(\cos\theta + i\sin\theta) + \rho(-\sin\theta + i\cos\theta)\frac{d\theta}{dx} = i\rho(\cos\theta + i\sin\theta).$$

Equating real and imaginary parts we have:

$$\frac{d\rho}{dx}\cos\theta - \rho\sin\theta\frac{d\theta}{dx} = -\rho\sin\theta$$

$$\frac{d\rho}{dx}\sin\theta + \rho\cos\theta\frac{d\theta}{dx} = \rho\cos\theta.$$

Solving, we obtain

$$\frac{d\rho}{dx} = 0; \qquad \rho\frac{d\theta}{dx} = \rho.$$

Hence: $\rho=1$ and $\theta=x$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 801. *Proposed by J. P. Ballantine, University of Washington*

How many different bidding sequences are possible in one hand of Bridge?

E 802. *Proposed by E. A. Nordhaus, Michigan State College*

(1) Find the smallest positive integer N having the property that the sum of its digits does not divide the sum of the cubes of its digits.

(2) Find the two consecutive positive integers each of which equals the sum of the cubes of its digits.

E 803. *Proposed by J. H. Butchart, Arizona State College at Flagstaff*

If the squares of the sides of a triangle form an arithmetic progression, the line joining the centroid and the symmedian point is parallel to one side of the triangle.

E 804. *Proposed by S. H. Gould, University of Wisconsin*

Denote by U the ellipsoid $a_1^2x_1^2+a_2^2x_2^2+a_3^2x_3^2=1$, by E_b the ellipse of intersection of U with the plane $b_1x_1+b_2x_2+b_3x_3=0$, by (p_1, p_2, p_3) a point variable on E_b, and by E_p the ellipse of intersection of U with the plane $p_1x_1+p_2x_2+p_3x_3=0$.

Determine (p_1, p_2, p_3) so as to minimize the major axis of E_p.

E 805. *Proposed by N. A. Court, University of Oklahoma*

If two coplanar edges of a tetrahedron are each equal to the respectively opposite edge, the remaining two opposite edges are each coplanar with the Euler line of the tetrahedron.

SOLUTIONS

Quadrangle Inscribed in an Equilateral Hyperbola

E 769 [1947, 224]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a plane quadrangle $ABCD$, the perpendicular at A to side AB cuts the opposite side CD in M, and the perpendicular at A to side AD cuts the opposite side BC in N. Show that the radical axis of the circles described on AM and AN as diameters coincides with the tangent at A to the equilateral hyperbola circumscribing the quadrangle.

Solution by L. M. Kelly, University of Missouri. Place the figure on a rectangular cartesian frame of reference so that the circumscribed equilateral hyperbola assumes the standard position $xy=k$, and let the coordinates of A, B, C, D be (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) respectively.

The circles on AM and AN as diameters intersect in a point on MN, so our problem is, evidently, to show that the tangent at A is perpendicular to MN. Draw lines parallel to the y-axis through B and D meeting AN and AM in P and P' respectively. Also draw lines through B and D parallel to the x-axis meeting AN and AM in Q and Q' respectively. The pencil $D(A\infty_yC\infty_x)$ is projectively related to the pencil $B(A\infty_yC\infty_x)$, whence the ranges $APNQ$ and $AP'MQ'$ are projectively related and, having A as a common corresponding point, are in perspective position. That is, PP', MN, and QQ' are concurrent. We propose now to show that the slopes of PP' and QQ' are both equal to x_1^2/k. This will establish the proposition.

The equation of AN is

$$\begin{aligned}(y-y_1)/(x-x_1) &= -(x_1-x_4)/(y_1-y_4)\\ &= -(x_1-x_4)/(k/x_1-k/x_4) = x_1x_4/k.\end{aligned}$$

Similarly, the equation of AM is

$$(y-y_1)/(x-x_1) = x_1x_2/k.$$

The coordinates P, P', Q, Q' are then found to be

$$P:\quad [x_2,\ x_1x_4(x_2-x_1)/k+y_1],$$

$$P':\quad [x_4,\ x_1x_2(x_4-x_1)/k+y_1],$$

$$Q: \quad [k(y_2 - y_1)/x_1x_4 + x_1,\ y_2],$$
$$Q': \quad [k(y_4 - y_1)/x_1x_2 + x_1,\ y_4].$$

The solution is now easily completed.

The proposer stated that this problem is a particular case suggested by a more general theorem given by M. A. de Majo, *L'Intermédiare des recherches mathématiques*, Paris, 1946, p. 45.

Derivatives of a Function

E 770 [1947, 224]. *Proposed by Norman Miller, Queen's University, Canada*

A function $f(x)$ has the following definition, n being a positive integer,

$$f(x) = x^n \sin \pi/x, \qquad x \neq 0,$$
$$f(0) = 0.$$

Show that, if $n=2m$ or $2m+1$ (m a positive integer), then at $x=0$ the function possesses a derivative of the mth but no higher order, and that, at $x=0, f^{(m)}(x)$ is continuous or discontinuous according as $n=2m+1$ or $n=2m$.

Solution by S. T. Thompson, Tacoma, Washington. We may show, by mathematical induction, that, for $x\neq 0$ and $0\leqq r\leqq m$,

$$f^{(r)}(x) = P_r \sin \pi/x + Q_r \cos \pi/x,$$

where P_r and Q_r are polynomials in x, in which the term of lowest degree in either polynomial is of degree $n-2r$. Again, by mathematical induction, we may easily show that

$$f^{(r)}(0) = 0, \qquad 0 \leqq r \leqq m.$$

(a) Suppose $n=2m$. Then, if we take $r=m$, either P_m or Q_m contains a term of degree zero, whence $\lim_{x\to 0} f^{(m)}(x)$ does not exist. Thus $f^{(m)}(x)$ is discontinuous at the origin, and, of course, $f^{(m+1)}(x)$ then cannot exist at the origin.

(b) Suppose $n=2m+1$. Then, if we take $r=m$, all terms of P_m and Q_m are of degree greater than zero. Hence $\lim_{x\to 0} f^{(m)}(x)=0$, and $f^{(m)}(x)$ is continuous at the origin. However,

$$\begin{aligned} f^{(m+1)}(0) &= \lim_{x\to 0} \frac{f^{(m)}(x) - f^{(m)}(0)}{x - 0} = \lim_{x\to 0} \frac{f^{(m)}(x)}{x} \\ &= \lim_{x\to 0} \left[(P_m/x) \sin \pi/x + (Q_m/x) \cos \pi/x\right]. \end{aligned}$$

Since P_m/x or Q_m/x contains a term of degree zero, this limit clearly does not exist, and therefore $f^{(m+1)}(x)$ does not exist at the origin.

Also solved by Elizabeth M. Huskey, E. S. Keeping, and the proposer.

Keeping pointed out that this problem, for $n=2$, is discussed in vol. 1 of de La Vallée Poussin's *Cours d'Analyse.*

The Most Likely Total

E 771 [1947, 280]. *Proposed by C. C. Carter, Bluffs, Illinois*

A die bearing the numbers 0, 1, 2, 3, 4, 5 on its faces is repeatedly thrown until the total of the throws first exceeds 12. What is the most likely total that will be thus obtained?

Solution by N. J. Fine, University of Pennsylvania. Consider the throw before the last; the total on that throw must be 12, 11, 10, 9, or 8. If the total is 12, the final result must be 13, 14, 15, 16, or 17, with an equal chance for each. Similarly, if the total is 11, the final result must be 13, 14, 15, or 16, with an equal chance for each, and so on. It is now clear that the most likely final result is 13.

Also solved by E. M. Berry, Monte Dernham, Abraham Golub, J. B. Kelly, J. M. Kingston, William Kruskal, and C. R. Phelps.

In addition to solving the given problem, Dernham considered the problem of computing the probabilities of obtaining the totals 13, 14, 15, 16, 17. These probabilities he found to be fractions with denominator 1,220,703,125 ($=5^{13}$) and respective numerators 408,335,461; 327,125,461; 243,735,961; 160,543,561; 80,962,681. He arrived at these results by considering the essentially equivalent problem: *From a bag containing* 5 *marbles, numbered respectively* 1, 2, 3, 4, 5, *a marble is drawn at random and replaced. This is repeated until the total of the numbers drawn exceeds* 12. *What are the respective probabilities of obtaining the totals* 13, 14, 15, 16, 17?

A number of solvers generalized the given problem by replacing 12 by N. The answer is then $N+1$.

Shortest Paths in a Lattice

E 772 [1947, 281]. *Proposed by D. H. Browne, Buffalo, N.Y.*

What is the number of shortest paths between two points in an n-dimensional lattice?

I. *Solution by Norman Miller, Queen's University.* Let each cell in the lattice have an edge of one unit. A shortest path between two points is made up of a number a_1 of units in a direction d_1, a number a_2 in a direction d_2, $\cdots$, a number a_n in a direction d_n. If we take a_1 identical letters d_1, a_2 identical letters d_2, $\cdots$, a_n identical letters d_n, then any distinct arrangement, in a line, of the $a_1+a_2+\cdots+a_n$ letters will represent a distinct one of the required paths. The number is therefore

$$(a_1 + a_2 + \cdots + a_n)!/(a_1!a_2! \cdots a_n!).$$

II. *Solution by J. B. Kelly, Kew Gardens, N. Y.* Let $R(a_1, a_2, \cdots, a_n)$ be the number of shortest paths between $(0, 0, \cdots, 0)$ and $(a_1, a_2, \cdots, a_n)$, where $a_1, a_2, \cdots, a_n$ are integers. There is no loss in generality in assuming that $a_1, a_2, \cdots, a_n$ are non-negative. Any shortest path between the two points must enter $(a_1, a_2, \cdots, a_n)$ from one of the points $(a_1-1, a_2, \cdots, a_n)$, $(a_1, a_2-1, \cdots, a_n)$, $\cdots$, $(a_1, a_2, \cdots, a_n-1)$. Hence

$$(1) \qquad R(a_1, \cdots, a_n) = R(a_1 - 1, a_2, \cdots, a_n) + \cdots + R(a_1, a_2, \cdots, a_n - 1).$$

Obviously

$$(2) \quad R(a_1, 0, 0, \cdots, 0) = R(0, a_2, 0, \cdots, 0) = \cdots = R(0, 0, 0, \cdots, a_n) = 1.$$

The unique solution of the difference equation (1) with the boundary conditions (2) is

$$R(a_1, \cdots, a_n) = (a_1 + a_2 + \cdots + a_n)!/(a_1!a_2! \cdots a_n!).$$

Also solved by Harry Goheen, R. B. Herrera, Leo Moser, C. R. Perisho, and the proposer.

Noughts and Crosses

E 773 [1947, 281]. *Proposed by A. L. Rubinoff, University of Toronto*

Suppose that noughts and crosses are played on an n-dimensional cube of side k. Show that there are precisely

$$\frac{(k + 2)^n - k^n}{2}$$

rows, columns, diagonals, . . . on which a win may be scored.

Solution by Leo Moser, University of Manitoba. Consider the "cube" of side k inside a cube of side $k+2$. Clearly, every win will determine exactly one pair of surface elements, while each surface element determines exactly one win. Hence the number of wins will be half the number of surface elements, which is the result stated.

Also solved by N. J. Fine and the proposer.

A Positive Determinant

E 775 [1947, 281]. *Proposed by R. P. Boas, Jr., Brown University*

Consider a determinant of order n whose elements are x on the main diagonal, ± 1 elsewhere. Find the smallest positive number a such that for $x > a$ the determinant is positive for all choices of the $\pm$ signs.

Solution by Olga Taussky (Mrs. John Todd), London, England. It is known that the determinant $d(a_{ij})$ is positive if $a_{ii} > \sum_{i \neq j} |a_{ij}|$, for $i = 1, 2, \cdots, n$. It follows that for $x > n-1$ the determinant in question is certainly positive for all choices of signs. We can take $a = n-1$ because the determinant clearly vanishes if $x = n-1$ and all the signs are negative.

The result quoted is due to P. Furtwängler (*Sitzunsber. Acad. Wiss. Wien*, Abt. II a, 145 (1936), p. 527) and can be proved by induction.

Related results have been discovered by Minkowski, Hadamard, Artin, and others. There recently were some notes about these in the *Mathematical Gazette* (XXVI, p. 191; XXVIII, p. 63; XXIX, p. 15).

Also solved (partially) by R. P. Brady and Norman Miller.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4280. *Proposed by Joseph Rosenbaum, Weatogue, Connecticut*

The points $A_1, A_2, \cdots, A_n$ are pursuing one another cyclically, the speed of A_i being proportional to the distance A_iA_{i+1}. If at time $t=0$ the points are the vertices of a given polygon, find the paths described by the points.

4281. *Proposed by M. S. Knebelman, Washington State College*

Given an integer n. Show that an integer can always be found which contains only the digits 0 and 1 (in the decimal scale) and which is divisible by n. Is there an algorithm for finding the smallest such number?

4282. *Proposed by F. W. Herlihy, Comstock, Michigan*

An elastic string (modulus λ, mass ma, unstretched length a) is confined within a straight tube to one end of which it is fastened. The tube rotates around that end with uniform angular velocity ω in a horizontal plane. Find the length of the string in equilibrium.

4283. *Proposed by E. P. Starke, Rutgers University*

The conjugate $\bar{z}$ of z, considered as a function of z, is nowhere analytic. Nevertheless, if C is an arbitrary circle or line, there exists a function $f(z)$ such that at every finite point of C, $f(z)$ is analytic and equal to $\bar{z}$. Consider also other curves for which a function exists having the same property.

4284. *Proposed by E. T. Bell, California Institute of Technology*

"Diophantus [II, 20 and IV, 45] proposed to find three squares such that

$$y^2 - x^2 : z^2 - y^2 = a:b,$$

where $a:b$ is a given ratio." (L. E. Dickson, *History of the Theory of Numbers*, vol. 2, p. 419.) For each of $a/b=3$, $1/3$ he gave one rational solution. Prove that the complete integer solution is given by

$$\begin{aligned} 2x &= k(a_1b_1g^2 + 2agh - a_2b_2h^2), \\ 2y &= k(a_1b_1g^2 + a_2b_2h^2), \\ 2z &= k(a_1b_1g^2 - 2bgh - a_2b_2h^2), \end{aligned}$$

where a_1, a_2 are any integers whose product is a, and b_1, b_2 are any integers whose

product is b, and k (for any pair of integer values of g, h) is an arbitrary integer multiple of the reciprocal of the G. C. D. of b_1g+a_2h, a_1g-b_2h, $a+b$.

SOLUTIONS

Special Tetrahedrons

4013 [1941, 639]. *Proposed by V. Thébault, Tennie, Sarthe, France*

Determine the kind of tetrahedron for which: (1) The circumcenter (or incenter) is on one of the medians. (2) The straight line joining the circumcenter to the centroid is perpendicular (or parallel) to one of the faces.

*Solution by R. Bouvaist, Vincelles (Saône-et-Loire), France.** Let $ABCD$ be the tetrahedron, O its circumcenter, G its centroid.

1. If O is on the median AG, the polar plane of G with respect to the circumsphere (O) is parallel to the tangent plane of (O) at A. The equation of (O) in tetrahedral normal coördinates (with $ABCD$ chosen as the reference tetrahedron) is†

$$\frac{a^2yz}{h_2h_3}+\frac{b^2zx}{h_3h_1}+\frac{c^2xy}{h_1h_2}+\frac{a'^2xt}{h_1h_4}+\frac{b'^2yt}{h_2h_4}+\frac{c'^2zt}{h_3h_4}=0,$$

where $BC=a$, $AD=a'$, $CA=b$, $DB=b'$, $AB=c$, $DC=c'$, and h_1, h_2, h_3, h_4 designate the altitudes drawn from A, B, C, D. The equations of the polar plane of G with respect to (O) and of the tangent plane of (O) at A are, respectively,

$$L_1 \equiv \sum (a'^2+b^2+c^2)x/h_1 = 0,$$

$$L_2 \equiv \frac{c^2}{h_2}y+\frac{b^2}{h_3}z+\frac{a'^2}{h_4}t=0.$$

These planes are parallel if

$$\frac{a^2-a'^2}{b^2+c^2-a'^2}=\frac{b^2-b'^2}{b^2-c^2-a'^2}=\frac{c^2-c'^2}{c^2-a'^2-b^2}\cdot‡$$

The incenter I will be on the median AG if $h_2=h_3=h_4$.

2. The line OG will be perpendicular to BCD if the polar plane of G with respect to (O) is parallel to BCD, that is, if

$$-(a^2-a'^2)=b^2-b'^2=c^2-c'^2.$$

* Translated and checked by W. E. Byrne, Virginia Military Institute.

† See Niewenglowski, *Cours de Géométrie Analytique*, t. III, Art. 120.

‡ The condition that L_1 and L_2 are parallel is that L_1, L_2, and L_3 be linearly dependent, where L_3 is the plane at infinity whose equation is

$$L_3 \equiv \frac{x}{h_1}+\frac{y}{h_2}+\frac{z}{h_3}+\frac{t}{h_4}=0.$$

We have

$$L_1-(a'^2+b^2+c^2)L_3 \equiv 2\lambda L_2$$

from which the above condition is easily obtained.

OG will be parallel to BCD if the polar plane of G with respect to (O) is parallel to the altitude drawn from A, that is, if

$$\frac{a'^2+b^2+c^2}{h_1}=\frac{a^2+b'^2+c^2}{h_2}\cos c'+\frac{a^2+b^2+c'^2}{h_3}\cos b'+\frac{a'^2+b'^2+c'^2}{h_4}\cos a,$$

where c', b', a are the plane angles of the dihedral angles of the edges DC, DB, BC.

Circles Related to the Triangle

4092 [1943, 457]. *Proposed by V. Thébault, Tennie, Sarthe, France*

For the triangle ABC let $(A_1B_1C_1)$, $(A_2B_2C_2)$, $\cdots$, $(A_nB_nC_n)$ be the centers of squares constructed exteriorly (or interiorly) on the sides (BC, CA, AB), (B_1C_1, C_1A_1, A_1B_1), $\cdots$, (B_nC_n, C_nA_n, A_nB_n) of the corresponding triangles. (1) Show that the center ω_1 of the circle orthogonal to the circles with centers A, B, C and radii B_1C_1, C_1A_1, A_1B_1 coincides with the center of the nine-point circle of ABC. (2) Find the locus of centers $\omega_2, \omega_3, \cdots, \omega_n$ of the circles, orthogonal to the circles with centers, A, B, C and with radii (B_2C_2, C_2A_2, A_2B_2), (B_3C_3, C_3A_3, A_3B_3), $\cdots$, (B_nC_n, C_nA_n, A_nB_n).

*Solution by R. Bouvaist, Vincelles, Saône-et-Loire, France.** Let M_1, M_2, M_3 be the midpoints of BC, CA, AB; H_1, H_2, H_3 the feet of the altitudes upon the same sides. We consider the squares drawn exteriorly, and suppose $a>b>c$. Then

$$\overline{B_1C_1}^2=\overline{AB_1}^2+\overline{AC_1}^2-2AB_1\cdot AC_1\cos B_1AC_1,$$

$$\overline{B_1C_1}^2=(b^2+c^2)/2+bc\sin A=(b^2+c^2+4S)/2,$$

where S is the area of ABC. Similarly

$$\overline{C_1A_1}^2=(c^2+a^2+4S)/2,\qquad \overline{A_1B_1}^2=(a^2+b^2+4S)/2.$$

The radical axis of the circles (B, C_1A_1), (C, A_1B_1) cuts BC in X_1. Then

$$\overline{A_1B_1}^2-\overline{A_1C_1}^2=(b^2-c^2)/2=M_1H_1\cdot BC=2a\cdot M_1X_1.$$

Hence $2M_1X_1=M_1H_1$, and similarly $2M_2X_2=M_2H_2$, $2M_3X_3=M_3H_3$. The radical center is therefore the center ω of the nine-point circle. Again

$$\overline{A_2B_2}^2-\overline{A_2C_2}^2=(\overline{A_1B_1}^2-\overline{A_1C_1}^2)/2=(b^2-c^2)/4.$$

If the radical axis of the circles (B, C_2A_2), (C, A_2B_2) cuts BC in X_1',

$$4M_1X_1'=M_1H_1,\qquad 4M_2X_2'=M_2H_2,\qquad 4M_3X_3'=M_3H_3.$$

Hence in general

$$2^nM_1X_1^{(n)}=M_1H_1,\qquad 2^nM_2X_2^{(n)}=M_2H_2,\qquad 2^nM_3X_3^{(n)}=M_3H_3.$$

* Translated and checked by C. J. Ramler, Catholic University of America, Washington, D. C.

The locus of the radical centers is then the segment ω_0 joining the center of the nine-point circle to the circumcenter of ABC.

Euler's Function

4221 [1946, 537]. *Proposed by P. Erdös, Syracuse University.*

Let $\phi(n)=n\prod_{p|n}(1-1/p)$ be Euler's function. Prove that for every k the equation $\phi(x)=k!$ is solvable.

Solution by J. Lambek, McGill University, Montreal. Consider the problem of solving the equation

$$\phi(x) = n \tag{1}$$

subject to the condition that x has precisely the same prime factors as n. The restriction on x implies that

$$\phi(x)/x = \phi(n)/n. \tag{2}$$

By (1) and (2) it follows that

$$x = n^2/\phi(n). \tag{3}$$

Let $n=\prod p^\alpha$. Then $x=\prod p^{\alpha+1}/(p-1)$. The integer x will have the same prime factors as n if and only if

$$\prod_{p|n} (p-1) \mid n. \tag{4}$$

Conversely (3) and (4) imply (1) and (2). Therefore a necessary and sufficient condition for (1) to be solvable under the given restriction is (4), the unique solution being given by (3).

If $n=k!$, (4) is clearly satisfied. An explicit solution of the proposed problem is therefore given by

$$x = (k!)^2/\phi(k!).$$

Solved also by P. T. Bateman, Alfred Brauer, N. J. Fine, William Gustin, R. B. Herrera, V. L. Klee, Jr., Emma Lehmer, Leo Moser, Ivan Niven, and W. V. Parker.

Volume of a Simplex in Terms of Its Edges

4222 [1946, 537]. *Proposed by J. H. Butchart, Grinnell College*

If points are numbered and if $\overline{12}$ denotes the distance from 1 to 2, then

$$\begin{vmatrix} 0 & \overline{12}^2 & 1 \\ \overline{21}^2 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2L^2,$$

where L is the length of the segment. Generalize this for the triangle and tetrahedron showing that the corresponding determinants are $-16A^2$ and $288V^2$

respectively, where A is the area of the triangle and V is the volume of the tetrahedron.

Solution by Leo Zippin and Joan Williamson, Queens College. We shall prove the following: *Let V_n be the volume of a simplex with vertices $P_1, P_2, \cdots, P_{n+1}$ in Euclidean space of n dimensions. If*

$$A = (a_{ij}), \qquad i, j = 1, 2, \cdots, n+1,$$

is the square matrix of order $n+1$ with $a_{ij} = \overline{P_iP_j}^2$ and if ϵ is the row vector $(1, 1, \cdots, 1)$ *of dimension $n+1$ and ϵ' is the column vector which is the transpose of ϵ, then*

$$\begin{vmatrix} A & \epsilon' \\ \epsilon & 0 \end{vmatrix} = (-1)^{n+1} 2^n (n!)^2 V_n^2.$$

In a rectangular coördinate system let the coördinates of P_i be given by the row vector $x_i = (x_i^{(1)}, x_i^{(2)}, \cdots, x_i^{(n)})$ and let C be the matrix of n columns and $n+1$ rows whose rows are the $n+1$ vectors x_i. Then the ordinary determinantal expression for V_n takes the form

$$(1) \qquad n!V_n = |C\epsilon'|.$$

Further

$$(2) \qquad CC' = B = (b_{ij}), \qquad i, j = 1, 2, \cdots, n+1,$$

where

$$b_{ij} = \sum_{k=1}^{n} x_i^{(k)} x_j^{(k)}.$$

If, O being the origin, we put $OP_i^2 = r_i^2$ and $\rho = (r_1^2, r_2^2, \cdots, r_{n+1}^2)$, we have

$$(3) \qquad r_i r_j \cos P_iOP_j = \sum_{k=1}^{n} x_i^{(k)} x_j^{(k)} = b_{ij}, \qquad i, j = 1, 2, \cdots, n+1,$$

and

$$(4) \qquad D = (d_{ij}) = \rho'\epsilon + \epsilon'\rho,$$

where $d_{ij} = r_i^2 + r_j^2$. Since

$$a_{ij} = \overline{P_iP_j}^2 = r_i^2 + r_j^2 - 2r_ir_j \cos P_iOP_j,$$

we have by (3) and (4)

$$(5) \qquad A = D - 2B.$$

If E is the unit matrix of order $n+1$, we have

$$\begin{pmatrix} E & -\rho' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & \epsilon' \\ \epsilon & 0 \end{pmatrix} \begin{pmatrix} E & 0 \\ -\rho & 1 \end{pmatrix} = \begin{pmatrix} A - \rho'\epsilon - \epsilon'\rho & \epsilon' \\ \epsilon & 0 \end{pmatrix} = \begin{pmatrix} -2B & \epsilon' \\ \epsilon & 0 \end{pmatrix}$$

by (5). Therefore

$$\begin{vmatrix} A & \epsilon' \\ \epsilon & 0 \end{vmatrix} = \begin{vmatrix} -2B & \epsilon' \\ \epsilon & 0 \end{vmatrix}. \tag{6}$$

Finally,

$$\begin{pmatrix} -2C & 0 & \epsilon' \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} C' & 0 \\ \epsilon & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2CC' & \epsilon' \\ \epsilon & 0 \end{pmatrix} = \begin{pmatrix} -2B & \epsilon' \\ \epsilon & 0 \end{pmatrix}$$

by (2). Therefore by (6)

$$\begin{aligned} \begin{vmatrix} A & \epsilon' \\ \epsilon & 0 \end{vmatrix} &= -\left| -2C \quad \epsilon' \right| \begin{vmatrix} C' \\ \epsilon \end{vmatrix}, \\ &= (-1)^{n+1}2^n \left| C \quad \epsilon' \right|^2 \\ &= (-1)^{n+1}2^n (n!)^2 V_n^2 \end{aligned}$$

by (1)

It is interesting to note that the proof can be considerably simplified if the assumption is made that $V_n \neq 0$. Then the origin may be taken as the center of the circumscribing hyper-sphere, whence $r_i^2 = r^2$, $i = 1, 2, \cdots, n+1$. Then (6) is obtained from the matrix equation

$$\begin{pmatrix} A & \epsilon' \\ \epsilon & 0 \end{pmatrix}\begin{pmatrix} E & 0 \\ -2r^2\epsilon & 1 \end{pmatrix} = \begin{pmatrix} -2B & \epsilon' \\ \epsilon & 0 \end{pmatrix}.$$

Editorial Note. W. V. Parker calls attention to the proof for the case $n = 3$ given in Kowalewski, *Einführung in die Determinantentheorie*, pp. 344–350.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, 27, N. Y. and not to any of the other editors or officers of the Association.

Differential and Integral Calculus. By F. D. Murnaghan. Brooklyn, The Remsen Press, 1947. 10+502 pages. $5.00.

In this book one finds an admirable, courageous, and, in the opinion of the reviewer, a completely successful attempt to present the calculus correctly, and at the same time, in a form intelligible to the beginning student.

The first departure from the methods to be found in the currently popular texts is the lucid explanation of the concept of a real number. Starting with the implicit and justifiable assumption that the student's intuitive notions concern-

ing counting numbers, integers, and rational numbers are correct, the author gives a brief (13 pages including 46 exercises) discussion of these concepts which states and explains the salient facts concerning them, as well as demonstrates the inadequacy of the rational number system for certain types of problems. In this discussion the student is asked to learn to distinguish between the positive integer 5 and the counting number 5, but the text is directed to the student and he is not subjected to the hair-splitting subtleties which often puzzle beginning graduate students. The exercises here are well graded in difficulty ranging from the first "Can the calculation indicated by the equation $x+7=11$ be performed in counting numbers?" to the last "Show that if the product ab of two positive integers is divisible by a prime number p then either a or b (possibly both) is divisible by p." Here and throughout the text hints are given when necessary and, in the case of exceptionally difficult problems, a complete solution is outlined.

The next twenty pages are devoted to the concept of a real number which is explained by means of nested sequences of rational intervals. Addition and multiplication of real numbers are defined and it is proved that $x(y+z)=xy+xz$, but the text is not encumbered with the tedious verification of all of the algebraic relations which define a field. Instead, the most important of these laws are stated as exercises. The student at this point cannot be expected to have much appreciation of the power of the concept of a real number, nor will he have mastered the technique of using this concept. However, after having used it many times at judiciously spaced intervals throughout the text even the mediocre student should appreciate its value. This technique appears for example in the proofs of the fundamental properties of continuous functions, in the proof of the integrability of continuous functions and in showing that local boundedness on a finite closed interval implies boundedness over the interval.

Another innovation consists of treating the concepts of differentiability, differential, and derivative, as well as formal drill on differentiation, before introducing the concepts of limit and continuous function. This is done by using the notion of null function (one whose limit is zero) as the basic idea.

Beside the topics usually found in a first course in the calculus which is designed to go through the differentiation and integration of the trigonometric, inverse trigonometric, logarithmic and exponential functions, the text contains a chapter on Special Plane Curves and one on Infinite Series and Integrals.

To the reviewer, who is accustomed to teaching "theory" to freshmen, it appears that there is no place in the text where the student gets an overdose of the abstract and more difficult parts of the calculus. The purely manipulative parts are well interspersed and in all there are over twelve hundred problems. It is apparent that the presentation of the calculus as found in this text has been well tested by classroom use and that Professor Murnaghan has spent much time and thought on the many problems involved in teaching undergraduate mathematics.

NELSON DUNFORD

Curso de Análisis Matemático. Tomo II. Por Cristóbal de Losada y Puga. Lima, Universidad Católica del Peru, 1947. 21+701 pages. $9.50.

The first volume was reviewed in this MONTHLY, vol. 53 (1946), pp. 268–279. The second volume continues in the same careful and clear style as its predecessor; the printing and illustrations are of the same high quality.

The volume is divided into three main parts: *Series, Geometric Applications of the Differential Calculus, Questions Concerning Integration.* The material is in general such as is customary in a course in advanced calculus.

The first part begins with standard material on convergence of series, with a chapter on uniform convergence; power series; double series; infinite products. The next two chapters are principally devoted to Taylor series in one and several variables, with applications. Weierstrass's nondifferentiable function is discussed in detail.

The second part discusses first plane curves, then space curves, and finally surfaces. The exposition is given in logical order rather than the usual textbook order, so that topics which are often treated separately as calculus and as differential geometry appear together.

The third part opens with a long chapter on numerical, graphical and mechanical integration. The error involved in Simpson's rule is discussed. Several types of planimeters are described and their theory is given. A chapter on improper integrals leads the author to give a proof of the transcendence of e. Chapters on the manipulation of integrals containing a parameter and on line integrals follow. The book ends with a chapter on multiple integrals and their transformation. The author gives Schwarz's example to show that the area of a surface cannot be defined as the limit of the areas of inscribed polyhedra.

R. P. BOAS, JR.

College Algebra. By M. Richardson. New York, Prentice-Hall, Inc., 1947. 16+472 pages. $2.85.

This book appears attractive, not too bulky, and clearly readable, as far as the printing and binding are concerned. It pioneers in a new road of explanation and arrangement of material, which may be a pattern for some courses in college algebra in the not too distant future.

An early definition of the usual mathematical symbols and Greek alphabet in the *Preface for the Student* is most timely and appreciated. An unusual effort in *The Number System of Algebra* (Chapter I) is made to acquaint the student with many of our usually accepted procedures and rules of high school algebra. To understand and appreciate these will strain the average freshman's intellect a trifle! Those who succeed will be well repaid for their effort. Such a fact as the irrationality of the square root of two comes in for several interesting pages of comment in this chapter. On page 27 the motivation for the definition of the product of two negative numbers is welcome for better college students. Algebra is treated as *a language, a logical science, a collection of techniques of calculation, a*

branch of human endeavor, as well as a collection of puzzles in Chapter II. It makes good reading material.

Exponents are appropriately explained in Chapter III. Wise emphasis on equations and identities (page 66), generous explanations of coefficient, degree of a monomial, and so on, are appreciated by the modern student. A thorough introduction to *Functions and Graphs* (Chapter IV) is followed by a discussion of the division algorithm (Chapter V) which makes factoring, as well as the least common multiple and highest common factor, in Chapter VI take on more meaning.

The analytic geometry of the linear function with slope and two point ideas attached, make Chapter VII read like part of a modern text-book on analytic geometry. Verbal problems get an extra dose of explanation before they are reached in the exercises. This has considerable merit in this decade of student weaknesses with such. In Chapter IX, the negative, zero, and fractional exponents come in for ample discussion before radicals are studied in the next chapter. *Quadratic Equations and Quadratic Functions* (Chapter XI) are handled thoroughly and *Systems of Equations in Two Unknowns* follow in the next chapter.

The usual treatment of *Ratio, Proportion, and Variation* in Chapter XIII and of *Complex Numbers* in Chapter XIV are given. In Chapter XV, a nice illustration of finding an irrational root of a polynomial equated to zero by use of synthetic division is given on pages 245 and 246. This required some tedious computation but students will appreciate its meaning. Horner's method was made easier by an explanation of linear interpolation.

Determinants and Elimination Theory of Chapter XVI have an unusually thorough treatment for a book of this nature, while *Permutations and Combinations* (Chapter XVII) and *Probability* (Chapter XVIII) remind one of the college algebra text-books of two decades ago. (The reviewer does not object to this, however.)

The soldier illustration of Chapter XIX makes mathematical induction take on real meaning for every reader. Much can be said in favor of discussing both this topic and the binomial theorem at this stage of the student's mathematical knowledge. Next follow *Progressions* (Chapter XXI), *Inequalities* (Chapter XXII), *Logarithms* (Chapter XXIII), and *Mathematics of Investment* (Chapter XXIV) with adequate treatments. Few college algebra books give a chapter to the *Euclidean Algorithm* (Chapter XXV), while *Partial Fractions* (Chapter XXVI) and *Series* (Chapter XXVII) are not uncommon topics for some of the older books as well as the newer better books.

Interpolation and Curve Fitting (Chapter XXVIII) includes the unusual treatment of the Lagrange interpolation formula, as well as other more common topics, apparently well discussed.

One is overwhelmed with a book of this type if he hopes an ordinary freshman will digest this material in a usual one semester course. It would seem more likely that this material would keep one busy for two semesters and then

he and the instructor would be grateful if an acceptable digestion process is realized! On the other hand, those topics which normally would be omitted in a one semester course will serve as valuable reference material for the student who continues with the study of mathematics.

W. R. HUTCHERSON

Calculating Machines. By D. R. Hartree. Cambridge, at the University Press; New York, The Macmillan Company, 1947. 40 pages. $.75.

This publication of the inaugural lecture of D. R. Hartree as Plummer Professor of Mathematical Physics in the University of Cambridge carries the subtitle, *Recent and Prospective Developments and their impact on Mathematical Physics*. In 35 small pages of text and two pictures (of the ENIAC), Professor Hartree provides a brief but comprehensive introduction to some of the problems associated with large scale computing machines.

After a comparison of the relative advantages of analogue and digital machines, the types of operations which components of a digital machine must perform are enumerated and discussed. The ENIAC provides examples of parts which perform these operations. Inadequacies of present machines and the consequent developments are mentioned. A set of simultaneous non-linear differential equations with two point boundary conditions on which Professor Hartree and the ENIAC worked during the summer of 1946 is used to illustrate the power of high speed calculating machines. This problem and the problem of solving simultaneous linear algebraic equations are discussed later in connection with the concept of the simplicity of any particular formulation of a mathematical problem; the simplicity of a formulation depends, of course, on the available aids to solution. The possibilities of important new lines of research in mathematics stimulated by shifts in the criteria of simplicity are emphasized. Mathematical physics will need to adjust to the new criteria of simplicity as well as to the possibility of dealing with problems heretofore involving prohibitive amounts of calculation.

K. J. ARNOLD

NEW BOOKS RECEIVED

Brief College Algebra. Revised Edition .By W. L. Hart. Boston, D. C. Heath and Co., 1947. 7+332 pages. $2.75.

Handbook of Engineering. Fourth Edition. By L. A. Waterbury. New York, John Wiley and Sons, Inc., 1947. 18+386 pages. $2.50.

Elementary Differential Equations. Third Edition. By L. M. Kells. New York, McGraw-Hill Book Co., 1947. 14+311 pages. $3.00.

An Introduction to Analytical Geometry. Vol. II. By A. Robson. Cambridge, at the University Press; New York, The Macmillan Company, 1947. 8+215 pages. $2.50.

College Algebra. By M. Richardson. New York, Prentice-Hall, Inc., 1947. 16+472 pages. $2.85.

Elementary Concepts of Mathematics. By B. W. Jones. New York, The Macmillan Company, 1947. 13+294 pages. $4.00.

Functions of a Complex Variable. Third Edition. By T. M. Macrobert. London, The Macmillan Company, 1947. 15+390 pages. $4.50.

Theory of Functions. Revised Edition. By J. F. Ritt. New York, Kings Crown Press, 1947. 10+181 pages. $3.00.

Lehrbuch der Darstellenden Geometrie. By E. Stiefel. Basel, Birkhauser, 1947. 173 pages. 28.50 sw.fr.

Vector and Tensor Analysis. By Louis Brand. New York, John Wiley and Sons, Inc., 1947. 9+439 pages. $5.50.

A Chapter in the Theory of Numbers. By L. J. Mordell. Cambridge, at the University Press; New York, The Macmillan Company, 1947. 31 pages. $0.40.

Fundamentals of Statistics. By T. L. Kelley. Cambridge, Harvard University Press, 1947. 16+755 pages. $10.00.

Elements of Mathematical Statistics. By C. V. L. Charlier. Translated and published by J. A. Greenwood (25 Winthrop St., Brooklyn 25, New York), 1947. 120 pages. $3.00.

Table of the Bessel Functions $J_0(z)$ and $J_1(z)$ for Complex Arguments. Second Edition. Prepared by the Mathematical Tables Project, National Bureau of Standards. New York, Columbia University Press, 1947. 44+403 pages. $7.50.

Manual of Mathematics and Mechanics. Second Edition. By G. R. Clements and L. T. Wilson. New York, McGraw-Hill Book Co., 1947. 9+349 pages. $3.25.

The Early Work of Willard Gibbs in Applied Mechanics. By L. P. Wheeler, E. O. Waters, and S. W. Dudley. New York, Henry Schuman, Inc., 1947. 7+78 pages. $3.00.

Elementary Nuclear Theory. By H. A. Bethe. New York, John Wiley and Sons, Inc., 1947. 6+147 pages. $2.50.

The Naming of the Telescope. By E. Rosen. New York, Henry Schuman, Inc., 1947. 16+110 pages. $2.50.

Yale Science, the First Hundred Years, 1701–1801. By L. W. McKeehan. New York, Henry Schuman, Inc., 1947. 8+82 pages. $2.50.

An Introduction to Definitive Philosophy and Basic Psychology. By Theodore Van Schelven. Kosmos, Amsterdam, 1947. 112 pages.

Teorica della Sopravvivensa. By F. A. Insolera. Turin, Giappichelli, Editore (Via Vasco 2 ang. Via PO 21, Torino, Italy), 1947. 7+242 pages.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

EDITORIAL NOTE. A request has come to the Editor of this section for a syllabus or bibliography on the construction of models and other visual aids in mathematics. In the belief that some mathematics clubs may be interested in such a syllabus, the Editor is asking those who know of sources of information on this material to kindly forward the references to him. A list will be compiled for dissemination to club members upon request.

CLUB REPORTS, 1946–1947

Pi Mu Epsilon, Montana State University

The fraternity made awards to the following outstanding freshman: Virgil Naumann, Barbara Dockery and Jean Popham. The mathematics award was won by Thomas Joyce and the physics award by Leonard Lust.

New members are Thomas Joyce and Robert Willson.

Officer: Director, David Bostwick.

Mathematics Club, New York University

In addition to several business meetings, numerous evening socials were conducted consisting of highly informative lectures by authorities in various fields of mathematical endeavor with accompanying discussion and refreshments. Among the lecturers were:

The theory of numbers, by Professor Bernard Friedman

Nim, by Professor Ralph Phillips

Some aspects of the concept of necessary and sufficient conditions, by Professor Louis Baron.

Some paradoxes and problems, by Professor Morris Kline.

Our annual Freshman Mathematics Contest, sponsored and completely administered by student Club members, was held in December. Cash prizes and awards of medals were given to the first place winner, Edmund Rosen, and to Edward Swell and Harvey Strizak, who were tied for second place.

Math X, the official spring publication, was a great success. Articles were contributed by the undergraduate student body, dealing with mathematical topics of interest both to the casual reader and the more exacting technical specialist.

Student tutoring was also conducted every afternoon from two to five o'clock by our members as a service to pupils in need of such help. This has proved a boon to students and faculty alike.

The main social event of the year, the annual boat ride to Bear Mountain, was held in June.

The officers for 1946–47 were: President, Bernice Goldberg; Vice-President, Selma Ring; Secretary, Claire Weber; Treasurer, Elizabeth Markowitz.

The officers elected for 1947–48 are: President, Sid Diamond; Vice-President, Eric Libori; Secretary, George Rubin; Treasurer, Rhoda Buser; Executive council members, Bernice Goldberg, Joseph Alper and Ruth Vure.

Pi Mu Epsilon, Northwestern University

The *Illinois Beta* Chapter of *Pi Mu Epsilon* held six regular meetings during the academic year 1946–47. Programs were as follows:

Mathematics as a language, by Professor H. T. Davis

Generalizations of $(-1)^n$, by V. C. Harris

The solar system, by Robert Howerton

Some theorems on projective geometry, by Miss D. M. De Witt

Dynamics of our solar system, by Professor Oliver J. Lee.

Francis Galton: pioneer statistician, by Mr. D. H. Leavens of the Cowles Commission for Research in Economics was presented at the initiation banquet in February at which time twenty-eight new members were initiated.

The annual undergraduate mathematical competition was held in May for which prizes consisting of books were awarded to Donald MacMillan, James Murrin, and Ernest Parker. The program at the last meeting of the year was presented by the winners of the contest and consisted of a discussion of the problems on this examination.

In addition to the regular meetings, *Pi Mu Epsilon* sponsored a "get acquainted" party in November for all mathematics students. Square dancing was the featured attraction of the evening. A joint picnic with the Mathematics Department was held in June.

Officers for 1946–47 were: President, Helen Gray; Vice-president, Roseann Grundman; Recording Secretary, Marjorie Smith; Corresponding Secretary, James Murrin; Treasurer, George Knapp. Officers for 1947–48 are: President, James Murrin; Vice-president, Donald MacMillan; Secretary, Roseann Grundman; Treasurer, Ernest Parker.

Kappa Mu Epsilon, Washburn Municipal University

Monthly meetings were held during the year 1946–47 by the *Washburn Mathematics Society*, which became the *Kansas Delta* Chapter of *Kappa Mu Epsilon* on March 29, 1947. The following programs were given:

Short cuts in simple mathematics, by Thomas Pirotte, John W. Nipps, Jr., and Harold Snider.

The binary system, by Ernest Dillman

Chinese rings and other puzzles, by Margaret E. Martinson

Russian multiplication system, by Mary Fix

Folding conics, by Frances Breneman

Tricks with a slide rule, by Dr. Paul Eberhart

Probability, by Beverly Brown

Mobius rings, by Harold Snider

Vectors, by William F. Seigle.

The history of mathematics in America by Dr. Ralph G. Sanger, Kansas

State College, was the title of an address given in March to a joint meeting of the *Society* with a group of mathematics teachers from Topeka and surrounding district schools.

The installation of the Society as the *Kansas Delta* Chapter of *Kappa Mu Epsilon* was conducted by Sister Helen Sullivan, O. S. B., National Historian of *Kappa Mu Epsilon.*

The officers for the year 1946–47 were: President, Terry D. McAdam; Vice-president, John W. Nipps; Secretary, Harold Snider; Treasurer, Miss Jean Badders; Corresponding Secretary, Miss Frances Breneman.

Those elected for the year 1947–48 are: President, Thomas Hotchkiss; Vice-president, Miss Jean Badders; Secretary, Miss Evelyn Hazlitt; Treasurer, Miss Jean Hobble; Corresponding Secretary, Miss Margaret Martinson; Sponsor, Miss Laura Z. Greene.

Mathematics Club, North Texas State Teachers College

Monthly meetings consisting of special programs, social activities and other functions were held by the *Mathematics Club* of North Texas State Teachers College during the year 1946–47.

The officers for 1947–48 are: President, Billy Workman; Vice-president, Kenneth Hannah; Secretary and Treasurer, Jack Standerfer; Reporter, Dan Spalding; Historian, Norma Harman; Sponsors, Dr. E. H. Hanson, Dr. J. B. Cooke, and Mr. Carl York.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

LECTURES BY DR. SYDNEY GOLDSTEIN

Dr. Sydney Goldstein, F.R.S., Professor of Mathematics at the University of Manchester, England, was presented at the University of Texas in a series of four lectures on the mathematics of transonic and supersonic airflow, on December 18, 19, and 20, 1947. Dr. Goldstein also delivered the Wright Brothers Lecture before the Institute of the Aeronautical Sciences in Washington, D. C., on December 17, 1947. Dr. Goldstein is Chairman of the Aeronautical Research Council of Great Britain.

OHIO CONFERENCE ON UNDERGRADUATE MATHEMATICS

Representatives of twenty liberal arts colleges met at Ohio Wesleyan University, Saturday, October 25, to hold a symposium on "Professional Problems in Teaching Undergraduate Mathematics in Liberal Arts Colleges." The con-

ference was organized by a small committee under the chairmanship of Professor Wayne Dancer of the University of Toledo. There were forty-three persons present, including representatives from Antioch College, Baldwin-Wallace College, Bowling Green State University, Capital University, University of Cincinnati, University of Dayton, Denison University, Heidelberg College, Hiram College, John Carroll University, Kent State University, Kenyon College, Marietta College, University of Michigan, Oberlin College, Ohio State University, Ohio Wesleyan University, University of Toledo, Western College, and Xavier University.

Professor S. A. Rowland presided during the first part of the conference when the group discussed the problem presented by the poor preparation in algebra of entering freshmen. Colleges which select their students carefully escape this problem, but schools which cannot apply selection, especially publicly-supported institutions, find that perhaps half the students enrolled in mathematics are not prepared to proceed with the traditional course in College Algebra. It seems to be common practice to give some sort of placement examination, requiring students making lower grades to take a course in intermediate algebra to correct their deficiency.

The next part of the discussion concerned the direction of work of superior students, Professor F. B. Wiley presiding. It was maintained that we do not properly challenge the ability of the better students. In some colleges these students are placed in separate sections for their freshman and sophomore mathematics, covering the same material as the other classes but doing a better job of it. Other ways suggested to develop the abilities of outstanding students were: an apprenticeship system in which a few students share the offices and certain tutorial duties with the faculty, enrollment in senior seminar courses, and encouragement to participate in undergraduate mathematics clubs. "Reading" courses seem to be out of vogue.

In the first part of the afternoon session Professor C. V. Newsom led the discussion on the ideal curriculum for a mathematics major. It was thought that the normal student specializing in mathematics should first take the customary sequence of courses through elementary calculus; then, in his junior and senior years study a year of advanced analysis (advanced calculus and differential equations), take a year's work in applications of mathematics (the exact courses to depend upon whether his interests lean towards physics, engineering, statistics, or teaching), some work in advanced algebra, a course in advanced geometry including the synthetic approach, and a course in the foundations or fundamental concepts of mathematics.

The group spent some time in discussing the problem: Should students who plan to take only one year of mathematics take the traditional courses in College Algebra, Trigonometry, and Analytic Geometry, or would they profit more from a one-year survey-type course covering a wider range of material? Although none present claimed to have found the complete answer to this question, there seems to be a trend toward the survey course for non-technical students.

Those with experience recommended that such a course include work on statistics and the mathematics of investment.

Professor Wayne Dancer took the chair for the discussion of opportunities for employment of young people who have specialized in mathematics for the bachelor's degree. It was recognized that the liberal arts college is not expected to prepare one for a vocation; nevertheless, students who want to major in mathematics also want to know how they can use their mathematics after graduation.

It was pointed out that mathematics is excellent training for subsequent professional study in many fields, and that an increasing number of professions are demanding a knowledge of advanced mathematics of their specialists. Several present reported that they had had calls from large industries and the Civil Service for college graduates who have majored in mathematics. Some corporations prefer to employ for many positions young people with a well-balanced liberal education, giving specific training themselves. In such cases a knowledge of mathematics is universally desirable.

To continue the study and discussion of problems raised, the group voted to ask the Ohio Section of the Mathematical Association of America to sponsor the work of this conference. An annual fall meeting for thi purpose met with general approval.

PERSONAL ITEMS

Fritz Herzog of Michigan State College has been promoted to an associate professorship.

Assistant Professor R. C. Hildner of the College of Wooster, Wooster, Ohio, has been appointed to an associate professorship at the University of New Mexico.

Dean C. C. Hurd of Allegheny College, Meadville, Pennsylvania, has accepted the position of theoretical research engineer at Carbide and Carbon Chemicals Corporation, Oak Ridge, Tennessee.

W. J. Jaffe of Newark College of Engineering has been promoted to an assistant professorship.

Assistant Professor Walter Jennings of Virginia Polytechnic Institute has been appointed to an assistant professorship at the Naval Postgraduate School, Annapolis, Maryland.

Associate Professor H. E. Jordan of the University of Kansas has been appointed to an associate professorship at Colby College.

Professor B. W. Jones of Cornell University will be on leave for the current academic year and will be at the California Institute of Technology.

A. R. Kirby of Fordham University has been promoted to an assistant professorship in the School of Medicine.

Dr. W. J. Kirkham of Oregon State College has been promoted to an associate professorship.

Professor H. D. Larsen of the University of New Mexico has been appointed to a professorship at Albion College, Albion, Michigan.

D. H. Lehmer, of the University of California has been promoted to a professorship.

Jerome C. R. Li of Oregon State College has been promoted to an assistant professorship.

Dr. W. S. Loud of the Massachusetts Institute of Technology has been appointed to an assistant professorship at the University of Minnesota.

Dr. A. K. Mitchell of Pratt and Whitney Aircraft Division of United Aircraft Corporation has been appointed to an associate professorship at the University of Maryland.

Dr. Josephine Mitchell has been appointed to an assistant professorship at Oklahoma A and M College.

Mr. A. G. Montgomery of the College of St. Thomas has been promoted to an assistant professorship.

Professor F. S. Nowlan of the University of British Columbia has been appointed to a visiting professorship at the Chicago Undergraduate Division of the University of Illinois for this year.

Assistant Professor R. E. Johnson of Mt. Holyoke College has been appointed to an associate professorship at Smith College.

Professor Rufus Oldenburger of the Illinois Institute of Technology is on leave of absence during the current year to complete a book on applied mathematics and to work as mathematician for the Woodward Governor Company, Rockford, Illinois.

Assistant Professor S. T. Parker of the University of Louisville has been appointed to an assistant professorship at Kansas State College of Agriculture and Applied Science.

Dr. P. K. Rees is now Associate Professor and Acting Head of the Department of Mathematics at Louisiana State University.

Associate Professor E. J. Purcell of the University of Arizona has been promoted to a professorship.

R. A. Rosenbaum of Reed College has been promoted to an associate professorship.

Assistant Professor C. E. Sealander of Iowa State College has been appointed to an assistant professorship at Ohio State University.

Abraham Seidenberg of the University of California has been promoted to an assistant professorship.

Assistant Professor W. S. Snyder of the University of Tennessee has been promoted to an associate professorship.

Professor C. F. Stephens of Prairie View University, Prairie View, Texas, has been appointed to a professorship at Morgan State College, Baltimore, Maryland.

W. M. Stone has been appointed to an assistant professorship at Oregon State College.

Otto Szasz of the University of Cincinnati has been promoted to a professorship.

H. P. Thielman of Iowa State College has been promoted to a professorship.

C. H. Vehse of West Virginia University has been promoted to a professorship.

Frantisek Wolf of the University of California has been promoted to an associate professorship.

The following appointments to instructorships are announced:

Barnard College: Grace L. Bolton

Berea College: Ruth E. Porter

Haverford College: Dr. D. L. Thomsen

Los Angeles City College: R. E. Horton

Modesto Junior College: A. J. Osner

Oregon State College: J. F. Price, Mrs. Hazel Thurman, Mrs. Evelyn Schroeder, R. J. Wise

Queens College, CCNY: Dr. J. E. Eaton

Triple Cities College, Syracuse University: E. O. Allen

University of Alabama: Irma Berkowitz

University of Buffalo: Jean Lee Blaney, B. J. Clark, F. P. Kowalewski, Jr., Jane L. Noller, D. D. Strebe

University of California, Extension Division: Frank Harary

University of Michigan: P. S. Jones

University of Nebraska: Dr. Edwin Halfar, Dr. W. G. Leavitt

University of Oregon: Mrs. Pearl Van Atta

University of Tennessee: Ralph Donnell, Clyde Miller, Jack Moshman, Frances Street, Dr. R. L. Wilson

West Virginia University: Thomas Bauserman

G. W. Evans, a charter member of the Association, died in February, 1947.

Professor T. W. Wiley died September 1, 1947.

Mr. A. Z. Skelding wishes to dispose of back volumes of the MONTHLY covering a large part of the period from 1916 to date, some bound and some unbound. Anyone interested in purchasing such volumes should write for further information to Mr. A. Z. Skelding, Actuary, 162 Hamilton Road, Hempstead, L. I., N. Y.

The winner of the prize scholarship in the 1947 William Lowell Putnam examination is Mr. William Turanski of the University of Pennsylvania.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

APRIL MEETING OF THE SOUTHEASTERN SECTION

The annual meeting of the Southeastern Section of the Mathematical Association of America was held at the University of South Carolina, Columbia, S. C., on Friday and Saturday, April 18–19, 1947. Professor Ruth W. Stokes, Chairman of the Section, presided at the Friday afternoon and Saturday morning meetings, and Professor W. L. Williams presided on Friday evening at the informal dinner given in honor of the visiting speaker, Professor L. M. Graves.

There were one hundred and forty-one present, including the following seventy-seven members of the Association: Louise Adams, E. A. Bailey, T. A. Bancroft, D. F. Barrow, Helen Barton, W. S. Beckwith, R. C. Blackwell, R. V. Blair, R. G. Blake, Floyd Bowling, M. G. Boyce, A. T. Brauer, T. C. Carson, B. G. Clark, F. E. Clark, E. G. Douglas, Nelle C. Douglas, Jeanette R. Durst, L. A. Dyke, F. A. Ficken, R. B. Folsom, Jack Frierson, L. L. Garner, W. H. Gaver, L. M. Graves, C. L. Hair, E. A. Hedberg, R. A. Hefner, Erik Hemmingsen, P. R. Hill, C. H. Holton, G. B. Huff, L. P. Hutchison, J. A. Hyden, L. W. LaGrone, G. B. Lang, J. W. Lasley, Jr., R. J. Levit, Anne L. Lewis, F. A. Lewis, R. A. Lytle, G. H. May, W. G. McGavock, S. W. McInnis, W. N. Mebane, Jr., H. A. Myer, D. D. Miller, W. B. Moye, H. M. Nahikian, Sara L. Nelson, J. D. Novak, Eugene Park, R. I. Pepper, I. E. Perlin, C. G. Phipps, Ellen F. Rasor, G. E. Reves, J. O. Reynolds, H. A. Robinson, L. B. Robinson, C. L. Seebeck, Jr., D. C. Sheldon, C. Eucebia Shuler, E. L. Stanley, L. W. Stark, A. L. Starrett, R. P. Stephens, Ruth W. Stokes, C. F. Strobel, Cora Strong, C. S. Sutton, Mary M. Templeton, J. M. Thomas, J. A. Ward, Betty R. Weber, W. L. Williams, Y. K. Wong.

At the business session the following officers were elected for the coming year: Chairman, J. W. Cell, North Carolina State College; Vice-Chairman, L. A. Dye, The Citadel; Secretary-Treasurer, H. A. Robinson, Agnes Scott College. The Section voted to hold its 1948 meeting at The Citadel.

The program consisted of the following papers:

1. *Generalized differential operators*, by Professor L. V. Robinson, University of South Carolina.

Taking for the argument the product of a function of x and the differential operator plus another function of x, several expressions involving functions of that argument operating upon other functions of x were derived. Likewise, corresponding expressions for inverse functions and functions of the inverse of the given argument were derived.

2. *On an equation all of whose roots are real*, by Professor Tomlinson Fort, University of Georgia, (by title).

The author discussed a little-known inequality that holds between the coefficients of an equation all of whose roots are real.

3. *The concyclic sets of points in the Morley configuration*, by Professor L. A. Dye, The Citadel.

The Morley configuration of a triangle consists of 27 points determined by 18 angle trisectors which lie six by six on nine lines. These lines are in three sets of parallels, making angles of 60° with one another. There are 144 circles, each of which passes through 4 or 5 of the 27+3 points of the figure. Professor Dye showed that there are 9 triples of Morley points concyclic with some pair of vertices of the triangle, and there are 54 other triples which are concyclic with one of the vertices. Further, there are 27 quadruples of Morley points which are concyclic with one of the vertices of the triangle and 54 other quadruples which are concylic.

4. *The non-existence of a certain type of odd perfect number*, by Dr. R. J. Levit, University of Georgia.

In this paper, Doctor Levit showed that no odd perfect number exists with the property that each of its prime power factors has a divisor sum which is itself a prime power, though all even perfect numbers do possess this property.

5. *Partial fractions with repeated linear factors in the denominator*, by Dean E. A. Bailey, LaGrange College.

Dean Bailey demonstrated certain properties of partial fractions, and the algebraic simplification of certain identities.

6. *A function with a finite discontinuity*, by Professor J. A. Ward, University of Georgia.

This paper appeared in the March, 1947, issue of this MONTHLY, page 162.

7. *Determination of a polynominal in several indeterminates by a finite set of its values*, by Mr. F. E. Clark, Duke University.

Mr. Clark proved that the coefficients of the complete polynomial f of degree p in n indeterminates are linear homogeneous polynomials in the values assumed by f at the set of points whose coördinates are the exponents of the various monomials of f. An application of this result to the proof of identities was given.

8. *A novel algorithm at the freshman level*, by Professor G. B. Huff, University of Georgia.

The algorithm presented was a device, discovered experimentally, for computing the values of a polynomial at a set of evenly spaced values of the arguments. The algorithm and its inverse apply to problems in theory of equations, interpolation and summation. Professor Huff stated that he had found the device useful as a challenge to enterprising undergraduates.

9. *A geometrical construction for a two star fix*, by Professor J. M. Thomas, Duke University.

Professor Thomas gave a ruler and compass construction in the plane for finding latitude and longitude from the declinations, Greenwich hour angles, and altitudes of two points on the celestial sphere.

10. *Undergraduate mathematics curricula*, by Professor L. M. Graves, University of Chicago.

In this paper, Professor Graves discussed some principles believed to be advantageous in the setting up of a curriculum in mathematics. Among these were: emphasis on the vector point of view in trigonometry and analytic geometry, as well as in later courses; emphasis on relations between different concepts and methods; and training in logic and postulation methods. Some changes and innovations already operating at the University of Chicago were mentioned.

11. *Exact probabilities for the common tests of statistical hypotheses*, by Professor T. A. Bancroft, University of Georgia.

Certain probability distribution functions were transformed into the incomplete beta-function ratio. A general formula and certain special cases derived by Professor Bancroft were given for obtaining values of the incomplete beta-function ratio outside the range of the existing tables.

12. *The jeep problem: a more general solution*, by Professor C. G. Phipps, University of Florida.

This paper appeared in the October, 1947, issue of this MONTHLY.

13. *An exact formula for the quoted price of a bond*, by Dr. P. M. Hummel and Dr. C. L. Seebeck, University of Alabama.

By the use of present values of future benefits, a formula was established for the quoted price of a bond to yield a given rate.

14. *A solution to the bracelet problem*, by R. G. Blake, University of Florida.

A solution of the bracelet problem was given by stringing together x_1 beads of one kind, x_2 of a second kind, and so forth. The problem was made to depend upon the number of cyclic permutations and upon the consideration of which permutations were symmetric.

15. *Functional analysis*, by Professor L. M. Graves.

After giving illustrative examples of functionals and functional equations, Professor Graves reviewed the differential and integral calculus in Banach spaces, and the fundamental theorems on implicit functions. A simple device was used to procure a theorem on the extent of the domain of functions defined implicitly. This theorem may be applied to show the existence of solutions, satisfying prescribed boundary conditions, of certain classes of nonlinear ordinary or partial differential equations.

16. *On Transon's formula for aberrancy*, by Professor J. W. Lasley, Jr., University of North Carolina.

Abel Transon in 1841 developed the concept of aberrancy, or deviation, called by him the second "affection" of curvature. In his analytical argument, Transon used the inclination of the tangent for the independent variable rather than the arc length of the curve. If the latter is used, Professor Lasley demonstrated it was possible by means of a one parameter family of equilateral hyperbolas to obtain Transon's formula by direct calculus methods, and in a simpler form.

17. *On the development of the idea of dimension*, by Dr. Erik Hemmingsen, University of Georgia.

The development of the concept of dimension was traced from the ideas implied by Euclid to the definitions used by the topologist. Each definition discussed was shown to have certain disadvantages.

18. *A derivation of the equations for vibrating string*, by Professor F. A. Ficken, University of Tennessee.

Under rather general assumptions, the derivation was examined in order to display what use was made of each hypothesis. The longitudinal component of the displacement satisfied an equation of Sturm-Liouville type in which each of the variable coefficients was present and had immediate physical significance.

19. *On the characteristic equation of certain matrices*, by Professor A. T. Brauer, University of North Carolina.

Professor Brauer gave an algebraic proof of a theorem on matrices obtained by von Mises from results in the theory of probability. The paper was published in the *Bulletin of the American Mathematical Society*, June, 1947.

20. *Derivation of the impedance circle diagram as a mapping problem*, by W. R. Mullin, Vanderbilt University, introduced by Professor J. A. Hyden.

The derivation of the impedance circle diagram for the computation of input impedance of a lossless transmission line was accomplished in this paper as a mapping problem rather than from electrical considerations. Certain facts concerning transmission lines were shown to follow directly from the mathematical derivation with the introduction of a minimum amount of electrical theory.

21. *Determinants whose columns (rows) are given sequences*, by Professor I. E. Perlin, Georgia School of Technology.

Professor Perlin proved certain theorems and gave certain interesting examples relative to determinants whose columns were composed of given sequences.

22. *A topic of infinite matrices*, by Professor Y. K. Wong, University of North Carolina.

This paper dealt with the reducibility of a linear transformation in Hilbert space in relation to its inverse. Examples were constructed to show that infinite matrices may have reciprocals without the property of reducibility.

23. *The plane projectivity between points connected by an associativity*, by Professor B. G. Clark, Vanderbilt University.

Using the fact that the vertices of two triangles in the plane are associated with their opposite sides and that two sets of points associated with the same set of lines are projective to each other, Professor Clark discussed the various types of non-singular collineations determined by six points in their several relative positions.

24. *A plane transformation determined by opposite edges of a tetrahedron*, by Professor J. M. Clarkson, North Carolina State College, introduced by the Secretary.

Professor Clarkson illustrated a method for a plane transformation by considering a tetrahedron whose three sets of opposite edges were parallel to the coördinate planes. The equations of transformation he used showed that the straight line loci of the xy plane were mapped into hyperbolas on the xz plane. Some of these hyperbolas were composite, and the points of the xy plane which caused the breakdown were singular points in the transformation.

25. *Maxima and minima of sums*, by Professor Tomlinson Fort, University of Georgia, (by title).

The calculus of variations is concerned with the problem of maxima and minima of definite integrals. The definite integral is the limit of a sum. The present paper treats maxima and minima of sums by methods not dissimilar to those commonly applied to the integral.

H. A. ROBINSON, *Secretary*

MAY MEETING OF THE NEBRASKA SECTION

The twenty-third meeting of the Nebraska Section of the Mathematical Association of America was held at the University of Nebraska, in Lincoln, on May 3, 1947. Professor Ralph Hull, Chairman of the Section, presided.

The attendance was twenty-four, including the following thirteen members of the Association: M. A. Basoco, H. W. Becker, E. M. Berry, A. K. Bettinger, W. C. Brenke, C. C. Camp, H. M. Cox, J. M. Earl, M. G. Gaba, C. B. Gass, E. H. Hadlock, Ralph Hull and Lulu L. Runge.

At the business meeting the following officers were elected for the coming year: Chairman, H. M. Cox, University of Nebraska; Secretary, Lulu L. Runge, University of Nebraska.

The following papers were presented:

1. *An application of the Laplace transformation to the solution of a certain boundary value problem*, by D. D. Rippe, University of Nebraska, introduced by the Chairman.

The speaker considered the differential equation

$$\theta_{xx}(x, t) = \frac{1}{K}\theta_t(x, t) \qquad t > 0, \ -a < x < a,$$

with boundary values

$$\theta(x, 0) = 0 \qquad -a < x < a, \qquad \theta(a, t) = \theta(-a, t) \qquad > 0$$

$$Q(a, t) = 2\int_0^a \theta(x, t)dx \qquad t > 0,$$

$$Q_t(a, t) = c[\theta_m - \theta(a, t)] \qquad t > 0.$$

He showed, by applications of the Laplace transformation and especially by the application of the theory of the complex inversion integral which arises therein, that it was possible to express the solution in the form

$$\theta(x, t) = \theta_m\left[1 - \sum \frac{e^{-kr_n^2/t}\cos(r_n x/a)}{r_n(1 + \delta + \delta^2 r_n^2)\sin r_n}\right]$$

where r_n is a root, z_n, of the transendental equation $\cot z = 2kz/ca$ and $\delta = 2k/ca$.

2. *On an integral equation with discontinuous kernel*, by C. C. Camp, University of Nebraska.

Reference was made to the paper by J. D. Tamarkin in Vol. 29 of the *Transactions of the American Mathematical Society*, in which he considered the equation $u(x) = \rho\int_0^1 k(x, \xi)u(\xi)d\xi$. In this the kernel has a jump of one across the line $x = \xi$. Recently the author has treated the case of such a jump across the secondary diagonal $x = 1 - \xi$ in addition, but in a partial derivative of $k(x, \xi)$. Professor Camp extended the theory to discontinuities in the kernel itself along both diagonals. He showed that, under certain restrictions, the associated expansion theory still holds.

3. *Problems in the algebra and topology of electrical circuits*, by H. W. Becker, Omaha, Nebraska.

The speaker showed the importance of the operation $a \| b = ab/(a+b)$ in electrical cultures. It generates one of an infinite number of functional arithmetics defined by $F(a, b) = a \oplus b$.

The easiest method yet devised for the solution of bridge circuits was established. The transfer

conductance G of any branch b is the fracto-sum of the G's of the two component circuits formed by opening and shorting any other branch b. That is,

$$G_b = G_{bo} \otimes G_{boo} = N_b/D_b = \frac{b'N_{boo} + N_{bo}}{b'D_{boo} + D_{bo}}.$$

The enumeration of bridge circuits is made difficult by the possibility of topological identity between apparently different configurations, and also of symmetries. Topologically generalized binomial coefficients are required, in which the combinations are reduced by any symmetries present. Symmetries also create topological products $C_m \otimes C_n \neq C_m \cdot C_n$ when $m=n$. The census of all bridge circuits of less than thirteen branches, basic or unrestricted, was tabulated under various classifications.

4. *A new approach to celestial navigation*, by O. C. Collins, University of Nebraska, introduced by the Secretary.

In this paper, it was remarked that for any two stars there exist on the celestial sphere two families of ellipse-like loci such that for the one family the sums of the star distances on the sphere are constants and for the other family the differences are constants. For special values of the parameters, these loci degenerate into a system of three mutually perpendicular great circles, one of which passes through the two star positions and another of which is equidistant from the two star positions. A point on the sphere at measured distances from the two star positions may be located with respect to the three-great-circle system of those two stars in terms of the lengths of two mutually perpendicular great circle arcs, whence its equatorial coördinates may be found if the positions of the stars are known.

5. *Geometry in college*, by C. A. Huck, Peru State Teachers College, introduced by the Chairman.

The speaker called attention to the increased attendance in mathematics classes following World War II. He remarked that, following World War I, mathematical study was placed in an unfavorable position, and urged that we now organize to avoid a recurrence of what happened at that time. He urged that mathematics be presented as a dynamic force in the development of all phases of life. Because of inadequate preparation, high school teachers are not capable of inspiring their students to continue the study of mathematics beyond the second year. In order to better prepare these teachers, colleges should offer a course in plane geometry, as well as college algebra.

6. *Note on the effect of postponement of college education by war emergency*, by H. M. Cox, University of Nebraska.

In this address Professor Cox reviewed the results compiled from the examinations routinely administered to all entering sophomore and freshman students, covering subjects such as mathematics, science, English usage, social studies, science information, and college aptitude. By analysis of variance, no significant differences on any one of the tests were found among either men or women students when they were grouped by year of graduation, namely 1940 through 1945.

There were, however, significant differences between the separate groups just described and the graduates of high school in 1946. Extreme variations occurred in all groups studied and on all of the examinations. The veterans group averaged lower than the normal group of entering freshmen, but an exploratory study of the achievement of veterans indicated equal chances of success for the older and younger veterans within the age range of 18 to 23.

7. *Sequential analysis in statistics*, by Ralph Hull, University of Nebraska.

Professor Hull gave an expository talk on recent developments in this field, due to Wald and others.

LULU L. RUNGE, *Secretary*

MAY MEETING OF THE WISCONSIN SECTION

The fifteenth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at the University of Wisconsin, Madison, on Saturday, May 3, 1947. Professor R. H. Bruck presided at the morning session, Professor R. H. Bardell at the afternoon session, and the Section Chairman, Professor H. P. Evans, presided at the business meeting.

There were forty in attendance, including the following twenty-four members of the Association: L. K. Adkins, K. J. Arnold, R. H. Bardell, Leon Battig, May M. Beenken, W. W. Begelow, R. H. Bruck, B. H. Colvin, H. P. Evans, E. G. Harrell, J. F. Kenney, S. C. Kleene, R. E. Langer, H. W. March, Morris Marden, Sister Mary Felice, F. E. Nemmers, Elli Otteson, G. A. Parkinson, H. P. Pettit, G. F. Rose, Abraham Spitzbart, J. I. Vass, and R. L. Wilson.

At the business meeting, the following officers were elected for the coming year: Chairman, R. C. Huffer, Beloit College; Secretary, P. L. Trump, University of Wisconsin; Program Committee, H. P. Pettit, Marquette University, Etylwynn Beckwith, Milwaukee-Downer College, E. G. Harrell, Platteville State Teachers College. The executive committee was instructed (1) to keep itself and the Section informed concerning proposed general changes in standards of mathematical curriculum or of teaching personnel; (2) to coöperate with other groups in promoting high educational standards in Wisconsin; (3) to use every suitable occasion for securing favorable publicity for mathematics in Wisconsin. It was voted to accept the invitation of Professor H. H. Conwell of Beloit College to hold the next meeting on May 8, 1948 at Beloit.

The following papers were presented:

1. *Industrial applications of high energy electron accelerators*, by Jack Wilson, Allis Chalmers Corporation, introduced by Dr. Louis A. Wolf.

2. *The word problem*, by Professor S. C. Kleene, University of Wisconsin.

The problem as proposed was based upon the assumption that a finite set of symbols, such as a, b, l, is given, with words constructed from them, such as *abbla*. Let a finite number of equations, such as $bbl = ab$, be postulated between parts of words, so that then $abbla = aaba$, $bblb = ablb$, and so on. Axel Thue in 1914 proposed the problem, to find a method by which one can always decide whether two given words are equal. To be certain that one can always decide means that the decision must depend solely on the outcome of a computation. Using the theory of computing machines given by A. M. Turing in 1936, Emil Post in the March 1947 *Journal of Symbolic Logic* proved that the above problem of Thue is unsolvable. Professor Kleene explained Turing's and Post's methods. The problem proved unsolvable is the same as the word problem for semi-groups, and the possibility appears herewith that the word problem for groups may likewise be unsolvable.

3. *Shock hydrodynamics*, by Professor J. O. Hirschfelder, University of Wisconsin, introduced by Professor H. P. Evans.

The equations of hydrodynamics for non-viscous fluids were derived from Newton's laws. It was shown that, whereas the Lagrangian form of the equations appears to be linear, the Eulerian forms are non-linear. In all practical problems it is the Eulerian form which is used. These equations permit two types of solutions, namely continuous and discontinuous. The possibility of the discontinuities of the solutions is due to the non-linearity of the Eulerian equations. Riemann developed a method of characteristics for solving one-dimensional hydrodynamical problems when-

ever the solution is continuous. The continuous solutions are characterized by a constancy of the entropy for the individual elements of the fluid. If the fluid is compressed, the Riemann characteristics come together and the solution becomes discontinuous. The planes of the discontinuities are known as shocks. There is a finite and calculable entrophy change of the fluid element in passing across the shock. A continuous pressure pulse will develop a discontinuity after a certain length of time. The mathematical behavior of the interaction of shock waves is still open for investigation, and many strange effects are known experimentally to occur which may be traced directly to the non-linearity of the hydrodynamical equations.

PAUL L. TRUMP, *Secretary*

MAY MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The May Meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the George Washington University, Washington, D. C., on Saturday May 3, 1947. Professor W. K. Morrill, Chairman of the Section, presided.

The attendance was sixty including the following forty-two members: T. E. Berry, W. E. Bleick, H. H. Campaigne, C. R. Clark, Abraham Cohen, J. F. Daley, Alexander Dillingham, J. A. Duerksen, E. J. Finan, N. J. Fine, Ansel Fisher, M. K. Fort, Jr., B. C. Getchell, Michael Goldberg, R. A. Good, D. W. Hall, G. A. Hedlund, M. A. Hyman, Ernest Johnston, F. E. Johnston, Sidney Kaplan, L. M. Kells, V. L. Klee, Jr., W. D. Lambert, A. E. Landry, Carol V. McCamman, Florence M. Mears, A. B. Mewborn, W. K. Morrill, W. H. Norris, Jr., E. K. Paxton, C. E. Rhodes, J. N. Rice, E. D. Schell, W. F. Shenton, A. D. Sollins, J. H. Taylor, J. Tyler, A. W. Tucker, W. R. Utz, A. L. Whiteman, G. T. Whyburn.

At a business meeting held during the afternoon session the following officers were elected for the coming year. Chairman, E. J. Finan, Catholic University; Secretary-Treasurer, M. H. Martin, University of Maryland; additional members of the Executive Committee, G. A. Bingley, St. John's College, and T. W. Hatcher, Virginia Polytechnic Institute. It was announced that D. W. Hall, University of Maryland, had been elected Sectional Governor.

The first four of the following papers were read at the morning session. Professor Tucker's paper was read at the afternoon session at the invitation of the Section.

1. *A geometrical approach to Frandtl-Meyer flow*, by Professor M. H. Martin, University of Maryland.

In two-dimensional, steady, irrotational, isentropic flows for which the velocity potential surface $\phi=\phi(x, y)$ is a developable surface, the Legendre transformation which carries the partial differential equation

$$(G^2 - u^2)\phi_{xx} - 2uv\phi_{xy} + (G^2 - v^2)\phi_{yy} = 0$$

into a linear partial differential equation (L) breaks down. In the investigation of this exceptional type of flow if the developable surface is viewed as the envelope of a parameter family of planes $\phi=ux+vy-\omega$, where u, v, ω are functions of a parameter τ, the following facts are easily established: (i) The curves $u=u(\tau)$, $v=v(\tau)$ in the hodograph plane are epicycloids, the characteristics of (L); (ii) The flow in the physical plane is of Pradtl-Meyer type, the Mach lines being the projections of the generators of the developable surface upon the physical plane; (iii) The angle between

the direction of flow and a Mach line is the Mach angle; (iv) The equipotential lines are presented parametrically by

$$Z = Me^{i\theta} + \frac{\phi}{l} e^{i(\theta-\alpha)}.$$

in which $M = W + iW$; where W is an arbitrary function of the parameter θ (the direction of flow) and l is a determined function of θ having geometrical significance.

2. *Some properties of pointwise periodic transformations*, by E. E. Floyd, University of Virginia, introduced by the Secretary.

Let X be a compact metric space and $f(X) = X$ a pointwise periodic transformation on X. We say that f on X gives a continuous orbit decomposition at $x \in X$ if $x_i \to x$ implies $\lim O(x_i) = O(x)$, where $O(y)$ denotes the orbit of y. It is shown that, if $x \in X$ is a non-isolated point, then there exists a sequence $x_i \to x$ such that $\lim O(x_i) = O(x)$. Using this as a lemma, it is shown that if f does not give a continuous orbit decomposition at $x \in X$, then there exists an invariant perfect set K, $K \subset X$ and $x \in K$, such that $x_i \to x$, $x_i \in K$, implies $\lim O(x_i) = O(x)$.

3. *Leonard and Thomas Digges—their contribution to practical measurement*, by Professor W. F. Shenton, American University.

This study of the lives and mathematical contributions of this father and son team of Elizabethan mathematicians springs from a consideration of the interesting woodcuts in the copy of the 1591 *Pantometria* from the Artemas Martin Library. It shows, by means of lantern slides, the construction and use of the quadrant, square, and theodolite in working simple surveying problems. As to their place in mathematical performance, Mr. Halliwell says "Thomas Digges ranks among the first English mathematicians of the sixteenth century. Although he made no great addition of science, yet his writings tended more to its cultivation than perhaps all those of other writers on the same subjects put together."

4. *Formulas for numerical quadrature*, by Professor E. J. McShane, University of Virginia.

By integration of well-known interpolation formulas, numerical quadrature formulas are developed for computing $y(x)$ from the tables of $y'(x)$, or of $y''(x)$, or of both. One of these is useful as a check formula when using Simpson's rule. Another (due to C. B. Morrey) is useful when both y' and y'' are tabulated; it is easy to use and is more accurate than Simpson's rule. Another gives the second difference of y in terms of y''. These formulas are combined to furnish a procedure for the numerical integration of second-order differential equations, or of systems of such equations.

5. *The use of transformations in college geometry*, by Professor A. W. Tucker, Princeton University.

Elementary transformations and their products can play the unifying role in college geometry that elementary functions and their compounds play in college algebra and calculus. Reflections, inversions, pole-polar reciprocations, and their products (operating on the material of ordinary plane geometry) can be used to construct various geometries that belong naturally to the *Erlanger Programm* of Felix Klein. Reflections generate the displacements and symmetries of the euclidean geometry of congruence; reflections and inversions generate the direct and inverse circular transformations of conformal geometry; reflections and pole-polar reciprocations (in circles) generate both the collineations and correlations of projective geometry. Of course, the geometric forms and quantities (such as cross-ratio) that are left invariant by a whole group of transformations are precisely those preserved by the elementary transformations used to generate the group. The need for a transformation to be one-to-one without exception leads to the invention of a single point at infinity in conformal geometry and of a line of points at infinity in projective geometry. Certain

"subgeometries" of conformal geometry and projective geometry arise very simply from restrictions on the transformations that leave the point at infinity fixed, hyperbolic non-euclidean geometry from the circular transformations that preserve the interior and circumference of one particular "fundamental" circle, affine geometry from the collineations that preserve the line of points at infinity, and elliptic non-euclidean geometry from collineations related by central projection to spatial transformations generated by reflections in planes through the particular center of projection.

E. J. FINAN, *Secretary*

MAY MEETING OF THE KENTUCKY SECTION

The annual meeting of the Kentucky Section of the Mathematical Association of America was held at the University of Kentucky on Saturday, May 10, 1947. Dr. Guy Stevenson, Chairman of the Section, presided.

There were fifty-four in attendance, including the following twenty-one members of the Association: M. C. Brown, P. P. Boyd, H. H. Downing, Clarence Ford, A. A. Grau, Charles Hatfield, Sr., Aughtum S. Howard, W. R. Hutcherson, S. J. Jasper, E. D. Jenkins, C. G. Latimer, D. J. Myatt, R. S. Park, Sallie E. Pence, D. W. Pugsley, G. G. Roberts, W. J. Robinson, F. V. Rohde, J. H. Simester, D. E. South, Guy Stevenson.

At the business meeting the following officers were elected for the next year: Chairman, D. W. Pugsley, Berea College; Secretary, Sallie E. Pence, University of Kentucky. The annual meeting in 1948 is to be held at Berea College.

The following papers were presented:

1. *Trisection of an angle by means of higher plane curves*, by P. P. Boyd, University of Kentucky.

Dean Boyd gave two methods for the graphical trisection of an angle by means of higher plane curves. He first reduced the problem of the intersection of the type cubic with a conic to the intersection of two conics, and next presented the solution of de Longchamp's (1888). He concluded by mentioning the nature of the problem and the proofs of Klein and Dickson.

2. *Development of the Frenet formulas for N-dimensions*, by S. J. Jasper, University of Kentucky.

The well known three dimensional vector forms of the Frenet formulas were extended to a flat space of N-dimensions. The speaker developed the method of finding the $(N-1)$ unit normal vectors and the $(N-1)$ curvatures associated with these vectors for a curve in N-dimensions.

3. *Contrasting two solutions of a certain problem in modern geometry*, by W. R. Hutcherson, Berea College.

Given any triangle ABC, through the three vertices lines AL, BM, and CN were drawn through any point O to points L, M, N on the opposite sides. Two proofs were given for the formula

$$\frac{OL}{AL}+\frac{OM}{BM}+\frac{ON}{CN}=1.$$

The first proof was a lengthy one depending upon Ceva's theorem. The second one was quickly accomplished by consideration of the areas of certain triangles.

4. *Applications of mathematics in meteorology*, by McClellan Cook, Jr., University of Louisville, introduced by Dr. Stevenson.

The speaker discussed practical applications of formulas for forecasting the movement of isobars, isallobars, pressure centers, and fronts, and for forecasting the acceleration of pressure systems and the intensifications of pressure systems.

5. *The graphic construction of a lunar eclipse*, by F. V. Rohde, University of Kentucky.

After defining the astronomical terms involved, the computations necessary for the graphic construction of an eclipse from elements, as given in the American Ephemeris and Nautical Almanac, were explained and an actual construction made for the eclipse of June 3, 1947. Eclipse limits and the conditions that must be satisfied before a lunar eclipse can occur were discussed briefly.

M. C. Brown, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirtieth Summer Meeting, Madison, Wisconsin, September 6–7, 1948.

Thirty-second Annual Meeting, Columbus, Ohio, December 31, 1948.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

Allegheny Mountain, Pennsylvania State College, May, 8 1948
Illinois
Indiana
Iowa, Fairfield, April 16–17, 1948
Kansas, Atchison, April 10, 1948
Kentucky, Berea, Kentucky, May, 1948
Louisiana-Mississippi, Lafayette, La., April 23–24, 1948
Maryland-District of Columbia-Virginia
Metropolitan New York, April 24, 1948
Michigan
Minnesota
Missouri
Nebraska, Lincoln, May 1, 1948
Northern California
Ohio
Oklahoma
Pacific Northwest, Eugene, Oregon, March, 1948
Philadelphia, Philadelphia, Pa., Nov. 27, 1948
Rocky-Mountain
Southeastern, Charleston, S. C., March 19–20, 1948
Southern California, Redlands, March 13, 1948
Southwestern
Texas
Upper New York State, Schenectady, N. Y., May 1, 1948
Wisconsin, Beloit, May 8, 1948

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THE SCHOLAR IN A SCIENTIFIC WORLD*

C. C. MacDUFFEE, University of Wisconsin

1. Introduction. Recorded history began with the scholar. Sometime in that great blank period from twenty to six thousand years ago, the art of writing was developed so that the scholar could record the legends and history of his nation. The scholar was then also the priest and the lawyer and the physician as he has been throughout most of recorded history.

Since earliest times scholars have lived in communities. These communities or monasteries, or musea, were the early colleges and universities. They have constituted the chief vehicle in the development of civilization. If these little bands of scholars had not existed, we should now be in the woods carrying stone axes.

A nation is just about as great as its universities. They are the ganglia in the central nervous system of the nation, whence come the nerves that stimulate its intellectual, industrial, and political life. Without the leadership of the universities, national life would probably continue for a while by inertia but would gradually slow down and succumb to the competition of rival nations.

The function of the university as a teaching institution is purely incidental. Its principal function is to keep alive the great wealth of knowledge and culture that past ages have collected, and to add to that wealth through the encouragement of scientific research and creative art. Knowledge and art can be kept alive only by the creation of scholars, and the scientific horizon can be extended only by the creation of scientists. To create scientists and scholars is the primary function of the university.

This statement of the function of the university is orthodox. It is as old as the universities themselves. It is the idea which induced Alexander to establish the Museum at Alexandria, the institution which shone out over the ancient world for the 600 years of its highest culture and where much of our mathematics was made. Alexandria was doubtless the greatest university of all time, and the burning of its library in 389 might well mark the beginning of the Dark Ages.

The Renaissance was marked by, and was in a very large measure due to, the establishment of the great universities in Padua, Paris, and Oxford, and soon in many other places. The study of the great legacy of Greece was brought into Europe by these universities, where it had been extinct for many years. Men had to learn again that it is not evil to think, and that we live in a reasonable world. The scientific developments of our generation could not have taken place if these medieval universities had not come into being.

2. Science in Modern America. American universities have progressed amazingly in a century, and so has the country. From the position of a backward nation we have come to the very front. The opportunity to establish undisputed

* Retiring address as President of the Mathematical Association of America, Athens, Georgia, January 1, 1948.

intellectual leadership is ours. Whether we can maintain this position when the influx of German scientists stops remains to be seen.

As time goes on, and more and more is known of science, archaeology, and history, it becomes increasingly difficult to keep some branches of our culture from becoming lost. The day has long since passed when one man could have a working knowledge of all branches of science. The goal of the philosopher to know all things and to correlate them into one harmonious whole now seems very difficult of fulfillment.

The last war has destroyed many of the cultural links between our age and the past, particularly by the wanton bombing of the libraries and museums by both sides. It has also reduced the number of scholars in Europe so that some links that we had with our past are probably now gone forever. We must be willing to support scholars in all fields without everlastingly measuring their output in terms of American dollars.

Everything that I have said so far is trite and generally admitted, but we in the United States do not make it part of our practical thinking. We do not believe that the production of scholars is in itself a worthy objective, nor that the scholar is worthy of his hire. Our colleges and universities are supported by state and private funds, but the stated objective is the instruction of vast numbers of young men and women in ways and means of earning a better living than their fellows. They are taught citizenship and the American way of life. I am not always clear just what this means. Occasionally it seems to be football and the cult of the gods and goddesses of Hollywood.

We are now living in the Age of Science. One might almost say that Science has become our national religion. It has revolutionized our daily lives, our knowledge of the past, and our hopes for the future. Its miracles are commonplace and we can perform them ourselves. The researchers in medicine are the angels of mercy, and in popular imagination the Atomic Scientists are the devils.

This elevation of Science to a god-like eminence creates many difficult problems for the scientist, who is not primarily interested in becoming a high priest. First, everyone wants to be a scientist. Astrologers, numerologists, even religious groups, conjure with the name "Science." On a little higher plane, we have Political Science and the Science of Economics, when everyone knows that these subjects are not yet truly scientific. Even Psychology has renounced its parent, Philosophy, and wishes to be known as a science.

Everyone is pathetically eager to understand science, and of course most persons are unable or unwilling to pay the price in long years of study. Our book stores are flooded with books on "Relativity in Five Easy Lessons," illustrated with doodles, and "How to Make an Atomic Bomb." Even the elementary schools have courses in General Science, and eight-year-olds lisp that Einstein discovered the relativity of time and space. It is questionable whether sophistry at this level does more harm than good, but probably nothing can be done about it. It is part of the ritual in the worship of Science. A classic example is the quotation from the school boy's paper: "Gravitation was discovered by Sir Isaac

Walton while he was digging for hook-worms under an apple tree. It is more noticeable in the fall than in the spring."

If it were necessary or desirable to draw a fine distinction between the scholar on the one hand, and the artist or scientist on the other, it would have to be that the scholar is primarily interested in perpetuating the known, the scientist in new creation. But they are not independent. New discoveries are now rarely made by attic inventors. The creator is so steeped in the knowledge of his predecessors that he can begin where they left off. Otherwise he merely rediscovers.

There are fashions and fads in intellectual pursuits as well as in politics or clothes. At the time of Socrates in Greece, speculative philosophy was the fashion, and so was sculpture in stone. In the Middle Ages about the only outlet was theology. There were later periods in which great music was produced; others were notable for great paintings. In sixteenth century Italy there was a remarkable flowering of painting and sculpture. It must have been true that the people of this period were enthusiastic about art, and that the great painters were honored and important people. The result was that the best minds of the period were attracted into this field, and that they were stimulated by knowing that their work was appreciated.

In this period lived Leonardo da Vinci, one of the greatest potential scientists of all time. It has been said that if he had published his scientific discoveries, science would have been revolutionized. But he could not have published, for he had no sympathetic audience. In his own period he was known and honored as an artist, and it is only now in the Age of Science that his scientific thoughts are appreciated. If he had lived today, it is probable that he would have become a great scientist and that his potentialities as an artist would have remained undeveloped.

Deans of graduate schools and those who have served on university committees of award for scholarships will testify that, by and large, the undergraduate records of those who apply for fellowships in mathematics and the natural sciences excel all others. It is all very well to argue that we need the best minds in Sociology and Politics. The fact remains that the country is worshipping at the shrine of the Natural Sciences.

3. Science in the curriculum. There is now a widespread movement in the colleges and universities to incorporate some science into the liberal arts program. It seems as if all the weight of logic is in favor of this. We study and admire the gems of various older cultures, their stories, poems, songs, pictures, statues, and philosophical concepts. Included in the education of the well-rounded scholar should be the gems of thought of modern man. These gems are largely the processes of thought of the modern scientist.

But it is not going to be easy to design a curriculum which includes little dabs of the sciences without creating a nation of sophomores. This is now being done to some extent by the courses in General Science in the grade schools where students learn that Newton discovered gravitation by being bonked on

the head by an apple. But this does no lasting harm because the better students unlearn the fallacies and the poorer ones forget them. But to create a nation of college graduates who believe they know all about science from a Basic Course in the Physical and Biological Sciences would be a pity.

There is of course the classic instance of the sweet girl undergraduate who sat next to a famous astronomer at a Boston dinner party. She asked him what his business was. "I study astronomy," was the reply. "Oh, you do!", the girl answered in surprise, "Why, I finished that last year!"

Let us grant that it is no longer possible to deny the natural sciences their rightful place in the education of the humanist. There remains the problem of what science shall be presented, how it shall be presented, and by whom.

Clearly the student of the arts is not going to get a very complete course in any one scientific subject under this plan. Perhaps he isn't going to get any laboratory work at all. He may get nothing but a huge mass of undigested and undigestible facts. If the instructor is an experimental scientist, and is intent upon making a chemist or biologist of the student in one semester, it is not difficult to imagine a course which would be a veritable monstrosity, capable of inducing a permanent allergy on the part of the student victim toward all science.

Equally unsatisfactory will be the other extreme where the instructor is a philosopher who knoweth all things, believeth all things, and seeth through a glass darkly. He can clothe the simplest concept in metaphysical terms, which mean nothing to anybody. The flights of fancy which such people have indulged in, using Relativity and the Heisenberg Principle as justification, are known to all of us.

If science is to be taught in the same spirit as the other humanities, it will be taught with the object of exhibiting the thought processes of the scientist. One currently studies Art Appreciation without attempting to model in clay, and the liberal arts student can gain an insight into modern science without acquiring sulfuric acid burns.

The study of the scientific method, then, is the proper objective for a liberal arts course in the sciences. But the scientific method is the mathematical method, for the discoveries in science are obtained either by a direct use of known mathematics, or by the same methods of careful logical thinking to which mathematics is the best introduction. Without the scientific method, science becomes mere classification and memory.

Physics is now riding a crest of popularity not unmixed with awe. It will have a prominent place in the program. The lecturer will have the choice of merely listing the triumphs of Physics, or pulling the curtain a little and showing the student how these results were arrived at. Pulling the curtain means using mathematics.

Astronomy is another science which deserves a prominent place in any cultural course. Unfortunately that seems to be its only place at present, for our industrialists have not yet found a means of developing the natural resources of

the stars.

Perhaps I have given an unnecessarily long preamble to a mathematical audience before stating my point, which is this: One of the brightest gems in the crown of achievement of civilized man is the Differential and Integral Calculus, and no man has the right to call himself a Humanist if he is completely ignorant of its meaning.

Many members of the Mathematical Association of America are members of the curriculum committees of their respective institutions. It is to be hoped that they will have enough appreciation of their own subject, and enough knowledge of the fundamental role which it plays in modern Science, to demand that an adequate exposition of the mathematical method be incorporated in every generalized university or college course. Plato's belief in the value of mathematical thinking is still justified. In the years to come, Economics and Political Science, even Chemistry and Physics, will change so that our present ideas will seem absurd. But, as J. W. A. Young once remarked, in ten thousand years the square on the hypotenuse will still equal the sum of the squares on the sides.

4. The demand for scientists. The production of an adequate number of scientists is a problem which is now of great concern to the United States Government, as well it may be. Recently a five-volume Report to the President on Science and Public Policy, by John R. Steelman, was released. The fourth volume is devoted to Manpower for Research, and contains a one hundred page report by the Cooperative Committee on Science and Mathematics Teaching. This Committee is now a standing committee of the American Association for the Advancement of Science, under the chairmanship of Professor Lark-Horovitz, Secretary of the A.A.A.S. Thirteen prominent scientific societies have representatives on this committee. Professor Raleigh Schorling is the representative of our Association.

This is neither the time nor the place to give a detailed account of this report, but it is to be hoped that eventually it will reach every member of this Association. The Government's aim is directed toward the creation of (1) a corps of effective research scientists, and (2) an organized group of discerning science educators. The portion of this report written by the Cooperative Committee is primarily concerned with the early identification of youth with special talent in science and mathematics and with the proper nurture of these gifted youth through school, college, and the university.

It is of course high time that the Government took some interest in solving a problem which was to a large extent of its own creation. The short-sighted attitude of the Army during the war toward higher education did more damage than even a five-volume report to the President can now repair. While Canada and Australia and England nurtured their scientists and future scientists, we sent our A. S. T. P. boys to the Rapido River and the Anzio Beachhead.

The shortage of young scientists is indeed alarming. While it is due in large part to the cessation of the graduate schools during the war, it is aggravated by the great demands for research scientists by the Government and by Industry.

The latter has suddenly discovered the science of statistics, for instance. If there are enough men to go around, this is all to the good. Scientists are getting better and better salaries, until now the best of them are earning almost as much as an average big-league baseball player. Of course when the depression comes both Industry and Government will willingly relinquish their scientists to the colleges.

While all of us are in favor of adequate salaries for scientists, we cannot close our eyes to the impact of this sellers' market upon the colleges. A large corporation may employ a handful of scientists, whose salaries are an unimportant item on its budget. A university with a faculty of hundreds feels this competition keenly, and in reality cannot meet it. Older men may elect to remain in their teaching positions in spite of offers of increased salaries from Government and Industry. The younger men cannot be expected to do this, and they are not doing it.

The direct result of this situation may be the deterioration of higher education in the United States. The universities cannot train first-rate scientists with second-rate staffs. The fountain of knowledge may be shut off at its source.

The Government seems at long last to have recognized this state of affairs in the Steelman Report, but it remains to be seen if anything effective will be done about it. One obvious immediate remedy would be to close down many of the overlapping projects now being carried on with lavish expenditure, but we may be sure that this will not be done.

5. The public schools and the training of scientists. As a long-range program, however, the Steelman Report seems excellent in most respects. It recognizes that all is not perfect with our elementary and high school systems, and considers at length the problem of the early identification of youth with special talent in science and mathematics, and with the proper methods of handling them so that this talent may be developed as quickly and completely as possible. As the report states, a year or two lost can never be regained, not even if the student turns out to be a top-flight scientist.

I feel sure that at this point the Steelman Report has put a finger squarely upon the greatest weakness of our high school system. In the matter of educating the average child to become a good citizen, the schools are doing a fine job. In training the gifted for leadership, the record is not so good.

The mass production of high school graduates has had some unexpected by-products. We believe so firmly in the equality of men under the law that we assume them to be equal emotionally and mentally. At first we tried to offer everyone a college preparatory education. Many failed and were frustrated, so this system was considered unsatisfactory. The only obvious way out was to water down the contents of the courses to the capacities of the least capable, and this has been one of the aspects of "progressive education" during the past couple of decades. Since the mountain could not be persuaded to go to Mahomet, Mahomet was obliged to go to the mountain. This method may

produce citizens but it does not produce scholars.

For many high school pupils, the education which they receive is quite satisfactory. It is all they can absorb, they are not frustrated, and they are kept off the streets. They learn citizenship, although it must be admitted that their courses on citizenship do not seem to prevent them from littering the streets around the school with paper, much as they did before the Social Sciences were introduced into the high school curriculum. They are indoctrinated in the "American Way of Life," which means a mild contempt for scholarship and the belief that football is the only worthy masculine activity. With the substitution of swordsmanship for football, we have the way of life of the Norman nobles of the Middle Ages.

It is quite possible that some of these lads would have been frustrated by the old fashioned grammar school where grammar was taught instead of Social Studies. They will, eventually, be frustrated by life, but they will not take it too hard. Extraverts are not easily frustrated.

The greatest failure of our secondary schools is in their unsympathetic handling of the potential scholars. These superior students will be the future leaders of America, and their education is of primary importance. Without them, our civilization will fall. But they are frequently sensitive creatures, slow to mature, and possibly somewhat unlovely in adolescence. They are frequently unsympathetically handled by extravert teachers, and many of them are definitely frustrated. All they ask is to be allowed to pursue their serious interests at their own normal speed, but this is denied them, and they are made to feel that this desire is in some way abnormal. Frequently they find no kindred spirit even among their teachers.

European schools, and the American schools of forty years ago, cultivated more ground and ploughed it deeper than most of our modern schools. They had a curriculum of solid material so that the pupil had to develop at an early age the habit of serious study. In the grades he really learned spelling and grammar and arithmetic, and with a feeling of accomplishment following effort, came a genuine liking for the subjects. In high school he learned two languages, he learned to read them and to write them, and he learned their grammar. Possibly he learned to speak them. He also learned a little English, Ancient and American History, and a dab or two of the sciences. He took Algebra and Geometry as a matter of course, and frequently he took four years of mathematics. This was a college preparatory course. A student unable to complete this course went as far as he could with it. Modern experts say that this was bad, that his entire curriculum should have been changed and watered down. I stoutly maintain that this judgment is still unproved.

The colleges of today are facing an almost impossible task in trying to make scholars in four years out of unprepared students. What should be a leisurely course in College Algebra has become a mad drive to teach high school algebra to students who have already passed it at 97% in high school. Of course, the casualties are high but the colleges are trying to hold the line. But unless

relief comes soon, they will yield, and will spend two years doing what the high schools should have done.

What is the basic cause of this condition, and what is the remedy? I can do no better than to quote Dr. John Guy Fowlkes, Dean of the School of Education of the University of Wisconsin, from an address to his faculty last month: "More than ever before in the history of our country it is clear that teachers in our public schools need to be scholars."

It was an evil day when the direction of policy in our schools passed out of the hands of the scholars. When the direction of our medical schools shall pass from the hands of our doctors of medicine, and the direction of our law schools from the hands of our lawyers, they too will become ineffective.

This state of affairs is clearly due to the great increase in size which our school systems have undergone in recent years. There are not enough scholars to go around, and there is not enough money available in some local units to pay teachers adequately. This is the crux of the whole difficulty. High school teaching as a profession must be made more attractive to persons of both sexes who have a liking for the scholarly life.

The obvious answer to this is to increase taxes. We all have seen statistics comparing what we pay for cosmetics, tobacco, and liquor with what we pay for education. But even without any further increase in total expenditures, many schools could, under the administration of scholars, become more effective. Too much money is now spent on vocational guidance, bands, plays, card index systems, business machines and shop equipment, to say nothing of gymnasia and athletic paraphernalia. Now that I have burned my bridges behind me and admitted being a reactionary, I might as well state the belief that the damage done by basketball and track in the way of enlarged hearts far outweighs the benefits to the participants. Death results usually at about the age of fifty.

The high schools contain many capable and devoted scholars. But they are now in the minority, either numerically or politically, and have pretty generally lost control of the policy-forming bodies. Scholarship is not now the high ideal of the schools. It has been replaced by "Americanism," which is a rather vague concept somewhat beyond my powers of analysis. Probably it is "Wisconsinism" in Wisconsin, and "New Yorkism" in New York. It looks to me startlingly like "provincialism" everywhere.

The scholar is a reasonable person. He sees both sides of every problem and is quite willing to grant the other fellow every courtesy in an argument. Being very realistic, he is not positive that he is always right in every argument, and is a bit cautious in his declarations and claims. Moreover, he generally does not like an altercation. The scholar, therefore, is frequently an easy opponent for the extravert. The extravert knows that he is always right, because he can make more noise than his opponent. His beliefs have a basis in self-interest, but the extravert honestly does not realize this, and he becomes quite emotional about anyone who opposes him. He knows so definitely what he wants, and is so contemptuous of opposition, that he is impressive. Usually he gets his way.

As the schools of this country became "big business," their control passed from the hands of the scholars. The buildings became larger, the bus rides became longer, the frills began to multiply. Athletics and bands became more important than studies. Typewriting, shorthand, shop work, millinery, and home economics began to drive out the scholarly courses. Every teacher desired to become a principal so that he would not have to teach any more but could surround himself with secretaries, file clerks, vocational guidance directors, and assistant principals. To quote the title of an article in the Wisconsin Journal of Education last month, "The principal is an important person." Nothing was said of the importance of the classroom teacher.

Within the great system of our public schools there are many pressure groups working for the furtherance of their own best interests. I do not refer to anything so crass as the teachers who struck last winter and tied up many schools—it seems to be the American way. But there are groups who are perpetually striving for the over-emphasis of their own particular subjects or side shows at the expense of scholarly subjects. They do not hesitate to demand, for instance, that mathematics be eliminated to make room for political indoctrination.

As might be expected, the proponents of physical education and compulsory athletics are the most noisy and unreasonable, for they are the farthest from the scholarly type. At present they are pushing a campaign in Illinois to require athletics in all schools, from the first grade up. In Iowa, the only required subjects in high school are American history and government, and physical education. As if the farm boys in Illinois and Iowa did not have enough fatiguing work, and as if Nature did not keep every small boy in perpetual motion! Large sums going to manufacturers of athletic equipment might better be spent for the salaries of teachers.

There was a time when the colleges exercised some control over the secondary schools by demanding a measure of accomplishment from entering students, *and let no one persuade you that the colleges have no right to make such demands.* Those students who were headed for college had to have adequate instruction. This frustrated the pressure groups in the schools, and became the object of a concerted and violent attack. Most legislatures were persuaded to force the state-supported colleges to allow the high school principals to decide who is prepared for college and who is not.

This seems to be the critical point at which the scholars completely lost control of the schools. Long before this they had ceased to be in the majority in their own schools, but they had support from the colleges, most of which still remain scholarly institutions. But with the loss of this last restraining influence, degeneration has followed rapidly.

6. Obligations of the Association. The Mathematical Association of America was founded in 1916 with the avowed purpose of assisting in promoting the interests of mathematics in America. During the thirty-one years of its existence

it has been influential and respected. The Young Report on the reorganization of mathematics in the secondary schools was and is respected, and there have been other projects to our credit. But we must look forward rather than backward.

The Association must not be just another pressure group whose only purpose is to further the interests of its own members, and I don't think it has ever been such. Mathematics can hold its own in any fair competition. We must, however, work for a sane and realistic program of education, a return to the scholarly ideal.

With the great American experiment of mass education we should have no quarrel. But this project is being quite adequately handled, and presents no critical problem. The adequate education of scholars by scholars is not being well handled and needs our best thoughts. Call it the double-track system or what you will, there must be some way to give a genuine education to those who want it, and to persuade all those who are competent to want it. The ideal of scholarship must be made more honorable and more attractive.

Working alone, the Association can accomplish little. If we make common cause with other scholarly and scientific groups, we can be effective. The legislatures and state boards are not as a rule vicious or unfriendly, but they are not composed of educational experts and are subject to influence by pressure groups. One may believe that they frequently welcome our advice and are glad to support us if they have any evidence at all that they are doing the will of the people thereby. It is up to us to supply this evidence.

Just to show that a little effort may produce results, I want to tell you of recent happenings in Ohio. About three years ago when Dean Brandeberry of the University of Toledo became Chairman of the Ohio Section of the Association, he appointed a committee to see what, if anything, could be done about the low state to which mathematics in the state had fallen. No mathematics was required for graduation from high school, no mathematics was required for admission to many teacher-training institutions, and little if any was taught to the future teachers in these institutions. Under the chairmanship of Professor F. B. Wiley of Denison University, and later of Professor H. P. Fawcett of the College of Education of Ohio State University, this Association committee drew up resolutions and detailed programs of study and submitted them to the State Department of Education in Columbus. Within the present month the committee has learned that in the future one year of high school mathematics will be required for graduation. This is not enough of course, but it is a great deal.

There is some evidence that the anti-scholastic movement in Education, at least as it relates to mathematics, is on the wane in most places, and that the pendulum is about ready to swing back. Sooner or later this must take place, for if it were nothing more, mathematics is the back-bone of science and engineering. A little missionary work on our part at this time might prove surprisingly effective.

It seems clear that most of the opposition to mathematics in the schools

has been because of the idea that it stood in the way of mass education. I think we have no quarrel with mass education provided it does not interfere with the training of scholars. Instead of trying to force a high mathematics requirement on all schools, I think we should demand instead that those students who are going to go to college shall be adequately trained, and that the colleges shall have the right to decide whether a student is fitted to enter. If the poor student has a right to an education which will not frustrate him, by the same precept the talented student has a right to an education which will develop his talents as quickly and completely as possible. If the Steelman report is adequately implemented by Congress, steps will be taken to see that such an education is available.

With such an education available, the vocational guidance directors will have to be fought off so that all capable students will take this program. My daughter, who is now taking a college preparatory course in high school, has with considerable effort withstood the pressure of a vocational guidance director who insists that her course is not well balanced and that she should take home economics.

The matter of adequate training for teachers of mathematics is now a critical problem. In some Teachers Colleges practically none at all is required. At the University of Wisconsin, 15 semester hours beyond the calculus is necessary for the University Teachers Certificate. A joint committee of the Association and the National Council of Teachers of Mathematics is now considering the problem of training for teachers, and will soon have some suggestions to offer.

But experience in Wisconsin proves that a high paper requirement is not enough to ensure adequately trained and scholarly teachers. The number of students working for a Teachers Certificate with mathematics as a major has dropped everywhere in recent years. Thus the University of Michigan with 20,500 students has 5 who are looking forward to teaching mathematics in high school. There has not been a single student teacher in physics at Michigan during the last five semesters.

While the official requirements for high school teachers are high in these states, the actual teaching is being done by teachers with temporary permits who may have no qualifications at all. The only alternative would be to close the schools. It is rather futile, then, to talk of raising the price of tickets when the multitude crawls in under the tent.

7. Conclusion. All of these considerations lead us back to our original text: *Scholars are important persons.* They are more important than principals, than athletic coaches, than vocational guidance directors. Scholars are worthy of public respect, and should occupy a respected social position in this country as they do in most other countries. They should be given adequate salaries so that they can live as well as the physicians and lawyers, whose equals in public esteem they should be. The schools should be under the direction and management of the scholars, and the growing gap in both prestige and salary between

the principal and the teacher should be closed.

If such a conversion of the American people could take place, our school problems, and most of our political, economic, and social problems as well, would be solved. It is of course an idle dream. But scholars have always been dreamers, for only their dreams have made life endurable for them. And only those small bits of their dreams which have come to pass have made life endurable for the rest of the world.

RECENT PROGRESS IN COMPRESSIBLE FLUID THEORY

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1. Introduction. Modern technical developments have made the need for a workable theory of compressible fluids imperative. The classical techniques that have at times yielded successful solutions to similar applied problems here fall way short of the requirements. Let us survey this situation and analyze the nature of the difficulties which cause it.

We confine our attention to two-dimensional irrotational flows of nonviscous fluids. If the assumption of incompressibility is added, the classical development leads rapidly to a harmonious theory whose salient points are the following:

A. The stream and potential functions satisfy differential equations for which the boundary value problem admits of a unique solution.

B. These differential equations are linear and homogeneous. Thus linear combinations of solutions give further solutions. Usually there exist basic sequences of solutions so that an arbitrary solution may be formed from the limits of linear combinations of them.

C. Each differential equation is a Laplace equation. Thus solutions may be put into correspondence with analytic functions of a complex variable by the operation:

"Take the imaginary part." (1)

Aspect **A** continues to hold for a larger class of problems. In fact, for any problem amenable to differential equation treatment, mathematical uniqueness should follow from physical uniqueness if the mathematical problem is correctly formulated from the (necessarily idealized) physical situation. Aspects **B** and **C** each lead us into well-studied mathematical domains, and it is not surprising that incompressible fluid theory should be rich in effective, and often elegant, results.

But even with such advantages, all the troubles of the applied mathematician are not dispelled. The boundary value problem in practice consists in being given certain numerical data and being required to supply other such data with

a (sometimes tacitly) prescribed degree of accuracy. The only means possessed by men to make the transition are the common arithmetic operations on integers. Thus the *method* is reduced to finding a program of such operations.

This can often be done. For example, we may dot the domain of the sought function with a lattice of points and replace the function by a set of numbers, one at each point. The differential equation is replaced by a suitable difference equation. The problem is replaced (usually) by an algebraic one where we have an unknown for each lattice point. If the original differential equations are linear, so will be the algebraic ones for the unknowns. Then we possess the required program of arithmetic operations.

The drawback to such methods is that the number of operations may be enormous; so much so, in fact, that the time and labor required may exceed the economic value of the solution. With such advantages as **B** and **C** present, the amount of labor can sometimes be reduced into the practical range by utilizing existing known properties and tabulations of functions. But even in cases where the pure mathematician has supplied a sophisticated approach with impeccable convergence proofs, the numerical calculation may still be prodigious.*

The most promising remedy, in the author's opinion, lies in the use of modern computing devices such as punch card machines, the Automatic Sequence Controlled Calculator at Harvard University, or Aniac, the Bell Telephone Computer. The principle of these machines is to compound arithmetic calculators so as to carry through a program of operations automatically; that is, the result of one operation can appear in the data of the next. Thus vast new possibilities in applied mathematics are opened up.

For non-linear problems there may be a still more fundamental difficulty. Even if the resources are at hand for carrying out a vast program of arithmetic operations, there is still the primary problem of formulating the program. The usual devices, some of which we have mentioned, no longer suffice. In fact, sometimes even the general qualitative nature of the solution may be obscure. Thus there are two stages in the difficulties of compressible fluid theory; only recently has effective progress been made in overcoming them. In the ensuing paragraphs, we outline the main steps.

2. The main steps. First we recall some terms. The *potential* is that function whose gradient is the velocity of flow. It exists when the flow is *irrotational*. Descriptively, this term means that small regions of fluid do not rotate; precisely it means the vanishing of a certain line integral for all closed curves. The dynamical justification of irrotationality can be found in the classic researches of Helmholtz. The chief property of the *stream function* is that it is constant along the stream lines; hence its knowledge leads to a picture of the flow pattern.

* For example, let the skeptic attempt the alternating procedure of Schwarz with two slowly convergent infinite series. See Courant-Hilbert, Methoden der Mathematischen Physik, T. II, p. 264.

For an incompressible fluid the stream function ψ and the potential function ϕ each satisfy Laplace's equation. In fact $\phi+i\psi$ is a monogenic function of $x-iy$, where x, y are the cartesian coördinates in the plane of the flow.

For the incompressible case we must assume an equation of state, which we take to be

$$p = A + \sigma \rho^k \tag{2}$$

where p is the pressure, ρ the density, and A, σ, and k are constants. The equations satisfied by ϕ and ψ are

$$a^2[\phi_{xx} + \phi_{yy}] = \phi_x^2\phi_{xx} + 2\phi_x\phi_y\phi_{xy} + \phi_y^2\phi_{yy}, \tag{3}$$

$$\begin{aligned}\psi_{xx}\left[1 - \frac{1}{(\rho_0 a_0)^2}\left(\frac{\rho_0}{\rho}\right)^{k+1}\psi_y^2\right] + \psi_{yy}\left[1 - \frac{1}{(\rho_0 a_0)^2}\left(\frac{\rho_0}{\rho}\right)^{k+1}\psi_x^2\right] \\ + \frac{2}{(\rho_0 a_0)^2}\left(\frac{\rho_0}{\rho}\right)^{k+1}\psi_x\psi_y\psi_{xy} = 0,\end{aligned} \tag{4}$$

together with

$$a = \left[a_0^2 - \tfrac{1}{2}(k-1)(\phi_x^2 + \phi_y^2)\right]^{1/2}, \tag{5}$$

$$\left(\frac{\rho_0}{\rho}\right)^{1-k} = 1 - \frac{k-1}{2(\rho_0 a_0)^2}\left(\frac{\rho_0}{\rho}\right)^2[\psi_x^2 + \psi_y^2]. \tag{6}$$

Here a is the local velocity of sound; a_0 and ρ_0 mean a and ρ at zero velocity. It is apparent that these equations are not only non-linear, but complicated. However, the conveniences of aspects **B** and **C** may be restored by means of the so-called hodograph plane. This plane is one in which the horizontal and vertical velocity components u and v are the coördinates instead of x and y. The functions ϕ and ψ may be regarded as functions of u and v, and hence the stream and equipotential lines may be depicted in the hodograph plane. This gives rise to a flow pattern (the hodograph), generally quite different in appearance from that in the physical plane (*i.e.*, the x, y plane). For the incompressible case, ϕ and ψ are harmonic in the hodograph plane and so it appears to offer little advantage. But, as Chaplygin [**1**] has shown, for compressible fluids the equations become linear in the hodograph plane.

Bergman [**3**, **5**, **7**] modifies this idea (in the subsonic case) by using what he terms the pseudo-logarithmic plane instead of the hodograph plane. Here the coördinates are λ and θ where

$$\lambda = \tfrac{1}{2}\log\frac{1-T}{1+T} + \frac{1}{2h}\log\frac{1+hT}{1-hT}, \tag{7}$$

$$h = \left(\frac{k-1}{k+1}\right)^{1/2}, \qquad T = (1-M^2)^{1/2},$$

$$M \text{ (the Mach number)} = \frac{V}{a_0}\left[1 - \tfrac{1}{2}(k-1)\left(\frac{V}{a_0}\right)^2\right]^{1/2},$$

$$V \text{ (velocity of flow)} = (u^2 + v^2)^{1/2},$$

and θ is the angle which the flow velocity makes with the horizontal axis.

The equation for ψ now assumes the form

$$\psi_{\lambda\lambda} + \psi_{\theta\theta} + 4N\psi_\lambda = 0 \tag{8}$$

where

$$N = -\frac{k+1}{8}\,\frac{M^4}{(1-M^2)^{3/2}}\,.$$

Bergman has developed a theory of operators which define a correspondence between analytic functions of complex variables and the solutions of certain second order partial differential equations [**2**, **4**]. Thus it is the analogue of the operator (1) for harmonic functions. Applying the appropriate operator to (8) yields the following result:

Let g be an arbitrary analytic function. Put

$$Z = \lambda - i\theta$$

$$g^{[0]}(Z) = g(Z), \qquad g^{[n]}(Z) = \int_0^Z g^{[n-1]}(z)dz.$$

Then

$$\psi(\lambda,\theta) = \mathfrak{J}_m\left\{(1-M^2)^{-1/4}(1+\tfrac{1}{2}(k-1)M^2)^{-1/2(k-1)}\left[g(Z) + \sum_{n=1}^{\infty}\frac{(2n)!}{2^{2n}n!}Q^{(n)}(2\lambda)g^{[n]}(Z)\right]\right\} \tag{9}$$

is a solution of (8).

In (9) the sequence of functions $Q^{(n)}$ is defined as follows:

$$(2n+1)Q^{(n+1)}(\lambda) = -\frac{d}{d\lambda}Q^{(n)}(\lambda) - 4\int_{-\infty}^{\lambda}F(\alpha)Q^{(n)}(\alpha)d\alpha \quad (n = 0, 1, \cdots), \tag{10}$$

$$Q^{(0)}_{(\lambda)} = 1;$$

$$F(\lambda) = \frac{1-T^2}{16(1-h^2)^2T^6}\left[5 - (1+6h^2)T^2 - (5-4h^2)T^4 + (1+2h^2)T^6\right]. \tag{11}$$

The function F is fully defined by the conjunction of (7) and (11).

3. Remarks. Thus the problem is solved in the sense that methods of solution have been found. These methods bear an obvious analogy to the incom-

pressible case. The chief computational task needed to open the way toward numerical usage is the evaluation of the $Q^{(n)}$. But these can be tabulated once and for all; with their aid all subsonic (and with modifications, supersonic) flows can be found.

One way of handling the problem is to replace the derivative and integral expressions in (10) by suitable interpolative formulas. The computation can then be done by mechanical devices of the type described earlier. For details of the technique, see [8].

Once the $Q^{(n)}$ are available, we can construct the aggregate of flow patterns. Let us suppose ourselves faced with a particular problem, for example, the flow around a given obstacle. For the incompressible case we solve this problem by one of the known methods and thus find the appropriate analytic function g. Using the operator (9) we can obtain the flow in the pseudo-logarithmic plane. A routine procedure [6] enables us to pass to the physical plane. The result is a flow about an obstacle slightly distorted from the given one. By making a suitable counterdistortion at the outset it is possible to obtain the true pattern.

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THE TEACHING FELLOW PROGRAM AT MICHIGAN

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The employment of part-time teaching assistants or fellows is now a widespread and necessary practice. It should be continued beyond this period of staff shortages for the sake of the teacher-training of future instructors and professors as well as for the sake of financial aid to struggling scholars.

In the October, 1946, issue of the MONTHLY B. W. Jones [1] summarized the replies to an inquiry which he addressed to several colleges and universities concerning their supervision of teaching assistants. At that time a revised and expanded program for the supervision and training of the teaching fellows in mathematics was being instituted at the University of Michigan. It is hoped that an account of this program at the beginning of its second year may be of interest, and may draw forth exchanges of information about similar programs.

The class of fellowships entitled "teaching fellowships" was established at the University of Michigan in 1932 to be filled by graduate students who "exhibit both research capacity and teaching ability." It was stated at that time that "the Graduate School assumes a supervisory function with respect to the inservice teacher-training which each department by its appointment of teaching fellows obligates itself to perform" [2]. With the sudden growth of the Mathematics Department's teaching fellow group to thirty-five in 1946–47 and forty-five in 1947–48 it became necessary to revise and expand the previously existing teacher-training program. The objectives of this program were considered to be: (1) to provide for the welfare of freshmen and sophomore students by maintaining a high quality of instruction, (2) to serve thereby the best interests of the University as an educational institution, (3) to help the teaching fellow, frequently inexperienced and sometimes lacking in confidence, by making his present job easier, and by providing training in the work to which many of his later years will probably be devoted.

The program for this year consists of three major parts, namely: (1) regular group lecture and discussion meetings, (2) a "consultation service," (3) classroom "visitations" followed by individual conferences. This is in general the same as last year's program, modified, however, in many details as the result of the suggestions, and vigorous, interested, and cooperative criticisms of last year's teaching fellows.

The group meetings are of two types, weekly meetings of the entire group and less frequent meetings of those teaching a particular course. At the weekly meetings of the entire group such topics as the following, some taking more and some less than one meeting, will be discussed: (a) administrative details of course schedules, texts, attendance rules and grading at this University, (b) general classroom teaching hints and advice, (c) different methods of teaching (*e.g.*, lecture, heuristic, and board work or laboratory methods), (d) suggestions for planning a class hour (*e.g.*, read the text, seek to show a motivation for your discussion, select examples, try to provide some variety in your procedure within

each class hour and from day to day), (e) typical student errors and teaching trouble spots to be anticipated, (f) the nature, sources, and use of enrichment materials (*e.g.*, historical ideas, models, applications or connections with other fields), (g) how to prepare, give and correct examinations. The word "discussed" was used advisedly in connection with these meetings because the group is encouraged to present questions, points of disagreement, or alternative procedures at any time. Actually our problem here, at the outset, was to limit discussions and digressions, not to encourage them. However, democratic procedures and discussions, properly controlled, increase interest and bring out profitable variations in viewpoint. After the topics of most significance for classroom application have been covered, the general meetings will be held less frequently and will consist of talks by faculty members, movies and slide films, criticisms of texts and the curriculum, and consideration of the literature of collegiate teaching.

The small groups, each of which includes those teaching a particular subject, consider the schedule for that course, what topics, if any, are optional, specific errors and teaching problems encountered in that course, the motivation and development of particular topics, devices and materials that may be used in the next few weeks of classes. Each of these groups meets once every four weeks.

The consultation service is represented by the posting of a rather complete schedule of office hours by the sponsor or supervisor. (He has no official title, but from the resemblance in function and name to a popular comic strip character, he has been termed "Available.") During these hours he checks, before they are administered, the first two or three examinations given by the new fellows, discusses with them the observations made upon visits to their classes, and more than this, tries to answer, as they are brought in, all types of questions ranging from "When and where do we get paid?" to "Why does the author of this text present his ideas this way?," and "What should one do for (or to) students who cheat, are absent unduly, are misclassified?"

As the third major part of this program, the supervisor attempts to visit, at least twice, each time for a full hour, the class of each teaching fellow, and to discuss his observations with the fellow soon after the visit. A more ambitious visiting program might be desirable but seems impossible now in terms of the time available. The entire program is administered by one person. An allowance is made for the time required to supervise the teaching fellows in determining his class load.

A minor feature of the program has been the grouping together of the teaching fellows' offices and the stocking of a nearby cupboard with old and new standard textbooks, teacher's handbooks, such as J. W. Cell, *Engineering Problems Illustrating Mathematics*, R. C. Yates, *Curves*, historical works by D. E. Smith, Florian Cajori, and R. C. Archibald, works related to teaching, such as Seidlin, *A Critical Study of the Teaching of Elementary College Mathematics*, Felix Klein, *Elementary Mathematics from the Advanced Standpoint*,

Pólya, *How to Solve It*, copies of pertinent articles from the MONTHLY, such as Campbell, "Advice to the Graduate Assistant" [3], "survey" texts such as those by Dresden, Cooley, et al., Merriman, Richardson, and finally some of the thought-provoking new texts such as Murnaghan's *Analytic Geometry* and Albert's *College Algebra*.

To evaluate the success of this program is difficult. One can not measure effectively how good the teaching has been, let alone know how much worse it might have been without such a program. The replies found on questionnaires filled out anonymously by the teaching fellows at the end of each semester indicated that by a large majority (but not unanimously) they felt all three phases of the program were helpful and worthwhile. The general meetings were considered least valuable. We are trying to correct those features of them which drew the most criticism. The general feeling with regard to "visitations" was that though a little unpleasant they were helpful and an especially good idea for the other teaching fellows. It is of interest that on several occasions people have asked for more visits. The supervisor feels that the program has at least resulted in an interest in and thought for teaching and curricular problems which, combined with the recognition of teaching as a real job in itself, have produced a desirable high morale among the teaching fellow group.

References

1. B. W. Jones, The Supervision of Teaching Assistants, This MONTHLY, Vol. 53 (1946) pp. 485–486.

2. University of Michigan, The Horace H. Rackham School of Graduate Studies Announcement 1947–48, p. 47.

3. A. D. Campbell, Advice to the Graduate Assistant, This MONTHLY, Vol. 45 (Jan., 1938) pp. 32, 33.

THE SHORT-CUT PROBLEM

CHANDLER DAVIS, Harvard University

1. Introduction. The following problem in plane geometry concerns the shortest path from one point to another along a network of streets. To make the problem definite, let the streets be a family of straight lines, arbitrary in position and direction, infinite in length, and finite in number.

We will use the following notation. Let axes be chosen so the starting point O is at $(-1, 0)$ and the destination O' at $(1, 0)$; it is of course assumed that at least one line passes through each of these points. We will discuss properties of paths from O to O', composed entirely of segments of the given lines. Let the angle from the positive x direction to the direction of a line a be designated by $\theta(a)$, where the function θ has values in the range $-\pi<\theta\leqq\pi$. If we are considering using a segment of a as part of a path from O to O', we will understand $\theta(a)$

to have the value corresponding to that sense in which we consider traversing a. It may happen that we will want to give the angle of a, not with respect to the x direction, but with respect to the direction AB; in this case we write $\theta_{AB}(a)$. Thus, for example, $\theta(a) \equiv \theta(AB) + \theta_{AB}(a) \mod 2\pi$.

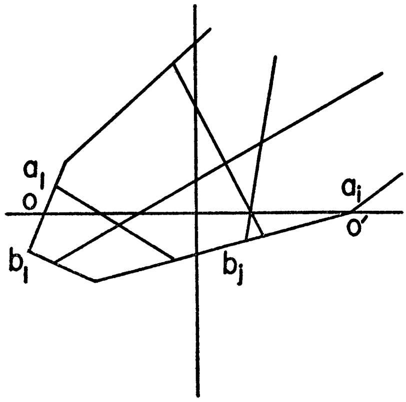

Fig. 1

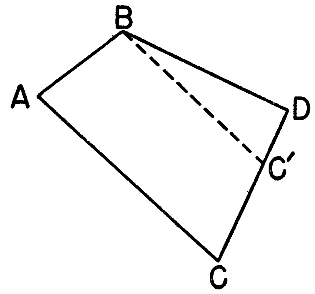

Fig. 2

Consider a line from the given family such that O and O' do not lie on opposite sides of it. Clearly the shortest path does not cross it, and we may for most purposes eliminate all parts of the plane on one side of it. After we have eliminated such irrelevant regions, we are left with a region like that in Figure 1, that is, a convex polygon, which however need not be closed. Call the segments of this polygon above the x-axis, $a_1, a_2, \cdots, a_i$, taken in order from O to O'; similarly call the segments below the x-axis $b_1, b_2, \cdots, b_j$. We have immediately

$$\pi > \theta(a_1) > \theta(a_2) > \cdots > \theta(a_i) > -\pi$$
$$-\pi < \theta(b_1) < \theta(b_2) < \cdots < \theta(b_j) < \pi$$
$$\theta(a_1) \geqq 0, \qquad \theta(a_i) \leqq 0$$
$$\theta(b_1) \leqq 0, \qquad \theta(b_j) \geqq 0$$
$$\theta(a_1) - \theta(b_1) \leqq \pi$$
$$-\theta(a_i) + \theta(b_j) \leqq \pi.$$

Those lines which are not of either type a or b we will call l-lines. In general the shortest path will include l-lines as well as a and b-lines.

A line m will be called *reflex* if $\theta(m) > \theta(a_1)$ and $\theta(m) > \theta(b_j)$, *or* if $\theta(m) < \theta(b_1)$ and $\theta(m) < \theta(a_i)$. Otherwise m will be called *direct*. Note that all "boundary" lines (types a and b) are direct.

The theorem to be proved can now be stated in the following form.

THEOREM. *The shortest path from O to O′ includes no reflex segments.**

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* This theorem, more general than the author's earlier result, is due to J. T. Tate, Jr., of Princeton.

2. Two lemmas.

LEMMA 1. *Suppose we have two alternative two-segment paths ABD and ACD from point A to point D, with* $\theta_{AD}(AB)>0$, $\theta_{AD}(AC)<0$. *Suppose further that* $\theta_{AD}(AB)<\theta_{AD}(CD)$. *Then if ACD is shorter than ABD we must have* $\theta_{AD}(AC)>\theta_{AD}(BD)$.

For, suppose instead $\theta_{AD}(AC)\leqq\theta_{AD}(BD)$. (See Figure 2.) Take the line through B parallel to AC; it meets CD in C', which lies between C and D (or coincides with D). Then comparing the lengths of the paths, we have $ACD>ABC'D\geqq ABD$, which proves the Lemma.

This Lemma and its various equivalent forms (for example, that in which $\theta_{AD}(AC)$ is taken less than $\theta_{AD}(BD)$, and $\theta_{AD}(CD)<\theta_{AD}(AB)$ is proved) will be used frequently in what follows.

LEMMA 2. *The first l-segment of a shortest path from O to O' cannot be reflex.*

Suppose without loss of generality that the path leaves O along the upper boundary segments $a_1, a_2, \cdots, a_\mu$. Then let it leave a_μ at A along the reflex segment AP_1, so that $\theta(AP_1)<\theta(b_1)<0$, $\theta(AP_1)<\theta(a_i)$. From P_1 it proceeds along l-lines through points $P_2, P_3, \cdots, P_\nu, B$, where B lies on the boundary. Then, $\theta(P_1P_2)>0$, $\theta(P_2P_3)<0, \cdots$. (See Figure 3.) For, if the lines do not alternate in this way between positive and negative θ, then obviously one of them is crossed more than once by the path in question.

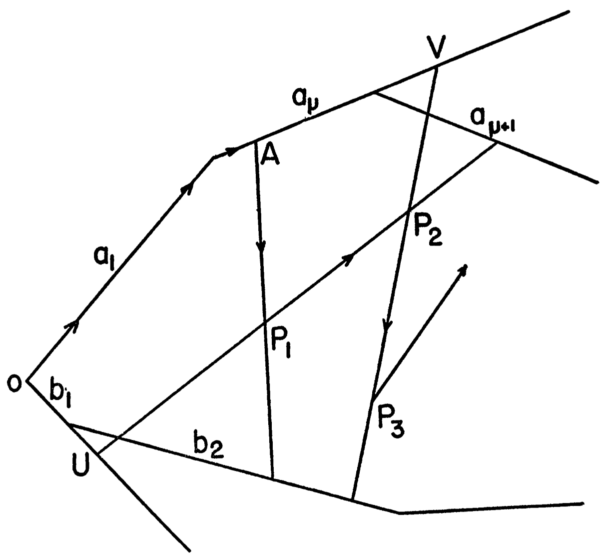

Fig. 3

Now let P_1P_2 meet b_1 at U, and imagine a line drawn from O to A. The path OAP_1 is shorter than the actual path $Oa_1a_2 \cdots a_\mu AP_1$, hence by hypothesis

shorter than OUP_1; also $\theta_{OP_1}(OU) > \theta_{OP_1}(AP_1)$. Hence Lemma 1, when applied to quadrilateral OAP_1U, gives $\theta_{OP_1}(P_1P_2) > \theta_{OP_1}(OA) > \theta_{OP_1}(a_\mu)$; $\theta(P_1P_2) > \theta(a_\mu)$.

Similarly, we take the intersection V of P_2P_3 with a_μ and apply Lemma 1 to quadrilateral AVP_2P_1 to obtain $\theta(P_2P_3) < \theta(AP_1)$. Successive applications of Lemma 1 give $\theta(P_3P_4) > \theta(P_1P_2)$, $\theta(P_4P_5) < \theta(P_2P_3)$, $\theta(P_5P_6) > \theta(P_3P_4)$, $\cdots$.

Take first the case where ν is odd; B must lie on an a-line, say a_k. Since $P_{\nu-1}P_\nu$ is reflex from the inequalities derived above, and a_k is direct, $\theta(P_{\nu-1}P_\nu) < \theta(a_k)$; also we know that $\theta(P_\nu B) > \theta(P_{\nu-2}P_{\nu-1})$. Then the assumption that the chosen path gives the shortest distance from $P_{\nu-1}$ to B leads, by Lemma 1, to a contradiction.

If on the other hand ν is even, B lies on a b-line, b_λ. In the event that $\theta(P_{\nu-1}P_\nu) > \theta(b_\lambda)$ we prove a contradiction as in the preceding case; so take $\theta(P_{\nu-1}P_\nu) \leqq \theta(b_\lambda)$. $P_{\nu-1}P_\nu$ clearly meets a_i in some point W above the x-axis, and application of Lemma 1 to quadrilateral $P_\nu WO'B$, in which $\theta(P_\nu W) \leqq \theta(BO')$ and $\theta(P_\nu B) < \theta(WO')$, gives a contradiction. (That part of the chosen path between B and O' has been replaced by the construction of a single straight line.)

Lemma 2 is therefore proved; note that the cases $\nu < 3$, though apparently not covered, are actually obtained by the same method.

3. The main theorem. The theorem now follows immediately by induction on the number n of l-segments in the path in question. For if $n=1$ then Lemma 2 is equivalent to the theorem. Also suppose the theorem true for all m which are less than or equal to $n-1$, and consider a path having n l-segments. The first segment AP_1 on which this path leaves a boundary (say the a-boundary) is, by Lemma 2, direct; thus $\theta(AP_1) \geqq \min\{\theta(b_1), \theta(a_i)\}$. But the portion of the path between A and O' must be a shortest path for those two points. Consider the short-cut problem related to A and O'; AP_1 will now be a boundary line, so there will be at most $(n-1)$ l-segments in this new problem. By the induction hypothesis, no segment is reflex with respect to AO'; but any segment reflex with respect to OO' would be so with respect to AO', so the theorem is proved.

It is clear that the proof has nowhere used the fact that the boundary is made up of straight-line segments; it could equally well be any convex curve, not necessarily closed, through O and O'. (All l-lines must still be straight.)

The following further problem has been suggested by J. T. Tate, Jr. Given some particular figure as boundary, to find for each point in the interior the maximum and minimum angles of lines through that point which can for some configuration of l-lines be portions of shortest paths. For some points these maximum and minimum conditions will be sharper than those obtained by the present theorem; indeed there can be points through which no shortest path can pass.

THE CHAUVENET PRIZE FOR MATHEMATICAL EXPOSITION

The Chauvenet Prize for Mathematical Exposition has been awarded to Professor P. R. Halmos of the Institute for Advanced Study for his paper entitled "The Foundations of Probability," published in this MONTHLY for November, 1944. This most recent award of the Prize "for a noteworthy expository paper published in English by a member of the Association" covers the three-year period, 1944–'46.

The Association first established the Chauvenet Prize in 1925. At that time it was specified that the award was to be made every five years for the best article of an expository character dealing with some mathematical topic, written by a member of the Association and published in English during the five calendar years preceding the award. The Prize was not to be awarded for books. Originally the amount of the award was fixed at one hundred dollars.

At a later date it was decided to award the Prize every three years, and the amount was changed to fifty dollars. In 1942, it was further specified that only such papers would be considered as "came within the range of profitable reading of Association members."

The publication program of the Association recognizes the importance of expository writing. The editors of the MONTHLY welcome short expository articles that treat mathematical concepts of modern importance. Longer manuscripts may be submitted to the Committee on the Slaught Memorial Papers or to the Editorial Committee on the Carus Monographs.

ERRATA, VOLUME 54

The following errata in Volume 54 have been called to the attention of the editors:

H. E. Salzer, *The approximation of numbers as sums of reciprocals.*

p. 135, at the bottom of the page, replace

$$\frac{1}{a_i^2 + a_i} \quad \text{by} \quad \frac{1}{a_i^2 - a_i}.$$

W. C. G. Fraser, *An inversion formula for an integral related to Dirichlet series.*

p. 586: The location of the author should be given as Rensselaer Polytechnic Institute instead of Rutgers University.

Kurt Godel, *What is Cantor's Continuum Problem?*

pp. 515–525: In the footnotes 1, 2, 20, 23 and in the first line of p. 521 the numbers of the footnotes quoted should read: 14, 15, 17, 20, 23, 26 instead of 12, 13, 14, 17, 19, 21, respectively.

In footnote 29, *l.c.* 6 should be replaced by *l.c.*[6].

In the second line of p. 524 Π_i should be replaced by $\prod_i$.

In the fifth line of p. 525 the word "at" should be replaced by "are."

MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California

Material for this department should be sent directly to E. F. Beckenbach, University of California, Los Angeles 24, California.

ROTATION OF THE TANGENT TO A HYPOCYCLOID

J. H. BUTCHART, Arizona State College

A surprising number of the properties of hypocycloids are related to the turning of the tangent, as the point of contact R between the rolling circle r and the fixed circle f moves from the cusp A to the position determined by the parametric angle θ. If C is the center of r and X is the point describing the hypocycloid, then the tangent to the hypocycloid at X always passes through S, the point of r opposite R. Also, from the equality of arcs RX and RA of r and f respectively, we have angle XCR equal to $a\theta/b$, where a and b are radii of f and r respectively. Then angle XSR is $a\theta/2b$, which means that the tangent SX has turned from its initial direction OA through the angle $\tau=(2b-a)\theta/2b$. The motion of the supporting point S about its circle s, accompanied by the uniform rotation of the line SX, determines this family of lines whose envelope is seen to be some hypocycloid or epicycloid.

We may evaluate τ for a deltoid upon replacing a by $3b$, thus obtaining $-\theta/2$. To identify the envelope of the Simson line, we may take S on the nine-point circle. Then we know (N. A. Court, *College Geometry*, p. 116) that the Simson line rotates half as fast in the opposite sense as S moves about the nine-point center. It would be hard to imagine a simpler proof for the famous theorem that the envelope of the Simson lines of a fixed triangle is a deltoid having the nine-point circle of the triangle as inscribed circle.

For the astroid, a equals $4b$, and τ becomes $-\theta$. Now S bisects the radius OR of f, and the tangent may be drawn joining the projections of R on the perpendicular cusp tangents. Then X is the projection of R on this tangent. The proof lies in noting that angle XSR is 2θ and angle XCR is 4θ. This not only furnishes a neat construction for the astroid but also proves very simply the textbook proposition concerning the constancy of the length of the segment intercepted by the axes on the tangent.

If $a=2b$, the hypocycloid has two cusps and τ vanishes, proving that the curve reduces to a diameter of f. If $a=b$, then $\tau=\theta/2$. Then S moves on f opposite R and the hypocycloid of one cusp reduces to the single point A. The general point of an epicycloid can be found by reflecting the point of the corresponding hypocycloid across the tangent at R. Thus the cardioid is the reflection of A across the variable tangent to f.

The deltoid has many interesting properties, some of which can be approached very directly from the present point of view. For instance, if the rolling curve r' has radius $b'=2a/3$, the point marked on r' and starting from A traces the same curve formed by letting $b=a/3$. To prove this, note that corresponding to this value of b', $\tau'=\theta/4$, and OS' is $-OR/3$. Then angle $PS'R$

is $3\theta/4$, where the moving point P is the projection of R on $S'P$. If S is such that OS is $OR/3$, it is the center of r' and angle PSR is $3\theta/2$, which shows that the arc RP of r' equals arc RA of f. That this point P coincides with a point of the deltoid with $b=a/3$ may be shown by drawing OT, where T is on f and angle AOT is $-\theta/2$. Let U be such that $OU=OT/3$. Since angle UOS is $3\theta/2$, $S'P$ passes through this same point U of the circle s on SS' as diameter. Thus angle TUP is $3\theta/4$, which implies that the line UP is a tangent to the deltoid generated by the circle on UT as diameter. Since UT and SR make numerically equal angles with $S'U$, the projections of R and T on this line coincide, which completes the proof. A result which is obvious at this point is the fact that the chord which f determines on the normal at P is trisected by P.

Another well known result which follows easily at this stage is the theorem that the segment determined on the tangent to one branch of the deltoid by the other two branches is constant. For angle PSR is $3\theta/2$, so that SP is tangent to one branch at X, where X is the projection of R on SP. When r' starts rolling with P at A, the point Q of r' opposite P lies on the deltoid and therefore stays on the same curve as r' rolls. Thus the segment PQ referred to in this proposition is a diameter of the circle r'. Also RP and RQ are perpendicular normals to the deltoid and $S'P$ and $S'Q$ are perpendicular tangents. We can accordingly announce the following results: Two of the normals from any point of f to the deltoid are orthogonal, and the feet of the three normals are collinear. Likewise, the intersection of orthogonal tangents to the deltoid describes the circle s (the orthoptic of the deltoid).

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Pennsylvania.

A GENERALIZATION OF THE RATIO TEST FOR SERIES

R. J. DUFFIN, Carnegie Institute of Technology

While teaching the chapter on series in the calculus, it occurred to the writer that the ratio test and the alternating series test could formally be combined into a single test. Thus, according to the ratio test, a series $a_1+a_2+\cdots$ (whose terms have like signs) is convergent if $0\leqq a_{n+1}/a_n\leqq r$ where $0<r<1$. According to the alternating series test, a series is convergent if $a_n\to 0$ and $-1\leqq a_{n+1}/a_n\leqq 0$. Is it possible, then, to combine these two tests into a single test which is valid regardless of the signs of the terms a_i, and which is more general than the ratio test for general series which states that convergence follows if $0\leqq |a_{n+1}/a_n|\leqq r$ where $0<r<1$? An affirmative answer is given by the following result:

THEOREM 1. *A series $a_1+a_2+\cdots$ of real numbers is convergent if $a_n\to 0$ and if the ratios a_{n+1}/a_n satisfy the inequality $-1\leqq a_{n+1}/a_n\leqq r$ where $0<r<1$.*

The following proof of this theorem should be comprehensible to many elementary students. Denote the ratio a_{j+1}/a_j by z, so $-1\leqq z\leqq r<1$. If z is a negative number, then $1+z=1-|z|$. Furthermore $1-r\leqq 1+r$. Forming the product of these relations between positive numbers gives

$$(1+z)(1-r)\leqq(1-|z|)(1+r). \tag{1}$$

The same inequality holds if z is positive or zero because in this case $1+z\leqq 1+r$ and $1-r\leqq 1-|z|$. Denoting $(1+r)/(1-r)$ by k we can express (1) in the form

$$|1+a_{j+1}/a_j|\leqq k(1-|a_{j+1}/a_j|).$$

Multiplying through by $|a_j|$

$$|a_j+a_{j+1}|\leqq k(|a_j|-|a_{j+1}|). \tag{2}$$

Let $R_{mn}=a_m+a_{m+1}+\cdots+a_{n-1}+a_n$. Then

$$\begin{aligned}
2R_{mn} &= a_m+(a_m+a_{m+1})+\cdots+(a_{n-1}+a_n)+a_n; \quad \text{or}\\
2|R_{mn}| &\leqq |a_m|+|a_m+a_{m+1}|+\cdots+|a_{n-1}+a_n|+|a_n|\\
&\leqq |a_m|+k(|a_m|-|a_{m+1}|)+\cdots+k(|a_{n-1}|-|a_n|)+|a_n|\\
&= (1+k)|a_m|+(1-k)|a_n|\\
&\leqq (1+k)|a_m|.
\end{aligned}$$

By hypothesis $a_m\to 0$ as $m\to\infty$, so $R_{mn}\to 0$ independent of n. Thus by the Cauchy-sequence criterion for convergence it follows that the series converges.

If S is the sum of the series then the above inequality gives $|S|\leqq|a_1|/(1-r)$. Note that this inequality includes the well known inequalities for geometric series and alternating series.

We now give a generalization which allows the a_n to be complex numbers.

THEOREM 2. *A series $a_1+a_2+\cdots$ of complex numbers is convergent if $a_n\to 0$ and if the ratios a_{n+1}/a_n lie in or on a polygon which is contained in the closed unit circle, and which touches the circle at one and only one point w, and $w\neq 1$.*

Proof. Again denoting the ratio by z we see that $1-|z|$ is the shortest distance from z to the unit circle and that $|w-z|$ is the distance from z to the point w on the unit circle. It is geometrically obvious that there is a constant $k>1$ (dependent on the shape of the polygon) such that $|w-z|\leqq k(1-|z|)$. Thus

$$|wa_j-a_{j+1}|\leqq k(|a_j|-|a_{j+1}|). \tag{3}$$

But $(w-1)R_{mn}=-a_m+\sum_m^{n-1}(wa_j-a_{j+1})+wa_n$.

$$|w-1||R_{mn}|\leqq|a_m|+k\sum_m^{n-1}(|a_j|-|a_{j+1}|)+|a_n|\leqq(1+k)|a_m|.$$

This completes the proof.

It is of some interest to obtain an inequality for the sum of the series in terms of the shape of the polygon. The relation $|w-z|=k(1-|z|)$ for $k>1$ defines a closed curve L of points z. If $z=x+iy$ and $w=a+ib$ the above relation becomes $\{(x-a)^2+(y-b)^2\}^{1/2}=k-k(x^2+y^2)^{1/2}$. The number of points of intersection of L with the straight line $y=c$ is the same as the number of points of intersection of the curve $y_1=\{(x-a)^2+d^2\}^{1/2}$ and the curve $y_2=k-k(x^2+c^2)^{1/2}$ where $d=b-c$. If neither d nor c are zero y_1 is concave upward and y_2 is concave downward so these curves may intersect in at most two points. If $d=0$ then y_1 is an angle function opening upward so if $c\neq 0$ the curve y_2 could intersect y_1 at most twice. If c is also zero then y_2 is an angle function opening downward but with slopes differing from that of y_1, so y_2 and y_1 could intersect at most twice. The remaining case is symmetrical. Thus L can intersect a line parallel to the x-axis in at most two points. Using a rotated coördinate system does not change the form of the equation of L but only the values of a and b. Thus L can intersect any straight line in at most two points, in other words L is a convex curve.

Let g_j be the distance of the jth vertex of the polygon to the circle and let h_j be the distance to the point w. Let $k=\max h_j/g_j=h/g$ (excluding the vertex at w). Clearly the points z' of the interior of L satisfy the relation $|w-z'| <k(1-|z'|)$ so the vertices of the polygon cannot lie outside L. Moreover, since L is convex the sides of the polygon must lie inside L. Thus we may use this value of k in the proof of Theorem 2 and obtain

THEOREM 3. *If S is the sum of the series $a_1+a_2+\cdots$, then it follows that* $|S|\leqq(1+h/g)|a_1|/|w-1|$.

Two suggested extensions of Theorem 2 are false. It is not permissible to have more than one vertex of the polygon touch the circle. It is not permissible to have the ratios lie within a circle inside but tangent to the unit circle.

The proof of the following theorem parallels that of Theorem 2.

THEOREM 4. *A series $a_1+a_2+\cdots$ of quaternions is convergent if $a_n\to 0$ and if the ratios $a_{n+1}a_n^{-1}$ lie in or on a four polyhedron which is contained in the closed unit four sphere, and which touches the sphere at one and only one point w, and* $w\neq 1$.

DERIVATION OF THE NORMAL FORM OF THE EQUATION OF THE STRAIGHT LINE

KENNETH MAY, Carleton College

The derivations of the normal form of the equation of the straight line which appear most frequently in elementary texts are based on the slope-intercept form, point-slope form, or on projection. The derivation from the intercept form, although less often found, has advantages of greater simplicity, symmetry and easy extension to three dimensions. It requires merely the observa-

tion that in all cases where the intercepts exist they are given by

$$a = p/\cos\alpha \quad \text{and} \quad b = p/\sin\alpha,$$

where p is the always positive length of the normal from the origin to the line and α is the positive angle through which the positive x-axis must be rotated to coincide with the positive direction on this normal. Substitution in the intercept form yields the normal form in a single step. Lines parallel to one of the axes or passing through the origin require, as usual, special treatment.

Similarly, the normal form of the equation of the plane in space may be derived from the intercept form by noting that each intercept (when it exists) is equal to the length of the normal from the origin to the plane divided by the corresponding direction cosine. The analogy may be emphasized by defining, for the two dimensional case, a second angle between the y-axis and the normal to the line.

These derivations are easy for the student to understand and remember. They facilitate unified treatment of plane and solid analytics in an elementary course without requiring explicit discussion of direction cosines in the plane. They lend themselves to efforts toward developing the student's appreciation of mathematical form in general and of the similarities between spaces of different dimensions in particular.

THE ELLIPSE AS A CIRCLE WITH A MOVING CENTER

F. H. YOUNG, Oregon State College

Our purpose is to examine the possibility of exhibiting the ellipse as the locus of a point on a rotating circle with a center moving on the focal axis. This condition may be satisfied if we can express the ellipse in the form $(x-x')^2+y^2=b^2$, for the symmetry of the figure requires that the radius be equal in length to b, half the minor axis. Throughout this discussion we shall assume that $a>b$.

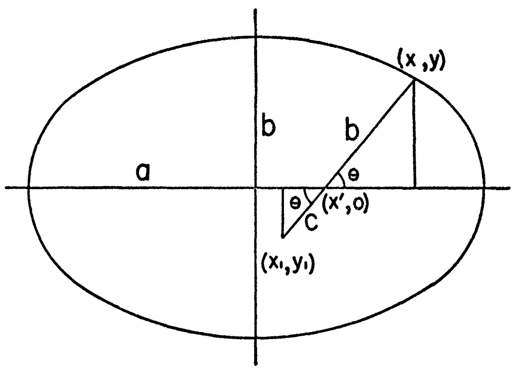

Fig. 1

In order to have x' move in such a way as to fit the ellipse $x^2/a^2+y^2/b^2=1$, we must let $x'=kx$ so that $x^2(1-k)^2/b^2+y^2/b^2=1$ where $b^2/(1-k)^2=a^2$, whence $k=(a-b)/a$.

Further, $x' = x - b\cos\theta$, but since $x' = [(a-b)/a]x$, then $x = a\cos\theta$. Also, $y = b\sin\theta$. Hence $x' = kx = (a-b)\cos\theta$. Therefore, $(x', 0)$, the center of the moving circle moves on the focal axis with simple harmonic motion over the range $-(a-b) \leqq x' \leqq (a-b)$ as θ increases constantly. Conversely, any such moving circle generates an unique ellipse, for the radius is of fixed length, b.

Let us now examine some of the properties revealed by this method of analysis of the ellipse. First, observe that the parametric equations $x = a\cos\theta$; $y = b\sin\theta$ can refer to the parameter θ, the angle the moving radius makes with the focal axis.

Consider next the situation arising when the radius is extended in the direction $\theta + \pi$ to give a point (x_1, y_1) lying the distance $b + c$ from (x, y) on the given ellipse. The point (x_1, y_1) then has the coordinates $y_1 = -c\sin\theta$ and $x_1 = x' - c\cos\theta = (a-b)\cos\theta - c\cos\theta = (a-b-c)\cos\theta$. Eliminating θ by squaring and adding, we obtain $y^2/c^2 + x^2/(a-b-c)^2 = 1$, again an ellipse, provided $b + c \neq a$. In the particular case in which $b + c = a$, we find that $x = (a-b-c)\cos\theta = 0$ and $y = (b-a)\sin\theta$, and as θ goes from 0 to 2π, we generate that section of the y-axis such that $-(a-b) \leqq y \leqq (a-b)$. If $b + c > a$, and we continue our assumption that $a > b$, then $c > a - b > 0$; hence $|a-b-c| > c$ and our resulting locus is an ellipse whose major axis lies along the y-axis. It is interesting to note that if $c = a$, we have our original ellipse with the axes interchanged.

Notice now that the slope of the generating radius is ay/bx. Hence, the radius is normal to the ellipse only at the endpoints of the axes. This result gives the answer to the problem of the type of curve generated by a normal of fixed length N moving about the ellipse. For a fixed N, $b - a$ remains unchanged. Thus, x' moves in the same manner as for the original ellipse, and the radius is now $b + N$. If (x_1, y_1) lies on an ellipse, its distance from $(x', 0)$ is $b + N$, and the line

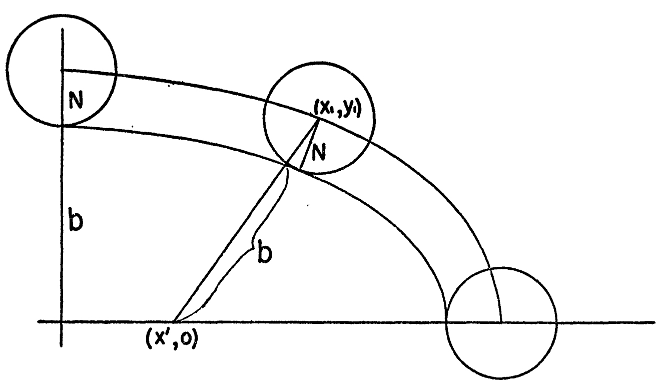

Fig. 2

joining these points is distinct from the normal. This requires the circle of radius N with center at (x_1, y_1) to be tangent to the ellipse externally in one place and to cut it in another, a contradiction. Hence, the curve generated by the moving normal is definitely not an ellipse. This can similarly be shown if N is measured in from the ellipse.

So far we have constrained b to be smaller than a. What results if we have our circle still moving on the x-axis, but $b>a$? This offers no difficulty, for from the relationship that $x'=(a-b)\cos\theta$, we can see that the center of the moving circle now must move from $[-(b-a), 0]$ to $[(b-a), 0]$, as θ increases positively, in describing the upper half of the ellipse whose equation is $x^2/a^2+y^2/b^2=1$, $b>a$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 806. *Proposed by Leo Moser, University of Manitoba*

Lewis Carroll once proposed the following problem.

"Two travellers spent from 2 o'clock till 9 in walking along a level road, up a hill, and home again; their pace on the level being x miles per hour, up hill y, and down hill $2y$. Find the distance walked."

In the original problem x and y were given integers. Deduce the solution to the original problem without *a priori* knowledge of what these integers are.

E 807. *Proposed by R. V. Andree, University of Wisconsin*

An elliptical endgate of a reservoir is to be mounted with the minor axis parallel to the water's surface and in such a manner that it will turn about a horizontal axis in the plane of the gate. Where should this axis be placed so that the gate will not tend to rotate in either direction when the water level is at a given distance above the top of the gate? Generalize to a gate of any shape.

E 808. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that the number

$$N = 1900045846159977680727716631$$

is a perfect cube and that the twenty-eight numbers which are formed by cyclic permutations of its digits are all divisible by the cube root of N.

E 809. *Proposed by P. L. Chessin, New York, N. Y.*

Show that $\sum_{n=1}^{\infty} n^{-(n+1)/n}$ diverges.

E 810. *Proposed by J. H. Butchart, Arizona State College*

Find the sum of the infinite series consisting of the following terms: the radius of the unit circle, the arc intercepted by the central angle x, the arc of the involute corresponding to this arc, the arc of the involute constructed on the first involute, and so on indefinitely.

SOLUTIONS

The Case of the Playful Children

E 776 [1947, 339]. *Proposed by L. R. Ford, Illinois Institute of Technology*

"Are those your children that I hear playing in the garden?" asked the visitor.

"There are really four families of children," replied the host. "Mine is the largest, my brother's family is smaller, my sister's is smaller still, and my cousin's is the smallest of all. They are playing drop the handkerchief," he went on; "they prefer baseball but there are not enough children to make two teams. Curiously enough," he mused, "the product of the numbers in the four groups is my house number, which you saw when you came in."

"I am something of a mathematician," said the visitor, "let me see whether I can find the numbers of children in the various families." After figuring for a time he said, "I need more information. Does your cousin's family consist of a single child?" The host answered his question, whereupon the visitor said, "Knowing your house number and knowing the answer to my question, I can now deduce the exact number of children in each family."

How many children were there in each of the four families?

I. *Solution by Monte Dernham, San Francisco.* The scene changes to 221*B* Baker Street.

"Well, Holmes, how on earth did you deduce 5, 4, 3, 2 children in the respective families? How could you guess that the house number was 120?"

"My dear fellow," answered Holmes, lighting his clay pipe, "as you know, there were not enough children all told to make up two nines; so it was obvious from the first that the smallest of the four families could not include more than two children. After a little figuring, I soon discovered that for two children in the cousin's family our inquiry would be restricted to seven possible products, only one of which, 120, might also fit the case of a single child. Consequently," he went on to explain, "had the house number been anything but 120, the visitor, familiar with the number and confessedly 'something of a mathematician,' would have known *without the need of more information* whether the cousin had one or two children.

"It is equally clear," Sherlock Holmes continued, "that if the reply to his question had been 'Yes' our visitor would have been confronted with an 'ambiguous case'; you see, there might have been 8, 5, 3, 1 children in the respective families, or again, 6, 5, 4, 1. But, since he was enabled to 'deduce the exact number of children in each family,' it is plain that the answer he received was 'No,'

consistent with but a single admissible combination. Rather elementary, my dear Watson!"

II. *Solution by N. J. Fine, University of Pennsylvania.* Let N be the house number, $a<b<c<d$ the numbers of children in the four families. Since $a+b+c+d<18$, we have the bounds $1\leqq a\leqq 2$, $2\leqq b\leqq 4$. Since the visitor asked whether $a=1$ or 2, N must have at least two admissible representations, with $a=1$ and $a=2$.

Using $a=1$ we have $c+d\leqq 16-b$, whence

$$4cd = (c+d)^2 - (d-c)^2 \leqq (16-b)^2 - 1.$$

It follows that $cd\leqq 48$, 42, 35 for $b=2$, 3, 4 respectively, so that $N=bcd\leqq 140$.

Now using $a=2$, $N\geqq 2\cdot 3\cdot 4\cdot 5=120$. Hence $120\leqq N\leqq 140$. Now if N is not divisible by 5, $N\geqq 2\cdot 3\cdot 4\cdot 6=144>140$. If N is not divisible by 8, $N\geqq 2\cdot 3\cdot 5\cdot 6=180>140$. Thus N is divisible by 40, from which $N=120$.

There are two admissible factorizations of 120 with $a=1$:

$$120 = 1\cdot 4\cdot 5\cdot 6 = 1\cdot 3\cdot 5\cdot 8.$$

There is only one with $a=2$:

$$120 = 2\cdot 3\cdot 4\cdot 5.$$

If the answer to the visitor's equestion had been $a=1$, he could not have obtained a unique solution. Hence

$$a = 2, \qquad b = 3, \qquad c = 4, \qquad d = 5.$$

Also solved by Murray Barbour, Max Beberman, Barney Bissinger, Ruth Campbell, J. L. Connors, R. E. Crane, J. S. Cromelin, J. H. Ferguson, C. W. Foard, C. O. Hines, William Kruskal, J. M. McLynn, Leo Moser, M. L. Oneil, Bart Park, C. F. Pinzka, W. M. Rust, Jr., G. W. Walker, Maud Willey, and R. H. Wilson, Jr.

The n-Kings Problem

E 777 [1947, 340]. *Proposed by C. R. Perisho, Nebraska Wesleyan University*

Find the number of permutations of n objects with the restriction that in no arrangement may an object be adjacent to either of its neighbors in the original order.

Note by Leo Moser, University of Manitoba. This problem is solved by I. Kaplansky in his paper *Symbolic solution of certain problems in permutations*, Bull. Amer. Math. Soc., vol. 50 (1944), pp. 911–913. The symbolic solution given there does not lend itself to expression by a simple explicit formula. As Kaplansky points out, the problem may be formulated as follows: In how many ways can n kings be placed on an n by n chess board, one on each row and column, so that no two attack each other?

Equal Shortest Paths

E 778 [1947, 340]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

For a given tetrahedron $ABCD$, find the point P in space such that the shortest paths separating P from each of the vertices A, B, C, D, after having touched the opposite faces BCD, CDA, DAB, ABC, are equal to each other.

Solution by G. W. Walker, Buffalo, N. Y. Locate A', the reflection of the vertex A across the plane of the opposite face, BCD, and locate B', C', and D' similarly. Then the shortest path from P to A touching the plane BCD will equal the distance PA'. (If O is the point of contact of the path with the plane, then $PO+OA=PO+OA'$ is a minimum when PA' is a straight line.) Thus the problem is simply to find the circumcenter P of the tetrahedron $A'B'C'D'$.

Editorial Note. It should be noted that the above solution tacitly assumes that the circumcenter of $A'B'C'D'$ is inside, or on a face, of $ABCD$. Under similar restrictions the corresponding problem in the plane is solved in an analogous fashion.

A Divisibility Problem

E 779 [1947, 340]. *Proposed by P. A. Pizá, San Juan, Puerto Rico*

Let x, y, z be three positive integers and set

$$a = x + y, \qquad b = x + z, \qquad c = x + y + z.$$

Show that for any prime exponent $p > 2$,

$$(ab)^p - (cx)^p - (yz)^p$$

is divisible by the product $pabcxyz$.

Solution by N. J. Fine, University of Pennsylvania. Let $ab=A$, $cx=B$, $yz=C$. Then $A=B+C$. We have

$$\begin{aligned}
A^p - B^p - C^p &= (A - B)\sum_{r=0}^{p-1} A^{p-r-1}B^r - C^p \\
&= C\left\{\sum_{r=0}^{p-1} A^{p-r-1}B^r - C^{p-1}\right\} \\
&= C\left\{A^{p-1} + B^{p-1} + AB\sum_{r=1}^{p-2} A^{p-r-2}B^{r-1} - (A-B)^{p-1}\right\} \\
&= C\left\{A^{p-1} + B^{p-1} + AB\sum_{r=1}^{p-2} A^{p-r-2}B^{r-1}\right. \\
&\qquad \left. - \sum_{r=0}^{p-1} (-1)^r \binom{p-1}{r} A^{p-r-1}B^r\right\}
\end{aligned}$$

$$= ABC\left\{\sum_{r=1}^{p-2} A^{p-r-2}B^{r-1} - \sum_{r=1}^{p-2}(-1)^r\binom{p-1}{r}A^{p-r-2}B^{r-1}\right\}$$

$$= ABC\sum_{r=1}^{p-2} A^{p-r-2}B^{r-1}\left\{1-(-1)^r\binom{p-1}{r}\right\}.$$

But, for $1 \leqq r \leqq p-1$ and p an odd prime,

$$\binom{p-1}{r} = \binom{p}{r} - \binom{p-1}{r-1} \equiv -\binom{p-1}{r-1} \pmod{p}.$$

Hence

$$\binom{p-1}{r} \equiv (-1)^r \pmod{p},$$

so that each factor

$$\left\{1-(-1)^r\binom{p-1}{r}\right\}$$

is divisible by p, and the theorem follows.

Also solved by Murray Barbour, P. A. Clement, F. J. Duarte, Leo Moser, and G. W. Walker. Clement pointed out that a calculation, taking $x=y=z=1$, shows the necessity of requiring $p>2$.

Cutting Out a Lampshade

E 780 [1947, 340]. *Proposed by G. Pólya, Stanford University*

A lampshade has the shape of a frustum of a right circular cone. Its perimeter is P at the bottom, p at the top, and its slant height is s. Show that such a lampshade can be cut out in one piece from a rectangular sheet of paper with dimensions

$$P \quad \text{and} \quad s + p(P-p)/8s.$$

You can even save paper for a flap to glue the ends together, except in the limiting case where $P=p$, when not a bit of paper is wasted.

Solution by Joseph Rosenbaum, Milford School, Conn. The lampshade *cannot* always be cut out in one piece from the given rectangle. One would suspect this conclusion from a consideration of the extreme case where $p=0$ and $P=2\pi s$. Here the shade is a circular disc of radius s, and obviously cannot be cut out in one piece from the rectangle, one of whose dimensions is now equal to s.

In order to obtain the facts for non-extreme situations we shall consider the two cases $P-p \leqq \pi s$ and $P-p > \pi s$. It will be shown that *in the first case the construction of the problem is always possible, but in the second case the construction is possible if and only if*

$$(1)\qquad p/P \geqq \left[2\cos\left(\pi - \frac{P-p}{2s}\right)\right] \Big/ \left[\left(\frac{P-p}{2s}\right)^2 - 2\right].$$

Let us, for the present, consider the piece for the shade as a sector of a circular ring, with R and r as the outer and inner radii, and α as half the central angle. It is easily shown that for the first case, α is non-obtuse. When $0<\alpha<\pi/2$, the piece can be cut from a rectangle $ABCD$ as shown in Figure 1, the piece being $EFGHIJE$.

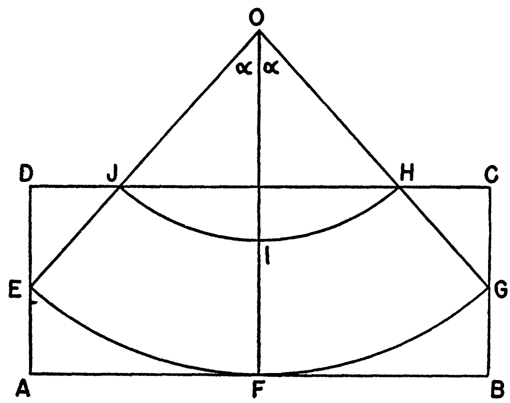

Fig. 1

Here we have

$$P = \text{arc } EFG > AB.$$

Also, from Maclaurin's expansion of cos x,

$$f(\alpha) = \alpha^2/2 - 1 + \cos\alpha \geqq 0.$$

Now, setting $R=s+r$, $P=2\alpha R$, $p=2\alpha r$, we have

$$s + p(P-p)/8s - R + r\cos\alpha = rf(\alpha) \geqq 0,$$

or

$$s + p(P-p)/8s \geqq R - r\cos\alpha = BC.$$

Thus, for $0<\alpha<\pi/2$, the dimensions of the given rectangle are never less than those of $ABCD$, and the lampshade can be cut out in one piece in the shape of a sector of a circular ring. It is easily shown that the given rectangle similarly suffices for the extreme cases where $\alpha=0$ (that is, where $p=P$) and $\alpha=\pi/2$. Thus the first part of the italicized statement is established.

Let us now consider the second case and still take the piece as a sector of a circular ring. Here α is obtuse and the piece can be cut from a rectangle $ABCD$ as shown in Figure 2. The dimensions of the rectangle $ABCD$ are now $2R$ and $R+R\cos(\pi-\alpha)$. Hence, for the given rectangle to be adequate, it is necessary and sufficient that

$$s + p(P-p)/8s \geqq R + R\cos(\pi-\alpha).$$

Substituting, as above, for the several quantities, but leaving the second R intact, this becomes

$$r\alpha^2/2 \geqq r + R\cos(\pi - \alpha)$$

or

$$r/R \geqq [2\cos(\pi - \alpha)]/(\alpha^2 - 2).$$

This, expressed in terms of the given quantities, is the condition (1) above.

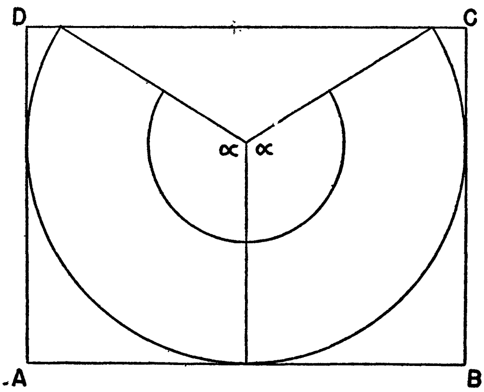

Fig. 2

It finally remains to show that the dimensions of the circumscribed rectangle $ABCD$ do not decrease when the piece which is cut out has any shape other than a sector of a circular ring. Toward this end we first observe that the general admissible shape for the piece in question is defined as follows:

Let M and N be any two points respectively on the lower and the upper circumferences of the frustum, and let L be any path, on the surface, joining M to N. If L has no points in common with the circumferences other than M and N, and if L does not cross itself, and if L is such that when the surface is cut along L and then flattened out, the resulting plane figure has no overlapping areas, then this plane figure is an admissible shape for the piece. (The last condition on L is necessitated by the fact that without it the piece cannot be cut from *any* plane sheet. That such pieces may exist is readily discovered by models.)

It is obvious that the above includes all admissible shapes, and it is further obvious that in any such shape a part of the perimeter of the piece will be the outer arc of the previous sector, and thus the dimensions of the corresponding rectangle cannot be less than those for the sector. This establishes the second part of the italicized statement.

Also solved, partially, by Ragnar Dybvik, R. B. Herrera, Irene Price, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4275 [1947, 601]. Corrected. *Proposed by Raymond Redheffer, Massachusetts Institute of Technology, Cambridge*

Let a_i represent any set of points in the complex plane, with sole limit-point at infinity, while b_i are any complex numbers. Prove there exists an integral function $f(z)$ such that $f(a_i)=b_i$.

4285. *Proposed by H. N. Davis, Victor Erikson, and Robert Nathans, United States Army*

A full reel of film (thickness τ) of original diameter A is being wound onto an empty spool of original radius a and rotating at a uniform angular velocity ω. How long does it take to unwind the first spool if its inner diameter is also a?

4286. *Proposed by H. F. Sandham, Trinity College, Dublin, Ireland*

Prove that

$$\int_0^\infty \frac{\cos x^2 - \cos x}{x}\, dx = \tfrac{1}{2}\gamma,$$

where γ is Euler's constant.

4287. *Proposed by C. R. Phelps, Rutgers University*

Show that for any given integer $k>1$, there are an infinite number of perfect kth powers which cannot be written as the sum of a prime and a kth power. (This disproves a conjecture of Hardy and Wright, *Introduction to the Theory of Numbers*, p. 19.)

4288. *Proposed by J. W. Campbell, University of Alberta*

In his Messenger Lecture on Probability (A. S. Eddington, *New Pathways in Science*, 1935, p. 121) Eddington referred to the following problem: If A, B, C, D each speak the truth once in three times (independently), and A affirms that B denies that C declares that D is a liar, what is the probability that D was telling the truth. He used the Exclusion Method of Solution and arrived at the numerical result 25/71.

Prove that the correct probability is 13/41, and that this is also the probability that each of A, B, and C told the truth.

4289. *Proposed by J. T. Culbertson, Southwestern University, Georgetown, Texas*

Let C be any finite class of n similar objects $(n>1)$ and call any subclass of C a singular class if it contains only one of the n objects and call it a plural class if it contains more than one of the n objects. Also let any set of classes consisting of C and all the singular subclasses of C, together with any other subclasses of C (or none other at all) be defined as a classification of C. Thus any classification has r members where $n<r<2^n$, and there are $M=2^n-n-2$ subclasses of C, each of which may or may not belong to any particular classification (all the plural subclasses except C itself). Therefore there are 2^M classifications of C.

Let K be any classification of C and a and b any two objects in C. By interchanging a and b we get a classification K' which may or may not be identical with K. Let us call K' an interchange of K. Also let this relation be transitive, that is, if K, K' and K'' are any three classifications of C, and K' is an interchange of K, and K'' is an interchange of K', then K'' is an interchange of K. Then any two or more classifications are defined as structurally similar if and only if any one of them is an interchange of each of the others.

The set of all classifications structurally similar to any given classification is an α-set, e.g., when $n=3$ there are 4 α-sets.

The problem is: How many α-sets does C have; that is, how many structurally different sets of classifications are there for any such class of n objects?

SOLUTIONS

Squares Constructed Within A Triangle

3990 [1941, 214]. *Proposed by V. Thébault, Tennie, Sarthe, France*

Let A', B', C' be the centers of squares $BCA_1'A_2'$, $CAB_1'B_2'$, $ABC_1'C_2'$ constructed interiorly on the sides of triangle ABC with the centroid G and the angle V of Brocard. If $\cot V=7/4$, show that: (1) The centers A'', B'', C'' of the squares constructed interiorly on the sides of $A'B'C'$ lie on a straight line through G. (2) The angle V' of Brocard of $A'B'C'$ is such that $\cot V'=2$. (3) The straight lines joining A, B, C respectively to the midpoints of $A_1'A_2'$, $B_1'B_2'$, $C_1'C_2'$ are parallel. (4) The distance of the circumcenter from the orthocenter of the orthic triangle is equal to one-fourth of the perimeter of the last triangle.

*Solution by R. Bouvaist, Vincelles, Saône-et-Loire, France.** Letting S represent the area of triangle ABC we have

$$\begin{aligned}\overline{B'C'}^2 &= \overline{AB'}^2+\overline{AC'}^2-2\overline{AB'}\cdot\overline{AC'}\cos\overline{B'AC'} = (b^2+c^2-2bc\sin A)/2\\ &= (b^2+c^2-4S)/2.\end{aligned}$$

* Translated and checked by O. J. Ramler, Catholic University of America, Washington, D. C.

If $(A'B'C')$ denotes the area of triangle $A'B'C'$,

$$(A'B'C') = (OB'C') + (OC'A') + (OA'B'),$$

where O is the circumcenter of ABC. Hence

$$\begin{aligned} 2(A'B'C') &= R^2\sum (\cos C - \sin C)(\cos B - \sin B) \sin A \\ &= R^2\sum (\sin 2A - \sin^2 A) = S(2 - \cot V), \end{aligned}$$

where R is the circumradius of ABC. The normal coördinates of A', B', C' are

$$\begin{array}{lll} x_{A'} = a/2, & x_{B'} = b(\sin C - \cos C)/2, & x_{C'} = c(\sin B - \cos B)/2, \\ y_{A'} = a(\sin C - \cos C)/2, & y_{B'} = b/2, & y_{C'} = c(\sin A - \cos A)/2, \\ z_{A'} = a(\sin B - \cos B)/2, & z_{B'} = b(\sin A - \cos A)/2, & z_{C'} = c/2. \end{array}$$

Whence

$$x_{A'} + x_{B'} + x_{C'} = h_1, \qquad y_{A'} + y_{B'} + y_{C'} = h_2, \qquad z_{A'} + z_{B'} + z_{C'} = h_3,$$

where h_1, h_2, h_3 are the altitudes of ABC. Therefore, $A'B'C'$ has G for its centroid. Since $2(A'B'C') = S(2-\cot V)$, it follows that $A'B'C'$ are collinear when $\cot V = 2$. This proves part (2), for the same relationships hold for triangle $A''B''C''$ in respect to triangle $A'B'C'$. Hence $\cot V' = 2$.

Now

$$\cot V' = 2 = \frac{\overline{A'B'}^2 + \overline{B'C'}^2 + \overline{C'A'}^2}{4(A'B'C')} = \frac{a^2 + b^2 + c^2 - 6S}{2S(2 - \cot V)}.$$

But $\cot V = (a^2+b^2+c^2)/4S$, hence $\cot V = 7/4$.

The midpoints of $A_1'A_2'$, $B_1'B_2'$, $C_1'C_2'$ have the normal coördinates

$$\begin{array}{ccc} a, & a(\sin C - 2\cos C)/2, & a(\sin B - 2\cos B)/2, \\ b(\sin C - 2\cos C)/2, & b, & b(\sin A - 2\cos A)/2, \\ c(\sin B - 2\cos B)/2, & c(\sin A - 2\cos A)/2, & c. \end{array}$$

The lines joining the midpoints of $A_1'A_2'$, $B_1'B_2'$, $C_1'C_2'$ to A, B, C respectively meet in the point whose normal coördinates are given by

$$x(\sin A - 2\cos A) = y(\sin B - 2\cos B) = z(\sin C - 2\cos C).$$

These lines will be parallel when

$$\sum \sin A(\sin B - 2\cos B)(\sin C - 2\cos C) = 0$$

or

$$7 \sin A \sin B \sin C - 2\sum \sin^2 A = 0,$$

which reduces to

$$a^2 + b^2 + c^2 = 7S, \qquad \cot V = 7/4.$$

If H_1, H_2, H_3 are the feet of the altitudes and H' is the orthocenter of $H_1H_2H_3$ (the orthic triangle of ABC) the normal coördinates of H' with respect to the triangle $H_1H_2H_3$ are given by

$$x \cos H_1 = y \cos H_2 = z \cos H_3.$$

The distance of H' to BC is proportional to

$$\frac{\cos H_2 + \cos H_3}{2 \cos H_2 \cos H_3 \sin (H_1/2)} = -\frac{\cos 2B + \cos 2C}{2 \cos 2B \cos 2C \cos A}$$
$$= -\frac{2 \cos (B + C) \cos (B - C)}{2 \cos 2B \cos 2C \cos A},$$

hence proportional to $\cos 2A \cos(B-C)$, since the equation of BC with respect to triangle $H_1H_2H_3$ is $y+z=0$. The actual normal coördinates of H' in triangle ABC are then readily found to be

$$x_{h'} = -R \cos 2A \cos (B - C), \qquad y_{h'} = -R \cos 2B \cos (C - A),$$
$$z_{h'} = -R \cos 2C \cos (A - B).$$

Now $\cos 2A \cos (B-C)$ can be represented as

$$-\tfrac{1}{2}[\sin A \textstyle\sum \sin 2A + \cos A(\sum \cos 2A - 1)], \quad \text{and}$$
$$\sum \sin 2A = \frac{2S}{R^2}, \quad \text{and} \quad \sum \cos 2A - 1 = 2 - \frac{a^2 + b^2 + c^2}{2R^2}.$$

If x_k and x_0 are the distances of the Lemoine point K and of the circumcenter O of ABC from BC we have,

$$x_k = R \tan V \sin A, \qquad x_0 = R \cos A.$$

Hence

$$x_{h'} - x_0 = S \cot V(x_k - x_0)/R^2$$

and

$$x_{h'} - x_k = \left(\frac{S \cot V}{R^2} - 1\right)(x_k - x_0).$$

Therefore

$$\frac{H'O}{H'K} = \frac{a^2 + b^2 + c^2}{a^2 + b^2 + c^2 - 4R^2}, \qquad \frac{H'O}{KO} = \frac{a^2 + b^2 + c^2}{4R^2} = \frac{S \cot V}{R^2}.$$

Now by a known formula $OK = R \tan V\sqrt{\cot^2 V - 3}$ and

$$OH' = \frac{S\sqrt{\cot^2 V - 3}}{R}.$$

If $\cot V = 7/4$, then

$$OH' = \frac{S}{4R}\cdot$$

Now $2S/R$ is the perimeter of the orthic triangle. Hence OH' is one-eighth of this perimeter instead of one-fourth as stated in the proposal.

Second Lemoine Point

4223 [1946, 537]. *Proposed by Victor Thébault, Tennie, Sarthe, France.*

In a tetrahedron the harmonic plane of the point L whose normal coördinates are proportional to the radii of the circumcircles of the triangles of the faces (second Lemoine point), coincides with the polar plane with respect to the circumsphere.

Solution by R. Goormaghtigh, Bruges, Belgium. Let $A_1A_2A_3A_4$ be the tetrahedron, a_{ij} the length of the edges A_iA_j, S_i and R_i the area and the circumradius of the face opposite to A_i.

The normal coördinates of L being proportional to $R_1, \cdots$, the barycentric coördinates are proportional to $S_1R_1, \cdots$, or $a_{23}a_{34}a_{42}, \cdots$.

The barycentric equation of the circumsphere being

$$\sum a_{ij}^2 x_i x_j = 0,$$

the polar plane of L is

$$a_{12}a_{13}a_{14}x_1 + \cdots = 0$$

or

$$x_1/a_{23}a_{34}a_{42} + \cdots = 0$$

and this is the equation of the harmonic plane of L.

Also solved by M. R. Blanchard.

Central Ellipse of Inertia

4225 [1946, 594]. *Proposed by H. F. Sandham, Trinity College, Ireland*

There are given N points in a plane, the roots of $f(z)=0$. A conic is constructed with center at the mean center, and is such that the sum of the squares of the perpendiculars from the points on to a line through the center is $N(N-1)$ times the square of the perpendicular from the center on to a parallel tangent. Prove that the foci are given by $f^{(N-2)}(z)=0$.

Solution by Fritz John, New York University. Let the N points

$$z_k = x_k + iy_k \qquad k = 1, \cdots, N$$

be the roots of the equation

$$f(z) = z^N - Az^{N-1} + Bz^{N-2} - \cdots = 0.$$

Without restriction of generality we can assume that the mean center O of the points is at the origin of the coördinate system. Then

$$\sum_k z_k = A = 0,$$
$$\sum_k z_k^2 = A^2 - 2B = -2B. \tag{1}$$

Let E be the ellipse with center O such that the sum of the squares of the perpendiculars from the z_k to a line through O is $N(N-1)$ times the square of the distance of the line from a parallel tangent of E. Let $ux+vy=1$ be the equation of a tangent of E. Then the distance of that tangent from the parallel diameter is $(u^2+v^2)^{-1/2}$, and the distance of a point (x, y) from that diameter is

$$\left| \frac{ux+vy}{\sqrt{u^2+v^2}} \right|.$$

The definition of E can then be written

$$\sum_k (ux_k + vy_k)^2 = N(N-1), \tag{2}$$

which can be interpreted as the equation of E in tangential coördinates.

The foci of E are the intersections of conjugate imaginary isotropic tangents of E. Here the isotropic tangents are those of slope $\pm i$. For an isotropic tangent of E of slope $+i$ we have $v=iu$, and hence from (2)

$$u^2 \sum_k z_k^2 = N(N-1). \tag{3}$$

The equation of that isotropic tangent is then $u(x+iy)=1$. Its intersection with its conjugate imaginary tangent, $\bar{u}(x-iy)=1$, is then represented by the complex number

$$z = x + iy = 1/u. \tag{4}$$

The two foci of E are then seen to be represented by the reciprocals of the roots of the quadratic equation (3) and are therefore the roots of

$$z^2 = \frac{1}{N(N-1)} \sum_k z_k^2,$$

which, by use of (1), reduces to

$$z^2 + \frac{2B}{N(N-1)} = 0,$$

or

$$f^{(N-2)}(z) = \frac{N!}{2} z^2 + (N-2)!B = 0,$$

so that the proof is complete.

From its definition E can be seen to be essentially the central ellipse of inertia of the z_i. (More precisely, the two ellipses are homothetic from O.) The statement of the problem can then be reduced to a theorem of Lucas, quoted in Appell, *Mecanique Rationelle*, t. 2, p. 16, ex. 13, and which is reproduced here freely in the notation used above: *The roots of* $f^{(N-2)}(z)=0$ *lie on the major axis of the central ellipse of inertia of unit masses distributed over the roots of* $f(z)=0$. *The difference of the squares of the radii of gyration of the masses about the principal axes of inertia is* $(N-1)/4$ *times the square of the distance of the two roots of* $f^{(N-2)}(z)=0$.

Also solved by H. E. Fettis and J. H. Simester.

Regions in which Polynomials Have Absolute Value Not Exceeding Unity

4229 [1947, 49]. *Proposed by Paul Erdös, Syracuse University*

Let $f(z)=z^n+\cdots+a_n$, $g(z)=z^m+\cdots+b_m$ be two polynomials. Denote by A the region where $|f(z)|\leqq 1$ and by B the region where $|g(z)|\leqq 1$. Prove that A cannot properly contain B.

Solution by Tibor Radó, Ohio State University. If $P(z)=z^k+\cdots$ is a polynomial of degree $k\geqq 1$, we shall denote by $D(P)$ the open set on which $|P(z)|>1$, and by $D^*(P)$ the (obviously unique) unbounded component of $D(P)$. Note that $D^*(P)$ contains a whole neighborhood of $z=\infty$, and that the frontier of $D^*(P)$ is not empty. Our problem is then equivalent to that of proving that $D(f)$ cannot be a proper subset of $D(g)$.

Let us now assume merely that $D(f)$ is a subset of $D(g)$ (not necessarily proper), in symbols $D(f)\subset D(g)$. Then clearly $D^*(f)\subset D^*(g)$. Consider the auxiliary function $\Psi(z)=F(z)/G(z)$ for $z\in D^*(f)$, where $F(z)=[f(z)]^m$ and $G(z)=[g(z)]^n$. Clearly $D(F)=D(f)$ and $D(G)=D(g)$. Then $\Psi(z)$ is analytic in $D^*(f)$, and, since $|G(z)|\geqq 1$ on the frontier of $D^*(f)$, $|\Psi(z)|\leqq 1$ on the frontier of $D^*(f)$. On the other hand, $\Psi(z)\rightarrow 1$ for $z\rightarrow\infty$. By the maximum-modulus theorem it follows that $\Psi(z)\equiv 1$ and hence $F(z)\equiv G(z)$ in $D^*(f)$ and consequently $F(z)\equiv G(z)$ throughout the whole plane. It follows that $D(f)=D(g)$ as required.

Note by Marshall Hall, Ohio State University. If $(m, n)=d$ and $m=dm_1$, $n=dn_1$, the above proof shows that $[f(z)]^{m_1}\equiv[g(z)]^{n_1}$; and since m_1 and n_1 are relatively prime, it follows from the uniqueness of factorization of polynomials over a field that there is a polynomial $h(z)$ such that $f(z)=[h(z)]^{n_1}$, $g(z)=[h(z)]^{m_1}$. Thus the relation $D(f)\subset D(g)$ holds if and only if $f(z)$ and $g(z)$ are positive integral powers of the same polynomial $h(z)$.

Also solved by Robert Breusch, Michael Golomb, Fritz Herzog, and the proposer.

Integers Reproduced in the Right-hand Digits of Their Squares

4230 [1947, 49]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In every system of numeration in which the base B is divisible by two or more

distinct primes but not by 3 or any prime of the form $6k+1$, the numbers which have the property that they are reproduced in the right-hand digits of their squares are the same as those which are reproduced at the right of their fourth powers. (For example: $B=10$, $76^2=5776$, $76^4=33362176$.)

Solution by Fritz Herzog, Michigan State College. The proposal is a special case (corresponding to $p=3$) of the following more general theorem.

Let p be an odd prime and let B be an integer greater than unity, not divisible by p nor by any prime of the form $pk+1$. In the system of numeration with B as base, a number is reproduced in the right-hand digits of its $(p+1)$st power if and only if it is reproduced in the right-hand digits of its square.

Let x be a number of m digits. We then have to show that $x^{p+1}\equiv x \pmod{B^m}$ if and only if $x^2\equiv x \pmod{B^m}$. It obviously suffices to show that the first congruence implies the second. Let q be any prime factor of B and let q^s be the highest power of q that divides B. From the first of the above congruences we have

$$x^{p+1} \equiv x \pmod{q^{sm}}, \tag{1}$$

and it remains to show that, for all prime factors q of B,

$$x^2 \equiv x \pmod{q^{sm}}. \tag{2}$$

Now if $(x, q)=1$, then we obtain from (1) the relation $x^p\equiv 1 \pmod{q^{sm}}$; hence the exponent of x modulo q^{sm} is either p or 1. But by hypothesis, q is not congruent to 0 or 1 (mod p), hence p cannot divide $\phi(q^{sm})=q^{sm-1}(q-1)$ and,therefore, we must have $x\equiv 1 \pmod{q^{sm}}$, which implies (2). On the other hand, if $x\equiv 0 \pmod{q}$ we write (1) in the form $(x^p-1)x\equiv 0 \pmod{q^{sm}}$ and, since in this case $(x^p-1, q^{sm})=1$, we have $x\equiv 0 \pmod{q^{sm}}$, which again implies (2).

Also solved by Free Jamison and the Proposer.

Editorial Note. The Proposer remarks that the condition that B be composite is not necessary for the proof, but if B is prime there are no numbers less than B^m except 0 and 1 which have the desired property. The proposed problem was suggested by a paper appearing in *Mathesis.*

Curves Having a Property of the Parabola

4231 [1947, 49]. *Proposed by Paul Nemenyi, Washington State College*

Show that the parabola $y=ax^n (a\neq 0, n>0)$ has the following property: If through the vertex any ray is drawn, the ratio of the area of the segment to that of the largest inscribed triangle is independent of the direction of the ray. Are there other curves with the same property?

Solution by Arthur Breusch, Amherst College. I. If $y=f(x)=ax^n+bx$, $n>0$, $n\neq 1$, then the area between the curve and the chord joining $(0, 0)$ to (x, y) is

$$\pm\left(\frac{1}{2}xy-\int_0^x ydx\right)=\pm\frac{n-1}{2(n+1)}ax^{n+1}.$$

The area of the largest inscribed triangle is

$$\pm \tfrac{1}{2}[x_1f(x) - xf(x_1)],$$

where $x f'(x_1) = f(x)$. Therefore $x_1/x = n^{1/(1-n)}$, and the area of the triangle is

$$\pm \frac{(n-1)ax^{n+1}}{2n^{n/(n-1)}}.$$

The signs will be both + or both − according as n is greater than or less than 1. The desired ratio is therefore

$$\frac{n^{n/(n-1)}}{n+1}.$$

II. Any curve given by

$$\alpha x + \beta y = (\gamma x + \delta y)^n, \qquad n > 0 \tag{1}$$

will also have the property; for a rotation of coördinate axes will reduce this equation to the form $\bar{y} = a\bar{x}^n + b\bar{x}$.

It can be shown that, under certain restrictions, if $y = f(x)$ is a function whose graph has the desired property, then y is given by a relation of the type (1).

III. Lemma 1: *If the curve given by $y = f(x)$ has the desired property, and if $f'(x)$ and $f''(x)$ exist in $0 < x \leqq h$, then $f(x)$ must satisfy in $0 < x \leqq h$ the functional equation*

$$f'\left[\frac{\int_0^x x^2 f''(x)dx}{k\int_0^x xf''(x)dz}\right] = \frac{f(x)}{x}, \tag{2}$$

where k is a constant.

Proof: Given

$$xf(x) - 2\int_0^x f(x)dx = k[x_1f(x) - xf(x_1)], \tag{A}$$

with

$$xf'(x_1) = f(x), \text{ and } k \text{ constant.} \tag{B}$$

Differentiating (A) with respect to x, we have

$$xf'(x) - f(x) = k[x_1f'(x) - f(x_1)] + k[f(x) - xf'(x_1)]dx_1/dx,$$

which by means of (B) reduces to

$$xf'(x) - f(x) = k[x_1f'(x) - f(x_1)].$$

From this last equation and equation (A) we can compute

$$x_1 = \frac{x^2f'(x) - 2xf(x) + 2\int_0^x f(x)dx}{k[xf'(x) - f(x)]} = \frac{\int_0^x x^2f''(x)dx}{k\int_0^x xf''(x)dx} \equiv \frac{\phi(x, f)}{k},$$

where we have introduced the symbol $\phi(x, f)$ to simplify the notation. Placing this value of x_1 in equation (B) we obtain (2).

IV. Lemma 2: *If $f(x)$ satisfies (2), then $g(x) = af(x) + bx$ does likewise.*

Proof: $g''(x) = af''(x)$, therefore $\phi(x, g) = \phi(x, f)$. Also $g' = b + af'$. Therefore

$$g'[\phi(x, g)] = b + af'[\phi(x, f)] = b + a\frac{f(x)}{x} = \frac{g(x)}{x}.$$

Thus if $y = f(x) = Ax + Bx^r + Cx^s + \cdots$ satisfies (2), then $\bar{y} = \bar{f}(x) = x^r + bx^s + \cdots$ does likewise, with $b = C/B$, $r \neq 1$, $s \neq 1$.

V. Lemma 3: *If $\bar{f}(x)$ satisfies (2) and is a sum of more than one non-linear terms: $\bar{f}(x) = x^r + bx^s +$terms of higher degree with $r < s$, $r \neq 1$, $s \neq 1$, $b \neq 0$, then s must equal $2r - 1$.*

Proof: By direct computation we obtain

$$\phi(x, \bar{f}) = \frac{\dfrac{r}{r+1}x + b\dfrac{s(s-1)}{(s+1)(r-1)}x^{s-r+1} + \cdots}{1 + b\dfrac{s-1}{r-1}x^{s-r} + \cdots}$$

$$= \frac{r}{r+1}x + b\frac{(s-1)(s-r)}{(r+1)(s+1)(r-1)}x^{s-r+1} + \cdots.$$

Substituting this in (2) and simplifying, we have

$$\frac{r}{k^{r-1}}\left[\left(\frac{r}{r+1}\right)^{r-1}x^{r-1} + (r-1)\left(\frac{r}{r+1}\right)^{r-2}b\frac{(s-1)(s-r)}{(r+1)(s+1)(r-1)}x^{s-1} + \cdots\right]$$
$$+\frac{bs}{k^{s-1}}\left(\frac{r}{r+1}\right)^{s-1}x^{s-1} + \cdots = x^{r-1} + bx^{s-1} + \cdots.$$

Comparing coefficients, we find

$$\frac{r^r}{k^{r-1}(r+1)^{r-1}} = 1, \qquad \frac{r^{r-1}}{k^{r-1}(r+1)^{r-1}}\frac{(s-1)(s-r)}{s+1} + \frac{sr^{s-1}}{k^{s-1}(r+1)^{s-1}} = 1.$$

Upon eliminating k and simplifying we obtain

$$(2r + 1 - s)r^{(s-r)/(r-1)} - (s + 1) = 0.$$

For a fixed positive value of $r \neq 1$, call the left member of this equation $g(s)$.

Then

$$g'(s) = r^{(s-r)/(r-1)}\left[-1+\frac{2r+1-s}{r-1}\log r\right]-1,$$

$$g''(s) = r^{(s-r)/(r-1)}\left[-2+\frac{2r+1-s}{r-1}\log r\right]\frac{\log r}{r-1}\cdot$$

Since the quantities in brackets are linear in s, therefore $g''(s)$ can vanish only once, $g'(s)$ only twice, and $g(s)$ only three times for $-\infty<s<+\infty$; but it is easily seen that $g(r)=g(1)=g(2r-1)=0$, and since $r<s$ and $s\neq 1$, s must be $2r-1$.

Corollary: *If* $\bar{f}(x)=x^r+bx^s+$*terms of higher degree, satisfies* (2), *and if* $r<s$, $s\neq 1$, *and* $r<1$, *then* b *must be* 0; for $2r-1$ would be less than r.

VI. Theorem: *If* $\bar{f}(x)=x^r+\cdots$ $(r\neq 1)$ *has no linear term and satisfies* (2), *and if* $\bar{f}(x)$ *is either regular in the neighborhood of* $x=0$, *or of the form*, x^r *times a regular function, with* $r>0$, *then* $y=a\bar{f}(x)+bx$ *is connected with* x *by a relation of the form* (1).

Proof: If $r<1$, then $\bar{f}(x)$ has only the term x^r, and therefore $y=ax^r+bx$, which is of the form (1). If $r>1$, then $\bar{y}=x^r(1+\cdots)=x^r$ times a regular function of x. Then $\bar{y}^{1/r}=x(1+\cdots)=$a regular function of x with non-vanishing derivative at $x=0$. Therefore x is also a regular function of $\bar{y}^{1/r}$:

$$x = g(\bar{y}) = \bar{y}^{1/r} + \text{higher powers of } \bar{y}.$$

Since $g(\bar{y})$ must satisfy equation (2), and since $1/r<1$, the only other possible term is a linear one:

$$x = \bar{y}^{1/r}+\beta\bar{y}, \qquad (x-\beta\bar{y})^r = \bar{y},$$

and finally

$$y = a\bar{y}+bx = a\left(x-\beta\frac{y-bx}{a}\right)^r + bx,$$

which is again of form (1).

VII. If $f'(x)$ does not exist everywhere in $0<x\leqq h$, then its graph may have the prescribed property even though x and y are not connected as in (1). For example, $y=|x-1|-1$.

Also solved (first part) by R. P. Peterson, Jr., and the Proposer.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Plane Trigonometry. By E. B. Mode. New York, Prentice-Hall, Inc., 1947. 10+216 pages. $2.40.

This enlargement and revision of an earlier lithoprinted work contains several features that should make it appeal to students and teachers alike. One of these is the set of three supplementary chapters on Approximate Computation, Logarithms, and The Slide Rule. The author refers frequently to the first two of these chapters and suggests that they be taken up before the main body of the text, and it might have been preferable to place them at the beginning of the book.

In addition to an adequate treatment of the solution of triangles, the author has indicated various other applications of the trigonometric functions. His introductory chapter contains a discussion of angles measured in degrees, radians, and mils, and radian measure is encountered frequently in the following chapters. The trigonometric functions are defined for the general angle and followed by the definitions for the angles of a right triangle. An unusual feature of the book is the use of the concept of even function and odd function in the derivation of several of the properties of the trigonometric functions.

The good features of this book are somewhat counterbalanced by a few undesirable ones. At least twice one finds "sin" at the end of one line and "θ" (or "β") at the beginning of the next. The section entitled "A common error" might better have been omitted. It seems unfortunate to suggest such a possibility and the explanation given is far from convincing.

E. S. SOKOLNIKOFF

Proceedings of the First Canadian Mathematical Congress. The University of Toronto Press, 1946. 44+367 pages. $3.25.

The First Canadian Mathematical Congress was held in Toronto during the week beginning on June 17, 1945. The papers presented at the meetings of this Congress, the program, the list of those who registered, and the Minutes of the General Meeting are published in the *Proceedings.*

The Congress encompassed a wide range of subjects. Four papers were devoted to a *Discussion of Secondary School Mathematics*, five to a *Symposium on Statistics*, four to a *Discussion of Engineering Mathematics*, two to a *Discussion on Research and Graduate Work in Canada* and six to *Short Research Papers.* Of the other papers presented, nineteen in number, thirteen were chiefly expository in nature and dealt with topics in both pure and applied mathematics. The remaining six were less technical in scope and might be characterized as dealing with the role of mathematics in various educational programs.

In view of the wide range of topics covered, and the interesting way in which they are presented, these Proceedings should be of interest to a wide circle of readers.

H. P. EVANS

Intermediate Algebra for Colleges. By E. B. Miller. New York, the Ronald Press Co., 1947. 10+361 pages. $2.50.

In every respect, this is a traditional, elementary algebra text. The recent trend to bring some of the basic concepts from foundations and abstract algebra into the educational picture at an ever earlier stage has been disregarded completely in this text. In the form, as well as in the content of the book, there appears to be no very marked liaison with advanced courses.

The author's aims include a careful and accurate exposition, with no sparing of detail. These aims seem for the most part to have been achieved. Detail, especially, has in fact become a nemesis. For example, the policy of including a multitude of negative hints may serve only to befuddle the student and to increase his already great capacity for inventing wrong procedures. Furthermore, the emphasis on techniques (which is heralded in the preface) has taken the form of cookbook-like sets of rules. This must sacrifice concept for the sake of method, and conceal the fact that algebra is based on a compact set of principles. Is the student then likely to reason his way through a problem? I fear that, on the contrary, he will have been encouraged to seek a pigeonhole in which the author has obligingly stored instructions for the situation at hand.

Certain features of the book might be attractive to some pedagogical tastes. For example, in the interests of plausibility, each formal proof is preceded by the treatment of a special case, and certain topics dealing with negative numbers are accompanied by counterparts from everyday experience, thus serving to shield the student from their frightening abstract character. "Story" problems are given considerable play throughout the book. In one chapter, there is a classified treatment of digit problems, "How old is Mary?" problems, coin problems, lever problems, work problems, and several other standard types.

The chapter on equations and their solution is quite well done.

This book seems to be directed at the student who must be nursed along, who finds logical thinking difficult, and who has little resourcefulness or imagination.

G. F. ROSE

Éléments de Calcul Infinitésimal. By Adrien Grosrey. Paris, Gauthier-Villars, 1945. 192 pages. 280 Fr.

This book is intended for use by young technical students in the cantonal schools of Switzerland rather than for college or university students. It is, nevertheless, on about the same level as the majority of calculus texts used in colleges in the United States. The treatment is much more concise, however, and the exercises are fewer in number and appear only at the ends of the chapters.

Fundamental concepts are treated quite briefly and the main emphasis is on techniques and applications rather than theory. Limits and continuity, for example, are disposed of in the first chapter of ten pages. The definition of a limit given in this chapter is unsatisfactory since, in common with many calculus texts, the "limit of a variable" is defined without reference to either a sequence or a function. The usual theorems on limits are stated without proof.

The second chapter disposes of the derivative, rules for differentiation, and differentiation of the elementary functions—all in fourteen pages. The succeeding chapters cover all the topics usually taken up in courses in elementary calculus plus brief discussions of Fourier series, curvilinear integrals, and the graphical solution of equations.

Many of the demonstrations given in this text should be regarded only as making plausible the theorems which they purport to prove. Perhaps one should not be overly critical of this, however, in view of the purpose of the book and since the author admits freely in the preface that his exposition contains gaps and that some of the demonstrations are lacking in rigor.

H. P. Evans

NEW BOOKS RECEIVED

College Algebra. By F. S. Nowlan. New York, McGraw-Hill Book Co., 1947, 14+371 pages. $3.00.

Intermediate Algebra. By R. S. Underwood, T. R. Nelson, and S. Selby. New York, The Macmillan Company, 1947. 7+283 pages. $2.60.

Five-figure Tables of Trigonometrical Functions. Prepared by H. M. Nautical Almanac Office. London, 1947. $4.00.

CLUBS AND ALLIED ACTIVITIES

Edited by L. F. Ollmann, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

Mathematics Club, Adelphi College

The projects for the last years had two aims. One was to give the mathematics majors insights into the various professions which are open to them. Representatives of the different professions spoke to the club in successive meetings and answered the questions which arose. The second aim was to arrange joint meetings with the mathematics clubs of neighbor colleges and universities. Since Adelphi College lies within the metropolitan area of New York City there are many contacts within easy reach.

The outstanding events of the last academic year were two joint meetings

with the *David Eugene Smith Mathematics Club* of Teachers College of Columbia University. In December the Adelphi *Mathematics Club* went to Teachers College and Professor H. von Baravalle gave his demonstration of moving skew curves with string models. In the spring the Teachers College *Mathematics Club* came to Adelphi and Professor W. D. Reeve spoke on the topic, *The future of mathematics education in America.*

The officers for 1945–46 were: President, Miss Alice Lawson; Secretary-Treasurer, Miss Agnes Cook. The newly elected officers for 1946–47 are: President, Miss Elsie Novy; Secretary-Treasurer, John Brull; Faculty Advisor, Professor H. V. Baravalle.

Pi Mu Epsilon, Ohio State University

During the Winter and Spring Quarters, *Ohio Alpha* chapter sponsored a series of five informal meetings, at which topics intended to be of interest especially to undergraduate mathematics students were discussed. Topics and speakers were:

Infinite sequences, by Professor Tibor Rado
A problem on squares, by W. R. Scott
The isoperimetric problem, by Professor E. J. Mickle
Certain geometric constructions, by Dr. L. H. Miller
Infinity, by Dr. R. L. Swain.

Average attendance at these meetings was approximately 50, most of them freshman and sophomores.

At a special meeting in February, problems comprising the Rasor Scholarship Examination, open to students not past first quarter calculus, were discussed by Professor Marshall Hall, Jr., and by Professor Mickle. Winners were announced, and B. L. Stradley, Vice President of Ohio State University, presented the awards, including a first prize of $50, won by Donald R. Kibbey.

In May the chapter sponsored a competition open to students not higher than third quarter freshman level. A committee consisting of Professor Hall, Mr. Herbert C. Parrish, and Mr. H. M. Fliess devised an examination. At the last meeting of the year, the problems constituting the examination were discussed by Mr. Parrish and Mr. Byron B. Dressler. Professor Hall announced the winners and presented the awards, including a first prize of $35 and a second prize of $15 won by Richard Rubenstein and Donald Ray McElwain, respectively.

At the annual initiation in May, Professor E. L. Pitcher of Lehigh University spoke, his topic being Quadratic Analysis. A total of 32 new members were initiated. At the banquet following the initiation, Professor Pitcher spoke briefly on *The ham sandwich problem.*

Officers for 1946–47 were: Director, William R. Scott; Vice-Director, Landon A. Colquitt; Secretary, D. Ransom Whitney.

Officers elected for 1947–48 are: Director, Ray E. Kidder; Vice-Director, Byron B. Dressler; Secretary, Faye Mozelle Rankin.

Mathematics Club, Case Institute of Technology

Five meetings were held during the year 1946–47; the speakers and their topics are as follows:

The harmonic analyser, by Professor S. W. McCuskey

Prime numbers, by B. Herzog

Numerical methods, by Professor O. E. Brown

Summability, by Professor L. J. Green

From Zeno's paradox to pursuit curves in modern warfare, by Dean Emeritus T. M. Focke.

The President of the Club was B. Herzog and the Faculty Adviser was Professor Max Morris.

Mathematics Club, Oberlin College

Bi-monthly meetings were held in the Physics building, the new location of the Mathematics Department. As had been the custom in former years, tea and cookies were served at the beginning of each regular meeting. Programs consisted of the following speeches:

The Möbius strip, by Ruth Berger

Linkages and straight lines, by Robert Graves

Hyperspace, by Frank Marzocco

Non-Euclidean Geometry, by Peter Manos

Soap bubbles, by Nancy Lowell

Opportunities for careers in mathematics, by Artha Jean Burington

Nomograms, by Margaret Waugh

Postulates of relativity, by Lester Arnold

Diphantine analysis, by Jean Wyre

Continued fractions, by Charlotte Peters

Postulational systems, by Elizabeth MacKay

Derangements, by Rosalind Monastersky.

In addition to the twelve regular meetings, two special meetings were held. Professor and Mrs. R. W. Wagner invited the Club to their home for a Christmas party. Mathematical charades drew dramatic and mathematical ingenuity into peculiar combinations. The annual banquet was open to all interested in mathematics. Professor L. M. Graves of the University of Chicago discussed *The spectra of proof*. Eighty guests attended the banquet.

Officers for 1946–47 were: President, Charlotte Peters; Vice-President, Frank Marzocco; Secretary-treasurer, Ruth Berger; Social Chairman, Artha Jean Burington; Publicity Chairman, Rosalind Monastersky; Faculty Advisor, Professor R. W. Wagner.

Officers elected for 1947–48 are: President, Lester Arnold; Vice-President Ruth Berger; Secretary-treasurer, Mary Wright; Social Chairman, Rosalind Monastersky; Publicity Chairman, Quentin Darmstadt; Faculty Advisor, Professor E. P. Vance.

Mathematics Club, Stanford University

The first postwar meetings of the Stanford *Mathematics Club* was held in February, 1947. It was decided that the Club should meet every three or four weeks, and that, in general, at each meeting an elementary and a more advanced talk should be given. Three meetings were held, at which the following papers were presented:

A theorem of Mertens, by Professor H. M. Bacon

Some problems in aerodynamics, by Professor J. G. Herriot

Application of geometrical optics to the calculus of variations, by Dr. R. Weinstock

Infinite sets, by Burnett Meyer

Geometric maxima and minima, by George Crane.

On May 17, a picnic was held at Searsville Lake with an attendance of about forty. During the year a new constitution was drawn up and adopted.

The officers for the year were: President, Albert Novikoff; Secretary-treasurer, Burnett Meyer; Faculty Adviser, Professor J. G. Herriot.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The Annual Meeting of the National Council of Teachers of Mathematics will be held in Indianapolis, Indiana, on April 2–3, 1948. Headquarters will be at the Hotel Claypool.

THE MATHEMATICS MAGAZINE

The Mathematics Magazine desires to call attention to a series of articles which will begin in its March–April issue. These articles will describe the general content of various courses in mathematics in terms that can be easily followed by the general reader. Articles on the more advanced courses will require of the untrained reader the reading of the preceding articles of the series.

The subscription price of The Mathematics Magazine is $3.00. Subscriptions should be sent to Professor Glenn James, University of California, Los Angeles 24, California.

TENTH INTERNATIONAL CONGRESS OF PHILOSOPHY

The Tenth International Congress of Philosophy will be held at Amsterdam, August 11–18, 1948. Information about the Congress and application forms for

membership may be obtained from Professor Dr. E. W. Beth, Bern Zweerskade 231, Amsterdam Z, Holland.

STATISTICAL SESSIONS AT ALABAMA POLYTECHNIC INSTITUTE

Professor G. W. Snedecor, President of the American Statistical Association and Research Professor of Statistics at Iowa State College, will be Visiting Research Professor of Statistics at Alabama Polytechnic Institute during the Spring Quarter, from March 22 to June 4, 1948. Professor Snedecor will lecture on Statistical Experimental Design and will be available for statistical consultations.

The newly formed Statistical Laboratory at A. P. I. will also offer a course in Survey Sampling during the Spring Quarter to be taught by the Director, Professor T. A. Bancroft. Conferences in applied statistics for research workers in the lower southeastern states are being scheduled during the time of Professor Snedecor's visit.

PERSONAL ITEMS

Dr. Leo A. Aroian of Hunter College has been promoted to an assistant professorship.

Associate Professor H. G. Ayre of Western Illinois State College has been promoted to a professorship and has become Director of the General College Division.

Assistant Professor S. Louise Beasley of Drury College has been appointed to an assistant professorship at Carleton College.

Assistant Professor W. R. Callahan of Northeastern University has been appointed to an assistant professorship of applied mechanics at New York University.

Dr. W. B. Caton of the University of Maine has been appointed to an assistant professorship at Washington State College.

Assistant Professor Francis L. Celauro of Lehigh University has been appointed Assistant Professor of Mathematics at the Newark College of Engineering.

Professor W. W. S. Claytor of Hampton Institute has been appointed to an associate professorship at Howard University.

Dr. J. D. Elder of the University of Michigan has been appointed to an associate professorship at St. Louis University.

Professor William Findlay of McMaster University has retired with the title of Professor Emeritus.

Dr. G. E. Forsythe of Boeing Aircraft Company has been appointed to an assistant professorship in meteorology at the University of California.

Dr. W. C. G. Fraser of Dartmouth College has been appointed to an assistant professorship at Rensselaer Polytechnic Institute.

Dr. H. E. Goheen of the Office of Research and Inventions, Navy Department, has been appointed to an assistant professorship at the University of Delaware.

Assistant Professor W. W. Gutzman of the United States Naval Postgraduate School has been appointed to a professorship at the University of South Dakota.

Assistant Professor Coleman Herpel of Pennsylvania State College, Altoona, Pennsylvania has been promoted to an associate professorship.

Assistant Professor I. M. Hostetter of Oregon State College has been promoted to an associate professorship.

Professor A. E. Johns of McMaster University has been appointed Head of the Department of Mathematics.

Mr. Sidney Kaplan of the Bureau of the Census has joined the Numerical Analysis Section of the Naval Ordnance Laboratory. In addition, he is serving as part-time lecturer in mathematics at the Catholic University of America.

Associate Professor P. E. Lewis of Oklahoma Agricultural and Mechanical College has been appointed to an assistant professorship at North Carolina State College.

Mr. H. C. McKenzie of the University of Wisconsin has been appointed to an assistant professorship at Western State College, Gunnison, Colorado.

Professor H. F. MacNeish of Brooklyn College has been appointed to a visiting professorship at the University of Miami.

Mr. C. J. Maloney has been appointed Chief, Statistics Branch, Camp Detrick, Frederick, Maryland.

Mr. B. L. Miller of Swarthmore College has accepted a position as physicist with the Bartol Research Foundation, Franklin Institute.

Mr. B. L. Moyls of Harvard University has been appointed Lecturer at the University of British Columbia.

Dr. E. N. Nilson of Pratt and Whitney Aircraft has accepted a position as analytical engineer at United Aircraft Corporation, East Hartford, Connecticut.

Assistant Professor J. M. H. Olmsted of the University of Minnesota has been promoted to an associate professorship.

Mr. R. R. Reynolds of Oklahoma Agricultural and Mechanical College has been promoted to an assistant professorship.

Dr. L. D. Rodabaugh of the Bureau of the Census has been appointed to an associate professorship at Southern Illinois University.

Mr. Peter Scherk of the University of Saskatchewan has been promoted to an associate professorship.

Professor Joseph Seidlin of Alfred University has been appointed Dean of the Graduate School.

Associate Professor J. Shibli of Pennsylvania State College has been promoted to a professorship.

Mr. E. J. Specht of the University of Minnesota has been appointed to a professorship at Emmanuel Missionary College.

Assistant Professor E. C. Stopher of State Teachers College, Brockport, New York, has been appointed to an assistant professorship at Miami University.

Mr. H. W. Syer of Boston University has been promoted to an assistant professorship.

Mr. John Todd, who is on leave of absence from Kings College, University of London, has been appointed to the staff of the National Bureau of Standards' Institute of Numerical Analysis.

Mrs. John Todd (Dr. Olga Taussky) is a guest worker at the National Bureau of Standards.

Dr. A. H. Van Tuyl of Stanford University has accepted a position as mathematician with the Naval Ordnance Laboratory, White Oak, Silver Spring, Maryland.

Associate Professor Alexander Weinstein of Carnegie Institute of Technology has accepted appointments as senior associate at the Naval Ordnance Laboratory and professor at the University of Maryland.

Assistant Professor J. J. Wheeler of the University of Kansas has retired with the title of Associate Professor Emeritus.

The following appointments to instructorships are announced:

Bethany College: Miss Catherine E. Moser

University of Chicago: Mr. I. R. Hershner

Hunter College: Mr. M. E. White

Johns Hopkins University: Mr. George Shapiro, Mr. D. R. Waterman

Oregon State College: Miss Ella Mae Sowder

University of Texas: Mr. W. L. Shepherd

Mr. James W. Dappert, a civil engineer of Taylorville, Illinois and a charter member of the Association, died December 10, 1947 at the age of eighty-eight years.

Assistant Professor U. P. Davis of the University of Florida died February 9, 1947.

Professor E. E. DeCou of the University of Oregon died October 15, 1947. He was a charter member of the Association.

Professor J. C. Fitterer of Colorado School of Mines died on March 12, 1947. He was a charter member of the Association.

Professor Emeritus William Gillespie of Princeton University died September 13, 1947, at the age of seventy-six years.

Dr. A. M. Harding, formerly president and professor of mathematics and astronomy of the University of Arkansas, died December 24, 1947.

Emeritus Professor G. H. Hardy of Cambridge University, Cambridge, England died December 1, 1947. He was the winner of the Chauvenet Prize for the period 1929–1931.

Dr. A. N. Whitehead, professor of philosophy emeritus of Harvard University, died December 30, 1947.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTY-FIRST ANNUAL MEETING OF THE ASSOCIATION

The thirty-first annual meeting of the Mathematical Association of America was held at the University of Georgia, Athens, Georgia, on Thursday, January 1, 1948, in conjunction with the annual meeting of the American Mathematical Society. About three hundred and thirty-six persons attended the meetings, including the following one hundred and ninety members of the Association:

LOUISE ADAMS, High Point College
V. W. ADKISSON, University of Arkansas
R. P. AGNEW, Cornell University
G. E. ALBERT, University of Tennessee
C. B. ALLENDOERFER, Haverford College
E. A. BAILEY, La Grange College
N. H. BALL, U. S. Naval Academy
J. C. BARNES, North Georgia College
C. F. BARR, University of Wyoming
D. F. BARROW, University of Georgia
H. M. BEATTY, Ohio State University
E. F. BECKENBACH, U.C.L.A.
W. S. BECKWITH, University of Georgia
E. G. BEGLE, Yale University
T. A. BICKERSTAFF, University of Mississippi
R. C. BLACKWELL, Furman University
R. G. BLAKE, University of Florida
L. M. BLUMENTHAL, University of Missouri
STANLEY BOLKS, Purdue University
M. G. BOYCE, Vanderbilt University
J. W. BRADSHAW, University of Michigan
A. T. BRAUER, University of North Carolina
H. E. BRAY, Rice Institute
FOSTER BROOKS, Kent State University
J. W. BROWN, Clemson College
N. R. BRYAN, University of Maine
R. S. BURINGTON, Bureau of Ordnance, Navy Department
L. P. BURTON, University of Alabama
L. E. BUSH, College of St. Thomas
MARGARET C. BYRNE, St. Joseph's College for Women
S. S. CAIRNS, Syracuse University
IRIS CALLAWAY, University of Georgia
E. A. CAMERON, University of North Carolina
C. C. CAMP, University of Nebraska
VIRGINIA CARLTON, Centenary College
W. B. CARVER, Cornell University
RANDOLPH CHURCH, U. S. Naval Postgraduate School
R. V. CHURCHILL, University of Michigan
F. E. CLARK, Duke University
A. C. COHEN, Jr., University of Georgia
J. B. COLEMAN, Presbyterian College
LENNIE P. COPELAND, Wellesley College
N. A. COURT, University of Oklahoma
W. H. H. COWLES, Pratt Institute
J. C. CURRIE, Louisiana State University
H. B. CURRY, Pennsylvania State College
J. H. CURTISS, National Bureau of Standards
C. H. DENBOW, Naval Postgraduate School
A. H. DIAMOND, Oklahoma A & M
R. D. DONER, Alabama Polytechnic Institute
NELLE C. DOUGLAS, University of South Carolina
H. H. DOWNING, University of Kentucky
W. L. DUREN, Jr., Tulane University
L. A. DYE, The Citadel
E. D. EAVES, University of Tennessee
F. A. FICKEN, University of Tennessee
FLOYD FIELD, Georgia Tech.
L. R. FORD, Illinois Institute of Technology
TOMLINSON FORT, University of Georgia
J. S. FRAME, Michigan State College
GORDON FULLER, Alabama Polytechnic Institute
H. K. FULMER, Georgia Tech
W. A. GAGER, University of Florida
H. M. GEHMAN, University of Buffalo
M. E. GILLIS, Blue Mountain College
J. W. GIVENS, University of Tennessee
MARY A. GOINS, Marshall College
J. S. GOLD, Bucknell University
MICHAEL GOLDBERG, Bureau of Ordnance, Navy Department
M. O. GONZALEZ, University of Alabama
S. T. GORMSEN, University of Florida
W. H. GOTTSCHALK, Institute for Advanced Study
S. H. GOULD, Purdue University
WALTER GRAHAM, Vanderbilt University
DOROTHY GREEN, Huntingdon College

R. E. GREENWOOD, University of Texas
W. C. GRIFFITH, Centenary College
W. S. GUSTIN, Indiana University
D. W. HALL, University of Maryland
E. A. HEDBERG, University of South Carolina
G. A. HEDLUND, University of Virginia
R. E. HENRY, Newark College, Rutgers University
G. W. HESS, Howard College
E. H. C. HILDEBRANDT, Northwestern University
T. H. HILDEBRANDT, University of Michigan
P. R. HILL, JR., University of Georgia
EINAR HILLE, Yale University
C. H. HOLTON, Georgia Tech.
C. W. HOOK, Georgia Tech.
G. B. HUFF, University of Georgia
ELAINE HUNDERTMARK, Florida State University
W. R. HUTCHERSON, Berea College
L. C. HUTCHINSON, Polytechnic Institute of Brooklyn
R. O. HUTCHINSON, Tennessee Polytechnic Institute
E. D. JENKINS, Kent State University
W. L. JOHNSON, Mississippi Southern College
K. R. JONES, NEPA Project, Oak Ridge, Tennessee
L. O. JONES, William Jewell College
H. T. KARNES, Louisiana State University
A. J. KEMPNER, University of Colorado
J. R. KLINE, University of Pennsylvania
F. W. KOKOMOOR, University of Florida
H. L. KRALL, Pennsylvania State
A. E. LAMPEN, Hope College
A. E. LANDRY, Catholic University of America
G. B. LANG, University of Florida
GILLIE A. LAREW, Randolph-Macon Woman's College
C. G. LATIMER, Emory University
H. L. LEE, University of Tennessee
SOLOMON LEFSCHETZ, Princeton University
R. J. LEVIT, University of Georgia
D. C. LEWIS, University of Maryland
F. A. LEWIS, University of Alabama
G. H. LUNDBERG, Vanderbilt University
R. A. LYTLE, University of South Carolina
C. C. MACDUFFEE, University of Wisconsin
E. L. MACKIE, University of North Carolina
A. C. MADDOX, Northwestern State College
J. D. MANCILL, University of Alabama
G. E. MARKLE, University of Detroit
W. A. MARTIN, Georgia Tech
W. T. MARTIN, Massachusetts Institute of Technology
W. L. MASSEY, University of Chattanooga
ELNA B. MCBRIDE, Memphis State College
DOROTHY MCCOY, Belhaven College
S. W. MCINNIS, University of Florida
E. J. MCSHANE, University of Virginia
L. E. MEHLENBACHER, University of Detroit
C. E. MELVILLE, Clark University
B. E. MESERVE, University of Illinois
H. A. MEYER, University of Florida
D. D. MILLER, University of Tennessee
D. C. MOORE, Emory University
T. W. MOORE, U. S. Naval Academy
W. B. MOYE, Georgia Teachers College
SARA L. NELSON, Georgia State College for Women
P. F. NEMENYI, Naval Ordnance Laboratory
C. V. NEWSOM, Oberlin College
MABEL I. NOWLAN, Trailways Bus
E. B. OGDEN, Union College
MORRIS OSTROFSKY, Duquesne University
E. R. OTT, Rutgers University
F. W. OWENS, Pennsylvania State College
Helen B. OWENS, Pennsylvania State College
W. V. PARKER, University of Georgia
I. E. PERLIN, Georgia Tech.
R. B. PLYMALE, Mercer University
W. G. POLLARD, Oak Ridge Institute of Nuclear Studies
G. B. PRICE, University of Kansas
J. F. RANDOLPH, Oberlin College
ELLEN RASOR, University of South Carolina
ADRIENNE S. RAYL, University of Alabama
L. M. REAGAN, University of Wichita
A. W. RECHT, University of Denver
MINA REES, Office of Naval Research
P. K. REES, Louisiana State University
G. E. REVES, The Citadel
P. R. RIDER, Washington University
R. F. RINEHART, Case Inst. of Technology
J. H. ROBERTS, Duke University
H. A. ROBINSON, Agnes Scott College
L. V. ROBINSON, University of South Carolina
W. J. ROBINSON, Centre College (Kentucky)
CAROL S. SCOTT, St. Petersburg Junior College
P. C. SCOTT, East Carolina Teachers College
W. E. SEWELL, U. S. Army
D. C. SHELDON, Clemson College
T. M. SIMPSON, University of Florida
H. L. SMITH, Louisiana State University
E. L. STANLEY, Clemson College
E. P. STARKE, Rutgers University

A. L. STARRETT, Georgia School of Technology
R. P. STEPHENS, University of Georgia
RUTH W. STOKES, Syracuse University
J. L. SYNGE, Carnegie Institute of Technology
OTTO SZÁSZ, University of Cincinnati
J. S. TAYLOR, University of Pittsburgh
H. P. THIELMAN, Iowa State College
J. M. THOMAS, Duke University
JOHN TODD, National Bureau of Standards
H. C. TRIMBLE, Florida State University
HENRY VAN ENGEN, Iowa State Teachers
T. L. WADE, Florida State University
A. D. WALLACE, Tulane University
J. A. WARD, University of Georgia
BETTY R. WEBER, University of South Carolina
K. W. WEGNER, Carleton College
G. T. WHYBURN, University of Virginia
W. L. WILLIAMS, University of South Carolina
R. L. WILSON, University of Tennessee

Rooms for members of the organizations and their families were provided in three of the University dormitories and meals were served in the cafeteria in Snelling Hall.

On Monday, members of the University faculty conducted tours of antebellum homes in Athens. Tea was served in Lyndon Hall on Tuesday afternoon by the ladies of the Department of Mathematics of the University of Georgia. On Tuesday evening a program of piano and violin music was presented in the University Chapel by Miss Despy Karlas and Mr. Robert Harrison, of the University's Department of Music. On Wednesday, the Department of Fine Arts offered an Art Exhibit in the Fine Arts Building, featuring selections from the paintings of Mr. Lamar Dodd and the collection of American paintings donated to the University of Georgia by Mr. Alfred Holbrook.

A dinner for members of the two organizations was held at 8 p.m. on Wednesday in Snelling Hall. Professor Tomlinson Fort, acting as toastmaster, sketched the history of Mathematics at the University of Georgia, and introduced the members of its Department of Mathematics. President Harmon W. Caldwell of the University welcomed the visiting organizations and spoke of the recent expansion of the University. After vocal solos by James Griffith, accompanied by Byron Walker, talks on Applied Mathematics were given by Professor J. L. Synge, Dr. J. H. Curtiss, and Professor H. B. Phillips. An amusing skit, entitled "Applied Mathematics" was presented by the members of the Department of Mathematics and their families. Professor Fort modestly appeared in answer to shouts of "Author! Author!" Group singing led by Professor J. S. Frame ushered in the New Year with the singing of the traditional *Auld Lang Syne.*

At the dinner resolutions were presented by Professor R. P. Agnew and adopted by a rising vote expressing profound appreciation to the members of the University of Georgia for their gracious hospitality.

The sessions of the American Mathematical Society were held from Monday to Wednesday, December 29–31, 1947. The twenty-first Josiah Willard Gibbs Lecture was delivered by Professor P. M. Morse of the Massachusetts Institute of Technology on Monday evening, the title being "Mathematical problems in operations research." On Tuesday afternoon, Professor E. F. Beckenbach, of the University of California at Los Angeles, gave an address by invitation on "Convex functions." Professor T. H. Hildebrandt of the University of Michigan

gave the retiring presidential address on Wednesday morning on the subject: "Integration in abstract spaces."

The Mathematical Association held its sessions on Thursday morning and afternoon in the auditorium of Connor Hall. The program was arranged by the following committee: Walter Leighton, Chairman, Herbert Busemann, and J. A. Cooley.

First Session of the Association

Retiring presidential address: "The scholar in a scientific world," by Professor C. C. MacDuffee, University of Wisconsin.

"On the moduli of the roots of the derivative of a polynomial," by Professor H. E. Bray, The Rice Institute.

"Some applications of topology," by Professor P. A. Smith, Columbia University.

Second Session of the Association

"Opportunities for advanced study offered by the Oak Ridge Institute of Nuclear Studies," by Dr. W. G. Pollard, Oak Ridge Institute of Nuclear Studies.

"The Mathematician in civil service," by Dr. J. H. Curtiss, National Bureau of Standards.

Symposium: "College entrance requirements in mathematics." Moderator: Professor C. V. Newsom, Oberlin College.

(a) "The problem from the point of view of the university teacher," by Professor A. J. Kempner, University of Colorado.

(b) "The problem from the point of view of the secondary school teacher," by Professor Carl N. Shuster, State Teachers College, Trenton, New Jersey.

Meeting of the Board of Governors

The Board met on Wednesday at 2:30 p.m. in Lyndon Hall. Seventeen members of the Board were present. Among the more important items of business transacted were the following.

The one hundred and twenty-two persons listed below, were elected to membership on applications duly certified.

Jane Ammerman, B.S. (Duke Univ.) Instr., Newark Coll. of Rutgers Univ., Newark, N. J.

H. M. Anderson, Ph.M. (Univ. of Wis.) Asst. Prof., Gustavus Adolphus Coll., St. Peter, Minn.

Fay Aufenberg, B.S. in Engg. (Univ. of Mich.) Asst., Univ. of Ill., Navy Pier, Chicago, Ill.

W. G. Bade, B.S. (Calif. Inst. of Tech.) Teaching Asst., Univ. of Calif. at Los Angeles, Los Angeles, Calif.

H. H. Barnett, A.B. (Univ. of Kansas) Asst. Instr., Univ. of Kansas, Lawrence, Kans.

Winifred V. Berglund, M.A. (Northwestern Univ.) Instr., Univ. of Ill., Navy Pier, Chicago, Ill.

Col. W. W. Bessell, Jr., C.E. (Rensselaer Poly. Inst.) Prof., U. S. Military Acad., West Point, N. Y.

Laura Blakeley, MA. (George Peabody Coll.) Teacher, Dayton High School, Dayton, Ky.

C. F. BRUMFIEL, S.M. (Univ. of Chicago) 306 N. Talley St., Muncie, Ind.

J. A. CARPENTER, A.B. (Univ. of North Carolina) Actuarial Clerk, Pilot Life Insurance Co., Greensboro, N. C.

SISTER M. ROSE AGNES CAVANAGH, M.A. (Catholic Univ. of America) Instr., Salve Regina Coll., Newport, R. I.

G. B. CHARLESWORTH, M.A. (Cambridge Univ.) Instr., Hofstra Coll., Hempstead, N. Y.

B. B. CLARK, B.S. (Univ. of New Mexico) Grad. Student, Oberlin Coll., Oberlin, Ohio

J. J. CLARK, M.A. (New York Univ.) Instr., Adelphi Coll., Garden City, N. Y.

MARGARET L. COMSTOCK, B.A. (Marygrove Coll.) Instr., Univ. of Detroit, Detroit, Mich.

V. F. COWLING, Ph.D. (Rice Inst.) Asst. Prof., Lehigh Univ., Bethlehem, Pa.

MARJORIE L. CROFT, M.A. (Loyola Univ.) Instr., Univ. of Ill., Navy Pier, Chicago, Ill.

RUTH L. DAVIDS, B.Sc. (New Jersey Coll. for Women) Instr., Newark Coll. of Rutgers Univ., Newark, N. J.

H. F. DEFRANCESCO, B.E.E. (Univ. of Va.) Grad. Student, Univ. of Virginia, Charlottesville, Va.

MAX EPSELL, B.S. (Coll. of the City of New York) 698 Ashford St., Brooklyn 7, N. Y.

P. L. EVANS, M.S. (Kansas State Coll.) Asst. Prof., Kent State Univ., Kent. Ohio

D. J. EWY, A.B. (Univ. of Calif). Instr., Bethel Coll., North Newton, Kans.

E. E. FLOYD, B.A. in Educ. (Univ. of Ala.) Student, Univ. of Virginia, Charlottesville, Va.

KATHERINE S. FOOTE, M.S. (Louisiana State Univ.) Critic Teacher, Miss. Southern Coll., Hattiesburg, Miss.

J. W. GADDUM, M.A. (Univ. of Mo.) Instr., Univ. of Mo., Columbia, Mo.

E. A. GOODHUE, M.S. (Mo. School of Mines) Asso. Prof., Missouri School of Mines, Rolla, Mo.

DOROTHY E. GREEN, M.A. (George Peabody Coll.) Chm. of Math. Dept., Huntingdon Coll., Montgomery, Ala.

R. L. GREENE, B.E.E. (Clarkson Coll.) Instr., Clarkson Coll., Potsdam, N. Y.

MADELEINE GRENARD, A.M. (Univ. of Nebr.) Instr., Univ. of Ill., Navy Pier, Chicago, Ill.

K. L. HANKERSON, M.S. in Educ. (Univ. of North Dakota) Asst. Prof., Univ. of N. D., Grand Forks, N. D.

DOROTHEA HARDER, M.A. (Univ. of Wash.) Instr., State Coll. of Wash., Pullman, Wash.

H. J. HARRJE, Architecture (Cooper Union) Civil Eng. (Coll. of the City of New York) P.O. Box 746, Jacksonville Beach, Fla.

R. E. HENRY, M.Sc. (New York Univ.) Asso. Prof., Newark Coll. of Rutgers Univ., New, ark, N. J.

ROSE L. HORNACEK, M.A. (Loyola Univ.) Instr., Univ. of Ill., Navy Pier, Chicago, Ill.

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E. H. WANG, Diploma Engineer, Techn. (Univ., Vienna) Instr., Univ. of Cincinnati, Cincinnati, Ohio

MARION L. WEST, Student, Univ. of Tulsa, Tulsa, Okla.

CHRISTINE WESTGATE, M.S. (Univ. of Chicago) Instr., Univ. of N. D., Grafton, N. D.

M. E. WHITE, B.A. (Wesleyan Univ.) Instr., Hunter Coll., New York City, N. Y.

R. M. WHITMORE, M.A. (Univ. of Texas) Asst. Prof., Southwestern Univ., Georgetown, Tex.

G. T. WILLIAMS, Brookhaven National Lab., Upton, N. Y.

C. C. WILSON, B.Ed. (Chicago Teachers Coll.) Instr., Univ. of Ill., Navy Pier, Chicago, Ill.

J. L. ZEMMER, JR., M.S. (Tulane Univ.) Grad. Asst., Univ. of Wis., Madison, Wis.

L. J. ZIMMERMAN, B.A. (Goshen Coll.) Instr., Goshen Coll., Goshen, Ind.

The Secretary reported the death of the following members of the Association:

H. M. ACKLEY, Professor, Western Michigan College. (February 9, 1947)
H. G. AVERS, Chief Mathematician, U. S. Coast and Geodetic Survey. (January 19, 1947)
ETTORE BORTOLOTTI, Professor, University of Bologna, Italy. (February 17, 1947)
J. F. BUTLER, Associate Professor, University of Detroit. (February 18, 1947)
C. C. CARTER, Attorney, Bluffs, Illinois. (May 5, 1947)
J. W. DAPPERT, Civil Engineer, Taylorville, Illinois. (December 10, 1947)
E. E. DECOU, Professor Emeritus, University of Oregon. (October 15, 1947)
G. W. EVANS, Retired, Charlestown High School, Charlestown, Massachusetts. (February, 1947)
B. F. FINKEL, Professor Emeritus, Drury College. (February 5, 1947)
J. C. FITTERER, Professor, Colorado School of Mines. (March 12, 1947)
WILLIAM GILLESPIE, Professor Emeritus, Princeton University. (September 13, 1947)
G. H. HARDY, Professor Emeritus, Cambridge University, England. (December 1, 1947)
CORA B. HENNEL, Professor, Indiana University. (June 26, 1947)
BERNARD MASON, Instructor in Physics, Hofstra College. (August 25, 1947)
W. A. MOODY, Professor Emeritus, Bowdoin College. (February 3, 1947)
WILLIAM ORANGE, Professor, Los Angeles City College. (December 9, 1946)

B. L. REMICK, Professor Emeritus, Kansas State College. (March 18, 1947)
W. C. RUFUS, Professor of Astronomy, University of Michigan. (September 21, 1946)
W. T. SHORT, Professor, Oklahoma Baptist University. (February 19, 1947)
S. A. SINGER, Professor, Capital University. (April 25, 1947)
G. L. WINKELMANN, Reverend, St. John's University, Minnesota. (January 23, 1947)

The Board voted to accept the invitations of Ohio State University to hold the Annual Meeting of 1948 at Columbus, Ohio. It was also voted to hold the Annual Meeting of 1949 in New York City.

On nomination by the Executive Committee, the Board voted to elect W. B. Carver a member of the Finance Committee for the four years, 1948–51. The Board approved the appointment by President Ford of a Nominating Committee for 1948 consisting of W. B. Carver, G. C. Evans, and C. C. MacDuffee, Chairman. President Ford also announced the appointment of E. J. McShane as a member of the Council of the American Association for the Advancement of Science for the two-year term 1948–49.

The retiring Secretary-Treasurer, W. B. Carver, was requested to express to the President of Cornell University the thanks of the Association for having contributed office space to the Association for the past five years.

Arrangements were authorized for the printing and distribution of the Eighth Carus Monograph, "Rings," by N. H. McCoy.

On the nomination of the Editor-in-Chief, C. V. Newsom, the Board elected the following Associate Editors of the MONTHLY for the year 1948:

C. B. ALLENDOERFER	H. P. EVANS	L. F. OLLMANN
E. F. BECKENBACH	HOWARD EVES	R. F. RINEHART
L. M. BLUMENTHAL	G. E. HAY	EDITH R. SCHNECKENBURGER
N. B. CONKWRIGHT	N. H. MCCOY	E. P. STARKE
H. S. M. COXETER	W. T. MARTIN	E. P. VANCE

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting of the Association was held on Thursday at 2:30 p.m., in Connor Hall, with President Ford presiding.

The tellers, H. E. Bray and P. K. Rees, announced the following results of the balloting for officers:

Saunders MacLane, University of Chicago, was elected First Vice-President for the two-years 1948–49.

C. R. Adams, Brown University, and W. L. Ayres, Purdue University, were elected Governors at Large for the three-year term 1948–50.

Announcement was made of the award of the Chauvenet Prize for the years 1944–46 to Professor P. R. Halmos of the Institute for Advanced Study for his paper: "The Foundations of Probability," this MONTHLY, Vol. 51 (1944), pp. 493–510. The Committee on the Chauvenet Prize consisted of R. P. Agnew, chairman, R. W. Barnard, and R. P. Boas, Jr.

On motion of C. G. Latimer and W. L. Duren, the following resloution was unanimously adopted:

Whereas, Professor W. B. Carver has just completed a five-year period as Secretary-Treasurer of the Association, and

Whereas, this was preceded by a two-year term as President and also by a five-year term as Editor-in-Chief of the MONTHLY,

Be it resolved, that the Mathematical Association of America hereby expresses to Professor Carver its profound thanks for the long, devoted, and efficient services which he has so freely given to this Association. His efforts during the difficult periods in which he has held office, have contributed in large measure to the functioning of the Association.

At the conclusion of the symposium on "College entrance requirements in mathematics," a resolution was adopted requesting the Board of Governors to study in cooperation with the National Council of Teachers of Mathematics and with other groups, the problem of the maintenance of standards in the preparation of students for college, including the study of certification of teachers, teacher-training programs, and curricular questions.

H. M. GEHMAN, *Secretary-Treasurer*

MAY MEETING OF THE ILLINOIS SECTION

The twenty-sixth annual meeting of the Illinois Section of the Mathematical Association of America was held at Wheaton College, Wheaton, Illinois, on Friday and Saturday, May 9–10, 1947. Professor C. N. Mills of Normal University presided at all sessions.

There were sixty-seven in attendance including the following thirty-seven members of the Association: D. L. Arenson, H. G. Ayre, J. K. Baumbart, S. F. Bibb, G. M. Bloom, Fanny W. Boyce, B. K. Brown, Laura Christman, E. G. H. Comfort, J. J. Corliss, W. H. Coulter, L. R. Ford, D. M. Friedlen, J. W. Givens, G. D. Gore, E. D. Hellinger, M. R. Hestenes, E. H. C. Hildebrandt, Norbert Kaufman, E. C. Kiefer, R. A. Kliphardt, W. C. Krathwohl, A. T. Lonseth, C. T. McCormick, Karl Menger, C. N. Mills, C. W. Moran, H. E. Nelson, Rufus Oldenberger, Margaret Olmsted, Gordon Pall, H. A. Poppen, W. T. Reid, Haim Reingold, M. Anice Seybold, Malcolm Smith, and L. R. Wilcox.

The officers for 1947–48 were elected as follows: Chairman, J. J. Corliss, Navy Pier, Chicago; Vice-Chairman, W. C. Krathwohl, Illinois Institute of Technology; Secretary, Earl C. Kiefer, Millikin University. The meetings for 1948 are scheduled to be held at the Illinois Institute of Technology, Chicago, on Friday and Saturday, May 8–9, 1948. A resolutions committee composed of L. R. Ford, Clyde McCormick, and Margaret Olmsted was appointed by the Chairman, and reported at the business meeting on Saturday morning.

The following papers were presented:

1. *Some construction problems with quadratic forms*, by Dr. Gordon Pall, Illinois Institute of Technology.

A genus of integral quadratic forms can be characterized by its determinant, index, and the residue of one of its forms modulo $8d$, where d is the determinant and k is the product of the distinct odd primes in the determinant. Conversely, a series of theorems of which the following is one of the

simplest can be proved: Let $n \geqq 1$, d and i be given integers, $(-1)^i d>0$, $0 \leqq i \leqq n$, and let ρ be a form with an integral matrix such that the determinant of ρ is congruent to d (mod $8d^2$), and such that

$$\prod_{p/2d} c_p(\rho) = (-1)^{(i-1)/2},$$

where q is a non-zero integer. Then we can construct an n-ary integral form f of determinant d and index i, satisfying $f \equiv \rho$ (mod $8dkq$) except that if n is even, ρ is the double of a non-classic form, and if d is odd, then the modulus is $4dkq$. As an immediate application we have a completely elementary proof of the theorem of Siegel that if one form represents another congruencially for every modulus, and in the field of reals, then some form in the same genus as the first represents the second form integrally.

2. *Artin's treatment of the gamma function*, by Margaret. S. Matchett, Illinois Institute of Technology, presented by Dr. L. R. Ford.

The Gamma Function can be characterized uniquely by the conditions that $\Gamma(1)=1$, $\Gamma(x+1)=x\Gamma(x)$, and that $\Gamma(x+1)$ is logarithmically convex. The function $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt$ is shown to satisfy these conditions. It is then shown that the conditions cited, and the existence of a function satisfying them, imply that

$$\lim_{n\to\infty} \frac{(n-1)!n^x}{x(x+1)\cdots(x+n-1)}$$

exists, and that any function satisfying the conditions is equal to this expression.

3. *The beginning teacher*, by Dr. Haim Reingold, Illinois Institute of Technology.

Due to the increased registration in American universities there is a large increase in the number of students taking mathematics. This has brought about a shortage of qualified instructors, and a large proportion of college mathematics classes throughout the country is being taught by graduate students, high school teachers, and others. Professor Reingold discussed the problems and difficulties which face these beginning teachers.

4. *Elasticity of demand and supply, a general study*, by Dr. Lewis A. Maverick, Southern Illinois Normal University, introduced by Dr. H. G. Ayre.

(The writer distributed to the audience a mimeographed exposition. This he did not read in full, but spoke from it, indicating the nature and significance of the study.)

There are here gathered together, from sundry sources, contributions to one or another aspect of elasticity. An explanation is given of the significance to economists of this concept. Several simple curves are employed to represent demand and supply; and for each the elasticity is investigated. General (non-economic) functions are examined. Functions of constant elasticity are of particular interest, they appear on log-log paper as straight lines; they may be employed to show "the" elasticity (the average elasticity) between two given points, and may profitably be fitted, by least squares, when more than two points are given. An empirical procedure is suggested for drawing a demand curve of "reasonable" or "not improbable" shape through a single point.

5. *Some aspects of the theory of traffic control*, by Dr. L. R. Wilcox, Illinois Institute of Technology.

The mathematical theory of "progressive" periodic traffic light networks on a rectangular lattice is developed. The concept of a network admitting non-stop travel at some speed in both directions along each path is formulated mathematically. The relation thus defined between (a) the geometric pattern of paths and intersections, (b) the behavior of the control lights and (c) the effective speeds along the various paths, is completely determined. Thus all lattices of intersections are found which admit the existence of suitable settings under (b) for which speeds (c) may be

found. For each such lattice, all settings of the lights admitting speeds (c) are determined, and for each such setting all speeds are found.

The condition on (a) is shown to be the commensurability of certain distances. A necessary condition on (b) is the equality of the periods of all the lights. Necessary and sufficient conditions on (b) and (c) are so formulated that either (b) or (c) may be presented within determined limitations and the other found.

The theory of periodic sets of real numbers which are sums of intervals is the basic mathematical tool, since such sets characterize (b) completely. The relations connecting (b) and (c) result from the solution of certain simultaneous congruences.

6. *The Life of Florian Cajori*, by Dr. H. T. Davis, Northwestern University, introduced by Dr. C. N. Mills.

The speaker was in a class under Dr. Cajori when in college. He gave many highlights of Cajori's teaching, as well as an historic biography.

7. *Special cryptoform*, by Mr. W. H. Coulter, Decatur, Illinois.

Mr. Coulter presented problem E-751 with a number of remarks and observations regarding such problems.

8. *Practical computational methods for the solution of equations*, by Dr. Rufus Oldenberger, Illinois Institute of Technology.

The methods of solving ordinary differential equations with constant coefficients, and the corresponding algebraic equations as taught in the colleges and universities, are too tedious and complicated for the engineer who needs to obtain his solution in a few minutes. The author has found that certain simple techniques involving modified synthetic division processes save a tremendous amount of time over well known methods. A typical involved known method is that of Graeffe's for finding the complex roots of an algebraic equation. If we teach simpler methods, the average engineer, who today does not use differential equations, may be encouraged to employ mathematics beyond the calculus, and will find that he can do his work much more efficiently by combining mathematical analysis with experiment. Teachers of mathematics are largely to blame for industry's failure to make more use of it. Instead of teaching many ways of solving many problems, thus confusing the student, we should emphasize the solution of a few types which are sure to occur in all fields of technical work.

9. *Self dual postulates in projective geometry*, by Dr. Karl Menger, Illinois Institute of Technology.

While the geometry of the projective plane satisfies the principle of duality, the traditional postulates which form its foundation are not self-dual. For example, the duals of the postulates that there are three non-collinear points, and that there are at least three points on each line (i.e., the statements that there are three non-current lines and at least three lines on each point) are not among the postulates because they can be derived from them. The following four self-dual independent propositions are equivalent to the traditional foundation of projective geometry: I. *There is at least one line on every two points, and at least one point on every two lines.* II. *No two points are on two lines.* III. *There are two points p_1, p_2 and two lines l_1, l_2 such that p_i is on l_j if and only if $i=j$.* IV. *There are two points, p, q and two lines l, m such that the intersection of l and m is on the join of p and q.* We obtain all the classical projective planes except Fano's seven-point-plane with three points on each line, if we replace III and IV by III*. III*. *There exist three non-collinear points p_1, p_2, p_3 and three non-concurrent lines l_1, l_2, l_3 such that p_i is on l_j if and only if $i=j$.* The laws of Desargues and Pappus are capable of self-dual formulations. Moreover, one can formulate a self-dual foundation of the geometry of the three-dimensional projective space in terms of points, lines, and planes. (Cf. *Reports of a Mathematical Colloquium*, Notre Dame, issue 8, 1947).

E. C. KIEFER, *Secretary*

MAY MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at the State Teachers College in St. Cloud, Minnesota, on Saturday, May 10, 1947. Two sessions were held in the forenoon, one at luncheon, and one in the afternoon. Sister M. Seraphim Gibbons, Professors C. O. Bemis, W. L. Hard, and K. H. Bracewell presided at the respective sessions.

Eighty persons attended the meeting, including the following twenty-nine members of the Association: K. H. Bracewell, R. W. Brink, L. E. Bush, W. H. Bussey, R. H. Cameron, E. J. Camp, C. S. Carlson, H. D. Colson, Walter Fleming, Gladys Gibbens, Sister M. Seraphim Gibbons, H. W. Godderz, W. L. Hart, D. A. Johnson, G. K. Kalisch, S. L. Mason, Kenneth May, Sister M. Joanne Muggli, M. J. Norris, J. M. H. Olmsted, Sister Claudette Scoblic, L. W. Sheridan, F. C. Smith, Marion Smith, R. C. Staley, F. J. Taylor, Takashi Terami, Ella Throp, and K. W. Wegner.

The following officers were elected for the coming year: Chairman, H. L.Turrittin, University of Minnesota; Secretary, L. E. Bush, College of St. Thomas; Executive committee, E. J. Camp, Macelaster College; Kenneth May, Carleton College; Charles Hatfield, Jr., University of Minn.

A tribute covering the life, work, and character of Professor Dunham Jackson, who died November 6, 1946, was read by Professor Bussey. A tribute covering the life, work, and character of The Reverend Gilbert L. Winkelmann, who died January 23, 1947, was read by Professor Taylor. Announcement was made of the election of Professor L. E. Bush, College of St. Thomas, as Sectional Governor for the term beginning July 1, 1947.

The Executive Committee was instructed to prepare a set of by-laws for the Section, to be presented for approval at the 1948 meeting of the Section. The Committee was further instructed to consider the advisability of enlarging the Section to include parts of the neighboring States, not now included in any Section of the Association.

By invitation of the Executive Committee, Professor Kenneth May delivered an address at the second morning session. The title of his address was *Probabilities of Certain Election Results.*

The speaker discussed the probability of a minority victory in an indirect or representative election. Given n districts, in each of which m votes were cast for one of the other of the two parties according to uniform independent distributions, it is desired to find the probability that one or the other party win a majority of the districts with a minority of the total popular vote. For small values of m and n, classical enumerative methods suffice. For large values of m and n, results can be obtained by utilizing character functions and Bernstein's central limit theorem. For three districts, the probability was shown to increase from $3/32$ for $m=3$, rapidly approaching $1/8$ with increasing m. Other values and formulas were found to complete the borders of a table of the probability as a function of m and n. With increasing m and n, the probability was shown to approach $1/6$ rapidly.

In addition to the lecture by Professor May, the following five papers were presented:

1. *Vector operations as matrices*, by Mr. Chih-yi Wang, University of Minnesota, introduced by Professor John M. H. Olmsted.

This paper was devoted to the representation of the gradient, diverence, and curl operators by means of the matric operators

$$\begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix}, \qquad \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), \qquad \begin{bmatrix} 0 & -\dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} & 0 & -\dfrac{\partial}{\partial x} \\ -\dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} & 0 \end{bmatrix}$$

respectively. It was shown that this representation permits a simple proof of the invariance of these concepts under a transformation from one rectangular coordinate system to another with the same orientation of axes, and their convenience was demonstrated for establishing familiar identities such as those involving the divergence of the curl, and curl of the curl.

2. *Some aspects of systems of partial differential equations*, Sister Joanne Muggli, The College of Benedict, introduced by Sister Claudette Schoblic.

The solution of the system of partial differential equations $D_i u = f_i(x_i, \cdots, x_n)$, where

$$D_i = \frac{\partial^{i_1+i_2+\cdots+i_n}}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}, \qquad (i = 1, 2, \cdots, k)$$

was considered. The monomial m_i was defined by $m_i = x_1^{i_1} \cdots x_n^{i_n}$. By use of the algebra of classes, recursion formulas were obtained to express all the monomials which are multiples of at least one m_i but separated into classes so that there is no overlapping of monomials. These recursion formulas lead to an explicit classification of either principal or parametric derivatives. From the formula for the principal derivatives, the integrability conditions are directly given, and from the formula for the parametric derivatives the arbitrary part of the solution is indicated.

3. *A statistical study of the frequency and intensity of storm centers*, by Professor L. W. Sheridan, The College of St. Thomas.

This was a presentation of the results of a study made as an aid to weather forecasting over regions critical to military operations during World War II. The study was illustrated by slides showing charts marked with the frequency and intensity distributions of centers of low pressure over the area investigated.

4. *Representations of real numbers by sequences of integers*, by Professor M. J. Norris, The College of St. Thomas.

The problem of finding a one-to-one mapping f of the real numbers into the family of infinite sequences of integers, such that given a natural number N and a real number x there exists an $\epsilon > 0$, such that $f(y)$ does not differ from $f(x)$ in the first N positions when y differs from x by less than ϵ was proposed. After imposing the Cartesian product topology on the family of sequences, the question was readily answered in the negative.

5. *Some remarks on Makeham's Law*, by Professor Franklin C. Smith, The College of St. Thomas.

After discussing the general problems of approximating mortality statistics with simple mathe-

matical functions, the author presented three famous approximations, known as Gompertz's law, Makeham's First Law, and Makeham's Second Law. Special attention was given to Makeham's First Law and the importance of this law from a practical point of view was stressed.

6. *Ptolemy's Theorem*, by Professor (Emeritus) W. E. Brooke, University of Minnesota.

A proof was given for the theorem: In any quadrilateral inscribed in a circle, the product of the diagonals is equal to the product of two opposite sides plus the product of the other two opposite sides. The following problem was proposed and solved: To construct the circle through two given points and through the extremities of a diameter of a given circle.

7. *A correlation of elementary methods of solving differential equations*, by Professor Neil Lockwood, Duluth State Teachers College (introduced by the Secretary).

The speaker started with the principle that if any operations, such as taking the derivative and multiplying by a constant, follow the commutative, associative, distributive, and other elementary laws of algebra, then these operations follow all the laws of algebra which depend upon these elementary laws, and the symbols of such operations may be treated as if they were ordinary algebraic quantities. It was shown how, by the methods of operators, a large number of types of solutions of differential equations which heretofore seem to have been treated as unconnected topics or as "lucky guesses" may be developed as a unified whole. Among other things, a simple *a priori* derivation was given for the Euler-d'Alembert solution of linear differential equations with constant coefficients, the method of undetermined coefficients, the method of variation of parameters, and the method of solution by infinite series.

The meeting concluded with a panel discussion on the training of secondary school teachers of mathematics. The members of the panel were Professor Kenneth Wegner, Carleton College, Moderator; Professor C. S. Carlson, St. Olaf College; Professor C. O. Bemis, St. Cloud State Teachers College; Sister M. Seraphim Gibbons, The College of St. Catherine. After brief statements by the members of the panel, the audience engaged in a lively discussion of the problem.

L. E. Bush, *Secretary*

MAY MEETING OF THE UPPER NEW YORK STATE SECTION

The annual meeting of the Upper New York State Section of the Mathematical Association of America was held at the University of Rochester, Rochester, New York, on Saturday, May 10, 1947. Dean W. H. Durfee of Hobart College presided at the morning session, and Professor C. W. Watkeys of the University of Rochester presided at the afternoon session.

About one hundred persons were present, including the following forty-one members of the Association: R. P. Agnew, R. G. Albert, E. B. Allen, H. T. R. Aude, P. R. Bartram, Dorothy L. Bernstein, William Betz, F. J. H. Burkett, S. S. Cairns, I. S. Carroll, W. B. Carver, Rachel Davison, F. F. Decker, E. J. Downie, Walter H. Durfee, C. W. Foard, A. H. Fox, A. S. Gale, H. M. Gehman, B. H. Gere, N. G. Gunderson, May N. Harwood, L. L. Lowenstein, R. R. R. Luckey, Sister Mary Michael (Maloney), L. L. Merrill, Harriet F. Montague, C. W. Munshower, W. V. Nevins, B. C. Patterson, L. R. Polan, Harry Pollard,

M. A. Scheier, Wladimir Seidel, Joseph Seidlin, Ruth W. Stokes, Mary C. Suffa, A. K. Waltz, J. F. Wardwell, C. W. Watkeys, G. M. Wing.

The following officers were elected: Chairman, D. S. Morse, Union College; Vice-Chairman, E. B. Allen, Rensselaer Polytechnic Institute; Secretary, C. W. Munshower, Colgate University. It was agreed to hold the 1948 meeting at Union College, Schenectady, New York, the 1949 meeting at the University of Buffalo, and the 1950 meeting at Syracuse University.

The following papers were presented:

1. *Electronic digital calculators*, By Dr. R. D. O'Neil, Eastman Kodak Co., introduced by Professor Seidel.

The present era, the speaker said, is witnessing a development of automatic calculators which will have a profound effect upon the evolution of mathematics and its applications. These calculators will relieve the mathematician of the burden of arithmetical computation, and will enable him to solve problems which would be impracticable to handle by any other available means. The principal features of the calculators are extreme speed, accuracy, flexibility, and the ability to handle a complete problem automatically. The designers of these computers have presented the mathematician with new responsibilities. He must develop new computing techniques which are better suited to automatic computation, and he must modify his research methods to take better advantage of the help which the computer offers.

2. *Absolutely and completely monotonic functions*, by Professor Harry Pollard, Cornell University.

A function is *absolutely monotonic* on an interval if it and all its derivatives are non-negative there; for example, $f(x)=\sin^{-1}x$, $(0\leqq x\leqq 1)$. Such a function is necessarily analytic in the interval. A function is *completely monotonic* on an interval if $(-1)^k f^{(k)}(x)\geqq 0$ there; if $f(x)$ is completely monotonic on (a, b), then $f(-x)$ is absolutely monotonic on $(-b, -a)$; for example a^{-x}, $a>0$ are completely monotonic on $(0, \infty)$. A function $f(x)$ is completely monotonic on $(0, \infty)$ if, representable in the form $f(x)=\int_0^\infty e^{-xt}d\alpha(t)$, where $\alpha(t)$ is increasing. The connection with the famous moment problem of Hansdorff was discussed. A function is *completely convex* if $(-1)^k f^{(2k)}(x)\geqq 0$ for all k; the Taylor series for such a function necessarily converges for all x, and hence represents an entire function.

3. *Eccentricity and slope*, by Professor Harriet F. Montague, the University of Buffalo.

The author establishes a correspondence between lines of slope m passing through the origin, and families of conics of eccentricity $e=m$ with axes along the coordinate axes and with centers (in the case of the central conics) at the origin, or vertices (in the case of the parabolas) at the origin. Lines of positive slope are associated with families of conics with foci on the x-axis; lines of negative slope are associated with families of conics with foci on the y-axis. The x-axis is associated with a family of circles with the centers at the origin. For the sake of symmetry, the lines of the slope ± 1 are associated with double families of parabolas. A particular member of any family of conics is determined by a pair of points on the associated line at a fixed distance from the origin. Special attention is given to intersections of the conics associated with lines whose slopes are reciprocals.

4. *Familiarity and understanding in mathematics*, by Professor S. S. Cairns, Syracuse University.

Familiarity, rather than understanding, is cultivated by most of our mathematical instruction in schools and colleges. Even the best students generally learn to do problems without knowing why the methods work; for technical skill is easier to teach, easier to grade and more useful in

outside applications than is the underlying theory. In the case of arithmetic, it is necessary to teach technique to children who are too young to appreciate theoretic considerations. The decimal system of notation is regarded by most people as something established by divine decree, rather than as a result of biological accident. With only the most elementary algebraic concepts, one can approach an understanding of arithmetic through a study of the four fundamental operations as they might be developed by creatures with eight or twelve (and so on) fingers instead of ten. One can then proceed to justify these techniques in terms of a general base. Other arithmetical devices, similarly studied, lead to amusing elementary consequences.

5. *The present mathematical situation and next steps in the teaching of mathematics*, by Dr. William Betz, Rochester, N. Y.

The speaker stated that the mathematical scene is still characterized by a confusion of aims, of content, and of organization. The fundamental thesis of the educator is that "academic" mathematics of the usual type is largely "non-functional," and that it does not "meet the needs" of the vast majority of our young people. The real cause of the present breakdown must be located in a widespread educational opportunism which is without an adequate philosophy in dealing with the problems of mass education. Among the most obvious steps needed to correct the present situation are the following: a nation-wide publicity campaign exposing the folly and danger of current policies; a clear and authoritative analysis of the real meaning of "functional competence in mathematics"; the general adoption of a "two-track program" in secondary mathematics; the planning of a continuous curriculum in secondary mathematics, which shall give due attention to understanding, significant application, and mastery.

6. *Panel discussion*: *Some problems relating to mathematics in New York Colleges*, by Professor Wladimir Seidel, the University of Rochester, Chairman; Professor J. M. Synnerdahl, Canisius College; Professor K. E. Bush, Mohawk College; Professor D. E. Kibbey, Syracuse University.

Several problems of the urban college were enumerated by the first speaker. Mr. Bush described several of the courses of instruction in mathematics in the Associated Colleges of Upper New York. He stated that engineering seemed to be the preferred field, and that about three-quarters of the students were studying mathematics. He outlined a number of problems which may arise when the students transfer to other institutions. Dr. Kibbey pointed out that the Associated Colleges and the branch colleges of Syracuse University will feed more students into the advanced courses of the established institutions, and increase the present demand for competent instructors; he predicted enlarged enrollments in such courses as advanced calculus, and raised the question of whether these courses meet the needs of students. Finally, Professor Kibbey discussed the possibility of teaching beginning students some of the fundamental concepts of mathematics to give them some sense of the structure of the science. A general discussion followed.

C. W. MUNSHOWER, *Secretary*

MAY MEETING OF THE INDIANA SECTION

The spring meeting of the Indiana Section of the Mathematical Association of America was held at Purdue University, Lafayette, Indiana on May 16–17, 1947. On Friday evening sixty members and guests attended a dinner in honor of President and Mrs. L. R. Ford. After the dinner Professor Ford gave a lecture entitled *Some Remarkable Theorems About Areas.*

Eighty-one persons attended the meetings, including the following thirty members of the Association: W. L. Ayres, L. G. Black, Stanley Bolks, I. W. Burr, G. E. Carscallen, K. W. Crain, P. D. Edwards, L. R. Ford, E. L. Godfrey,

Michael Golomb, G. H. Graves, W. R. Hardman, C. T. Hazard, H. K. Hughes, Rufus Isaacs, H. F. S. Jonah, M. W. Keller, E. L. Klinger, M. M. Lemme, F. C. Leone, G. T. Miller, P. M. Nastucoff, Ivan Niven, P. M. Pepper, J. C. Polley, C. K. Robbins, M. E. Shanks, L. S. Shively, R. B. Stone, M. S. Webster, K. P. Williams.

Professor G. H. Graves, Chairman, presided at the business meeting. Professor P. M. Pepper, Notre Dame, was elected Secretary-Treasurer to succeed Professor M. W. Keller who resigned after serving for six years. It was decided to hold a fall meeting again on October 17, 1947 at Ball State Teachers College, Muncie, Indiana, in conjunction with the fall meeting of the Indiana Academy of Science.

The following papers were presented:

1. *Geometries and their terminology*, by Sister Gertrude Marie, O.S.F., Marian College.

In this discussion the speaker traced in parallel the evolution and naming of the various types of geometry, and the history and significance of the names. The purpose of the study was to contribute to an evaluation of current geometric terminology on scientific and linguistic grounds.

2. *A five significant figure slide rule for plane and spherical trigonometry*, by Professor P. M. Pepper, University of Notre Dame.

Professor Pepper demonstrated a slide rule for performing the computations of plane and spherical trigonometry with an accuracy comparable to that obtained by using five-place logarithmic tables. Besides performing all the usual operations, the speaker illustrated how the rule could be used as a five-place table of reciprocals of numbers, of natural trigonometric functions, of logarithms of numbers, and of logarithms of the trigonometric functions.

3. *A simple proof that π is irrational*, by Professor Ivan Niven, Purdue University.

The results obtained in this paper were published in *The Bulletin of the American Mathematical Society*, vol. 53, p. 509.

4. *Remarks on the construction of tables of functions*, by R. D. Gordon, Indiana University, introduced by Professor K. P. Williams.

The speaker discussed briefly the principal devices which make possible the constuction of finite tables so as to accommodate "infinitely many" possible calculations. He also indicated how this subject could be introduced in elementary courses.

5. *A congruence on the sums of powers*, by Gordon Overholtzer, Indiana University, introduced by Professor M. W. Keller.

The methods of investigation of the Schur derivate of a sequence were applied to the summation of the kth powers (k being any integer) of the integers from 1 to p^n (p an odd prime) in a single residue class modulo p.

6. *Functional iterates of half-order*, by Professor Rufus Isaacs, University of Notre Dame.

Let g be a functional mapping any set E into itself. Professor Isaacs discussed the existence of a function f of similar type such that for all x in E, one obtains $f(f(x))=g(x)$. The speaker showed

that the existence criterion is that each linkage can be classified either into a matable pair or as self-matable.

7. *Some consequences of Sterling's formula for log* $\Gamma(z)$, by Professor H. K. Hughes, Purdue University.

In this paper, the speaker derived some series developments which he had occasion to use. The function

$$\frac{\Gamma(\alpha z + a)\Gamma(\beta z + b)}{\Gamma(\gamma z + c)\Gamma(\delta z + d)}$$

is typical of those developed. Here α, β, γ, δ are positive and such that $\alpha+\beta=\gamma+\delta$, and a, b, c, d are any real or complex numbers such that $a+b=c+d$. This function was expanded in the form of a factorial series multiplied by an exponential factor. The results obtained follow from Sterling's formula for log $\Gamma(z)$.

M. W. KELLER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirtieth Summer Meeting, Madison, Wisconsin, September 6–7, 1948.

Thirty-second Annual Meeting, Columbus, Ohio, December 31, 1948.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pennsylvania State College, May 8, 1948
ILLINOIS, Illinois Institute of Technology, Chicago, May 14–15, 1948
INDIANA, Purdue University, West Lafayette, May 8, 1948
IOWA, Fairfield, April 16–17, 1948
KANSAS, Atchison, April 10, 1948
KENTUCKY, Berea, May, 1948
LOUISIANA-MISSISSIPPI, Southwestern Louisiana Institute, Lafayette, La., April 23–24, 1948
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, United States Naval Academy, Annapolis, Maryland, May 8, 1948
METROPOLITAN NEW YORK, Washington Irving High School, April 24, 1948
MICHIGAN, University of Michigan, Ann Arbor, April 3, 1948
MINNESOTA, College of St. Thomas, St. Paul, May 8, 1948
MISSOURI, University of Kansas City, Kansas City, April 23, 1948
NEBRASKA, University of Nebraska, Lincoln, May 1, 1948
NORTHERN CALIFORNIA
OHIO, Ohio State University, Columbus, April 3, 1948
OKLAHOMA
PACIFIC NORTHWEST, Eugene, Oregon, March 26–27, 1948
PHILADELPHIA, Philadelphia, Pa., Nov. 27, 1948
ROCKY MOUNTAIN, April 23–24, 1948
SOUTHEASTERN, The Citadel, Charleston, South Carolina, March 19–20, 1948
SOUTHERN CALIFORNIA, Redlands, March 13, 1948
SOUTHWESTERN, New Mexico Highlands University, Las Vegas, New Mexico, May 3–6, 1948
TEXAS, Rice Institute, Houston, April 23–24, 1948
UPPER NEW YORK STATE, Schenectady, May 1, 1948
WISCONSIN, Beloit, May 8, 1948

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CONTENTS

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PROBABILITIES OF CERTAIN ELECTION RESULTS

KENNETH MAY, Carleton College*

1. Introduction. In any indirect electoral system or in the election of any representative body, the results will not in general correspond exactly to the votes of the electorate. In particular, it is possible for a candidate or a party to win a majority of the electors or representatives with a minority of the popular vote. What is the probability of such a result?† The answer depends, of course, upon many factors, including the method of election, the number of districts, voters and parties, and the frequency distributions of election outcomes in the districts.

In this paper we confine ourselves to certain simple cases in order to suggest a tentative answer to the above question. We assume n districts, m voters per district, and two parties. In each district all voters cast votes for one or the other party according to uniform, independent distributions. Ties are excluded by taking m and n odd. We are interested in $P(m, n)$, the probability that the party which wins the majority of the districts gains only a minority of the total vote mn. Expressions are found for $P(m, n)$, $P(m, 3)$, $P(3, n)$, $P(\infty, n)$, and $P(m, \infty)$. The results are summarized in the last section.

2. Preliminary results. Let x_j be the vote of one of the parties in the jth district. The vote of the other party is $m-x_j$. Since the problem is symmetric with respect to the two parties, we need to consider only the x_j. We assume that x_j takes integral values $0 \leqq x_j \leqq m$, with equal probabilities $1/(m+1)$. The smallest x_j to win a district is $\mu=(m+1)/2$. The party's total vote is $X=\sum_1^n x_j$. Let y_j be defined as follows: $y_j=0$ when $x_j<\mu$ and $y_j=1$ when $x_j \geqq \mu$. Then the number of districts won is $Y=\sum_1^n y_j$, and the minimum number of districts for a majority of the districts is $\nu=(n+1)/2$. The greatest X for which $X<mn/2$ is given by $\overline{X}=(mn-1)/2=2\mu\nu-\mu-\nu$. On the other hand, in order to win a majority of the districts it is necessary to have at least μ votes in at least ν districts. Hence the minimum total vote for which it is possible to win a majority of the districts is given by $\underline{X}=\mu\nu$. Thus the probability that the party wins a majority of the districts with a minority of the total vote is the probability of the simultaneous occurrence of $\underline{X} \leqq X \leqq \overline{X}$ and $Y \geqq \nu$. Since the other party has an equal chance of winning in the same way, we have $P(m, n)=2P(\underline{X} \leqq X \leqq \overline{X}, Y \geqq \nu)$.

The problem may be restated in classical terms as follows: Given n urns, in each of which are $m+1$ balls marked $0, 1, 2, \cdots, m-1, m$; when a ball is drawn from each, what is the probability that the sum of the numbers drawn is less than $mn/2$ (or greater than $mn/2$) while at the same time the number of

* Presented in part to the Minnesota Section of the Mathematical Association of America at St. Cloud, Minnesota, May 10, 1947.

† The question was proposed to the author by Professor G. L. Field of the Department of Government, Wayne University. The writer is indebted to Professors J. Neyman and H. Lewy of the University of California for helpful suggestions.

balls drawn with markings greater than $m/2$ is greater than $n/2$ (or less than $n/2$). The problem is also equivalent to finding the number of compositions (partitions where account is taken of the order of the parts) of all numbers between $\underline{X}$ and $\overline{X}$ into n parts greater than or equal to zero and less than or equal to m and such that at least ν parts will be at least equal to μ.

An interesting quantity is the minimum total vote with which a majority of the districts can be won. We have $\underline{X}=\mu\nu=(m+1)(n+1)/4$ and $\underline{X}/mn=\frac{1}{4}+(m+n+1)/4mn$. Evidently the minimum proportion of the total vote is larger than $\frac{1}{4}$ but approaches this value as m and n become large. For values of m and n of the order usually encountered in elections, it is thus possible, although obviously unlikely, that a party may win a majority of the districts with only slightly more than one quarter of the popular vote. The result may be generalized. In order to win more than rn of the districts, a party must get at least $([m/2]+1)$ votes in at least $([rn]+1)$ districts, where the square bracket indicates the largest integer less than or equal to its argument. The minimum proportion of the total vote will be $([m/2]+1)([rn]+1)/mn$, a quantity which is greater than $r/2$ and approaches it with increasing m and n. Hence it is possible to win an r-majority (for example a 2/3 majority) with only slightly more than $r/2$ (for example 1/3) of the popular vote.

3. Three districts. For small values of m and n, the desired probability can be found by classical enumerative methods. Since there are $m+1=2\mu$ equally likely outcomes in each of n independent districts, there are $(2\mu)^n$ equally likely outcomes for the whole election. If we can enumerate $F(m, n)$, the number of favorable cases for one of the parties, the required probability will be given by $P(m, n)=2F(m, n)/(2\mu)^n$. Various systematic arrangements of the favorable cases may be used to evaluate $F(m, n)$. A convenient method is to arrange the favorable outcomes according to the party's total vote, which varies from $\underline{X}$ to $\overline{X}$. If we designate by $F_k(m, n)$ the number of favorable cases for a total vote of $\underline{X}+k$ ($k=0, 1, 2, \cdots, D$ where $D=\overline{X}-\underline{X}=\mu\nu-\mu-\nu$), then $F(m, n) = \sum_{k=0}^{k=D} F_k(m, n)$.

These observations enable us to find a simple formula for $P(m, 3)$. For $n=3$, we have $\nu=2$, $\underline{X}=2\mu$, $\overline{X}=3\mu-2$, and $D=\mu-2$. For a favorable case the party must win just two of the districts. Consider a favorable case in which the party's total vote is $2\mu+k$. We can achieve a favorable distribution of these votes by assigning μ votes to each of two of the districts and then distributing the remaining k votes in any way over the three districts. The two districts can be chosen in 3 ways. The number of ways of distributing k votes to 3 districts is $\binom{2+k}{k}$.* Since k is always less than μ for the favorable cases, none of these distributions can lead either to duplication in counting or to the assignment of more than m votes to a district. Hence

$$F_k(m, 3) = 3\binom{2+k}{k} \quad \text{and} \quad F(m, 3) = 3\sum_{k=0}^{k=\mu-2}\binom{2+k}{k}.$$

* W. A. Whitworth, Choice and Chance, N. Y., 1927, p. 97.

By a well known identity,* the summation reduces to $\binom{\mu+1}{3}$. It follows that

$$P(m, 3) = \frac{2 \cdot 3\binom{\mu+1}{3}}{(2\mu)^3}$$

$$= \frac{\mu^2 - 1}{8\mu^2}. \tag{1}$$

4. The general case. Pleasing arrays result from the arrangement of the favorable cases according to the total vote and the number of districts having μ or more votes. But for all except the smallest values of m and n it is more convenient to use a method originally due to Euler. Consider the generating function $f(z) = (1+z+z^2+z^3+ \cdots +z^{m-1}+z^m)^n$. In the expansion of $f(z)$, before collecting terms, each term arises from a choice of one term from each of the n factors. If we let the choice of z^{x_j} in the jth factor correspond to a vote of x_j in the jth district, there will be a one-to-one correspondence between the election outcomes and the terms of the uncollected expansion. We wish to enumerate those terms in which the exponents of z are less than $\overline{X}$ and which at the same time are obtained by choosing from the n factors at least ν terms whose exponents are at least μ. The generating function may be rewritten as follows:

$$f(z) = \sum_{p=0}^{p=n} \binom{n}{p} \left(\frac{1 - z^\mu}{1 - z}\right)^n z^{p\mu}. \tag{2}$$

This form of $f(z)$ corresponds to an arrangement of the election outcomes according to the number of districts in which the party has a majority. The terms arising from a given p correspond to outcomes in which exactly p of the districts have μ or more votes. We wish to count those terms which arise from $p \geqq \nu$ and whose exponents are less than or equal to $\overline{X}$. In fact,

$$F(m, n) = \sum_{s=\underline{X}}^{s=\overline{X}} C_s,$$

where C_s is defined by

$$\sum_{p=\nu}^{p=n} \binom{n}{p} \left(\frac{1 - z^\mu}{1 - z}\right)^n z^{p\mu} = \sum_{s=\mu\nu}^{s=mn} C_s z^s. \tag{3}$$

In order to evaluate C_s, we consider

$$\left(\frac{1 - z^\mu}{1 - z}\right)^n = \sum_{r=0}^{r=(\mu-1)n} H_r z^r. \tag{4}$$

A comparison of (4) and (3) shows that $C_s = \sum \binom{n}{p} H_{s-p\mu}$, where the summation

---

* $\binom{A+B}{B} = \sum_{s=0}^{s=A} \binom{B+s+1}{s}$. (*Ibid.*, p. 79.)

extends over p satisfying $\nu \leqq p \leqq s/\mu$. Hence

$$F(m, n) = \sum_{s=\mu\nu}^{s=\overline{X}} \sum_{p=\nu}^{p=[s/\mu]} \binom{n}{p} H_{s-p\mu}, \tag{5}$$

where the square bracket indicates the greatest integer contained in the argument. Reversing the order of summation in (5) and introducing new subscripts defined by $p=\nu+i$ and $s-p\mu=r$, we find

$$F(m, n) = \sum_{i=0}^{i=[D/\mu]} \binom{n}{\nu+i} \sum_{r=0}^{r=D-i\mu} H_r. \tag{6}$$

The summation involving H_r may be evaluated by expanding (4) in the form

$$(1-z^\mu)^n(1-z)^{-n} = \left\{ \sum_{j=0}^{j=n} (-1)^j \binom{n}{j} z^{j\mu} \right\} \left\{ \sum_{t=0}^{t=\infty} \binom{n+t-1}{t} z^t \right\}. \tag{7}$$

The desired terms may be obtained by multiplying the first term in the first bracket by the first $D-i\mu$ terms in the second bracket, the second term in the first bracket by the first $D-(i+1)\mu$ terms of the second bracket, and so on, the process stopping with the last term in the first bracket for which $j\mu \leqq D-i\mu$. Hence

$$\sum_{r=0}^{r=D-i\mu} H_r = \sum_{j=0}^{j=[D/\mu]-i} (-1)^j \binom{n}{j} \sum_{t=0}^{t=D-(i+j)\mu} \binom{n+t-1}{t}. \tag{8}$$

The summation over t reduces to $\binom{n+D-(i+j)\mu}{n}$ by means of the identity used in Section 3. With this simplification, (6), (8) and the remarks at the beginning of Section 3 imply that $P(m, n)$ is given by the expression

$$\frac{1}{2^{n-1}\mu^n} \sum_{i=0}^{i=[D/\mu]} \sum_{j=0}^{j=[D/\mu]-i} (-1)^j \binom{n}{j} \binom{n}{\nu+i} \binom{n+D-(i+j)\mu}{n}. \tag{9}$$

It can easily be verified that (9) reduces to (1) for $n=3$. For $m=3$, it reduces to

$$P(3, n) = \frac{1}{2^{2n-1}} \sum_{i=0}^{i=[\nu/2]-1} \sum_{j=0}^{j=\nu-2(i+1)} \binom{n}{\nu+i} \binom{n}{j}. \tag{10}$$

For particular values of m and n, other special formulas may be derived, but none of these results are convenient for large values of the arguments.

5. Large number of voters in each district. Let $u_j = x_j/m$, where x_j is defined as in Section 2. Then $u_j > \frac{1}{2}$ is equivalent to a majority in the jth district, and $\sum_1^n u_j < n/2$ is equivalent to a minority total vote. Let P_k be the probability of a favorable case in which $u_j > \frac{1}{2}$ for $j=1, 2, \cdots, k$, and $u_j < \frac{1}{2}$ for $j > k$. Since we can choose k districts in $\binom{n}{k}$ ways, $\binom{n}{k} P_k$ is the probability of the favor-

able cases in which just k of the districts have a majority. Hence

$$P(m, n) = 2 \sum_{k=\nu}^{k=n-1} \binom{n}{k} P_k, \tag{11}$$

where P_k is the probability that the variables determine a point lying in the region R_k defined by $\sum_1^n u_j < n/2$, $\frac{1}{2} < u_j \leqq 1$ $(j \leqq k)$, $0 \leqq u_j < \frac{1}{2}$ $(j > k)$. As m becomes large, the probability law of the u_j approaches the continuous uniform distribution between zero and one. Hence P_k approaches the volume of the n dimensional region R_k, and $P(\infty, n)$ will be given by (11) with P_k replaced by the volume of R_k.

In order to find the volume of R_k we perform the translation $u_j = \alpha_j + \frac{1}{2}$ $(j \leqq k)$, $u_j = \alpha_j$ $(j > k)$. The equations defining the region become $\sum_1^n \alpha_j < (n-k)/2$, $0 \leqq \alpha_j \leqq \frac{1}{2}$ for all j. Evidently R_k is now the region under the hyperplane with intercepts $(n-k)/2$ and within the hypercube bounded by the axis hyperplanes and the hyperplanes $\alpha_j = \frac{1}{2}$. The planes $\alpha_j = \frac{1}{2}$ divide the region into a number of similar regions. Let V_i $(i = 0, 1, 2 \cdots, n-k-1)$ stand for the volume of the region given by $\sum_1^n \alpha_j < (n-k)/2$, $\alpha_j > \frac{1}{2}$ for $j \leqq i$, and $\alpha_j > 0$ for $j > i$. There are $\binom{n}{i}$ equal regions of this type in each of which *at least* i of the α_j exceed $\frac{1}{2}$. We desire the volume of the region under the plane for which none of the α_j exceeds $\frac{1}{2}$. The situation is exactly analogous to that covered by Whitworth's Proposition XIV.* Hence we may write

$$\begin{aligned} \text{Vol. } R_k = V_0 - nV_1 + \binom{n}{2} V_2 - \cdots + (-1)^i \binom{n}{i} V_i + \cdots \\ \pm \binom{n}{n-k-1} V_{n-k-1}. \end{aligned} \tag{12}$$

To evaluate V_i, we move the origin to the point $\alpha_j = \frac{1}{2}$ for $j \leqq i$ and $\alpha_j = 0$ for $j > i$. The region is now bounded by the axis hyperplanes and the hyperplane with intercepts $(n-k-i)/2$. A simple iterated integration shows that the volumes is given by $(n-k-i)^n/2^n n!$ Substituting this value for V_i in (12), we have

$$\text{Vol. } R_k = \frac{1}{2^n n!} \sum_{i=0}^{i=n-k-1} (-1)^i \binom{n}{i} (n-k-i)^n.\dagger \tag{13}$$

Replacing P_k in the right member of (11) by this expression for Vol. R_k we get a formula for $P(\infty, n)$ which, after the introduction of new indices of summation

* *Ibid.*, p. 73 ff.

† This result may be found analytically by constructing, with the aid of Fourier integrals, a function which takes the value 1 in the region R_k and zero elsewhere. Integration of this function over the whole space yields (13).

defined by $r=n-k-i$ and $s=i$, reduces to

$$P(\infty, n) = \frac{1}{2^{n-1}n!} \sum_{r=1}^{r=\nu-1} \sum_{s=0}^{s=\mu-1-r} (-1)^s \binom{n}{s} \binom{n}{s+r} r^n. \tag{14}$$

6. Large numbers of districts. In order to find $P(m, \infty)$ we apply a central limit theorem to the X and Y of Section 2. We consider the normalized variables

$$\xi_n = \frac{X - \mathcal{E}(X)}{\sigma_X} \quad \text{and} \quad \eta_n = \frac{Y - \mathcal{E}(Y)}{\sigma_Y},$$

where $\mathcal{E}(\ )$ signifies the mathematical expectation and σ stands for the standard deviation. It can easily be shown that R, the correlation coefficient of ξ_n and η_n, is independent of n. In fact it equals the correlation coefficient of x_j and y_j. The definition of R and the probability law of the x_j assumed in Section 2 yield

$$R = \frac{\mu\sqrt{3}}{\sqrt{4\mu^2 - 1}}. \tag{15}$$

From a central limit theorem due to Bernstein* we know that as $n \to \infty$ the joint elementary probability law of ξ_n and η_n tends uniformly to the law

$$\Phi(\xi, \eta) = \frac{1}{2\pi\sqrt{1 - R^2}} e^{-(\xi^2 - 2R\xi\eta + \eta^2)/2(1-R^2)}. \tag{16}$$

Hence the probability of the point (ξ_n, η_n) lying in any region will approach the integral of Φ over the region. But from the probability law assumed for x_j and the definition of y_j it follows that $\mathcal{E}(X) = n\mathcal{E}(x_j) = mn/2$ and $\mathcal{E}(Y) = n\mathcal{E}(y_j) = n/2$. Hence $\xi_n < 0$ is equivalent to a minority total vote, and $\eta_n > 0$ is equivalent to a majority of the districts. Hence $P(m, n)$ is the probability of $\xi_n/\eta_n < 0$, and $P(m, \infty)$ is the integral of Φ over the region $\xi/\eta < 0$. The integration yields

$$P(m, \infty) = \frac{1}{2} - \frac{1}{\pi} \arctan\left(R/\sqrt{1 - R^2}\right). \tag{17}$$

Making use of (15), we have

$$P(m, \infty) = \frac{1}{2} - \frac{1}{\pi} \arctan\left(\mu\sqrt{3}/\sqrt{\mu^2 - 1}\right). \tag{18}$$

7. Summary. In the following table are displayed numerical values of $P(m, n)$ for certain values of μ and ν. The numbers in parentheses indicate the formulas used for the computations.

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* S. N. Bernstein, Theory of Probability, Moscow, 1934 (in Russian), pp. 241 ff., 317 ff. Bernstein gives a simple sufficient condition on the moments which is easily seen to be satisfied in this case. See, alternatively, J. V. Uspensky, Introduction to Mathematical Probability, N. Y., 1937, pp. 318 ff.

The stability of $P(m, n)$ is striking. It appears to be an increasing function of its arguments, but the slow rate of increase suggests that it would be difficult to prove this conjecture. For moderate values of m and n it is sensibly equal to $1/6$. It is hard to say how more realistic hypotheses would affect the results.

$\mu \backslash \nu$	2	3	4	5	$\cdots$	ν	$\cdots$	∞
2	3/32	.117	.126	.131	$\cdots$	(10)	$\cdots$	.148
3	.111	.132	.140	.145				.159
4	.117	.137	.145	.149				.162
5	.120	.139	.147	.151				.164
$\vdots$	$\vdots$					$\vdots$		$\vdots$
μ	(1)				$\cdots$	(9)	$\cdots$	(18)
$\vdots$	$\vdots$					$\vdots$		$\vdots$
∞	1/8	.143	.150	.154	$\cdots$	(14)	$\cdots$	1/6

Dropping the assumption of independence should make the probability smaller, but the introduction of a more centrally concentrated probability distribution would make it larger, since unfair outcomes are more likely for nearly even elections. On the basis of the simple case here considered, the best we can do is to suggest that for the values of m and n usually encountered the probability is of the order of $1/6$.

THE QUADRILATERALS OF PASCAL'S HEXAGRAM

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1. Introduction. Pascal's hexagram has inspired more interest and investigation than almost any other geometric configuration. Many intriguing figures have been found and brought from their hiding places in the background of this hexagram. Pascal's famous theorem states that if a hexagon is inscribed in a conic the three pairs of opposite sides meet in three points on a straight line. This theorem was first published by Pascal on a little placard in 1640 when he was sixteen years old [1]. This line is known as Pascal's line, and we shall call the points on the line Pascal points.

Steiner was the first to direct attention to the complete hexagram formed by joining the six points on a conic in every way. In this there are sixty Pascal lines passing by four's through each of the forty-five Pascal points. Steiner also showed that the sixty Pascal lines pass by three's through twenty g points which may be divided into ten pairs such that the members of each pair are harmonic conjugates with respect to the conic, and these g points lie four by four on fifteen lines [2].

Kirkman extended Steiner's theorem to show that the Pascal lines intersect three by three, not only in Steiner's twenty g points but also in sixty other h points [3]. Cayley and Salmon showed that there are twenty G lines, each of which pass through one g and three h points, and that these G lines pass four by four through fifteen i points. Various proofs and demonstrations of these theorems on Pascal's hexagram as well as additional properties have been discussed by Salmon [4], Cayley [5], Veronese [6], Cremona [7], and Christine Ladd [8].

Brianchon developed the dual of Pascal's theorem by the method of polar reciprocals. His theorem states that the three lines joining the three pairs of opposite vertices of a hexagon circumscribed about a conic meet in a point. This point is the pole of a Pascal line and is known as a Brianchon point. The three lines are the polars of three Pascal points on a line and we shall call them Brianchon lines. The complete figure connected with this theorem contains sixty Brianchon points and forty-five Brianchon lines. Ann and Elizabeth Linton made some remarkable drawings of the results mentioned above [9], and the author made a study of the hexagram with certain elements in motion, and used the machinery thus set up to make certain straight line constructions on unicursal cubics [10].

The purpose of this paper is to show that the 60 Pascal lines form 15 distinct quadrilaterals whose vertices are Pascal points and whose diagonals are the 45 Brianchon lines. We shall also show the relationship of these quadrilaterals to the results obtained by others.

2. Notation. We shall designate the six vertices of the hexagon by the digits 1, 2, 3, 4, 5, and 6. The notation, 12, will mean the line joining points 1 and 2,

and 12,45 will be the intersection of lines 12 and 45. Any such combination representing the intersection of nonconsecutive sides will be a Pascal point. If we choose any permutation of the six digits, say, 354126, then pair the first and second with the fourth and fifth positions, 35,12, and the second and third with the fifth and sixth, 54,26, and the third and fourth with the sixth and first positions, 41,63, we have three points on a Pascal line. We shall, therefore, designate a Pascal line by the permutation which locates the three points on that line.

3. The main theorems.

THEOREM 1. *The sixty Pascal lines are divided into fifteen sets of four lines, such that each set forms a quadrilateral whose six vertices are Pascal points, and the quadrilaterals are distinct in the sense that no Pascal line is used to form the side of more than one quadrilateral.*

Proof. Consider the Pascal lines, (a) 123456, (b) 213546, (c) 132465, and (d) 624351. We find that the six points 12,45; 23,56; 34,61; 13,46; 35,62; and 24, 51 are the only points on all four lines. Also two and only two of these lines meet in each point. Hence the four lines form a quadrilateral whose vertices are the given points. We shall call this quad. A.

This accounts for two Pascal lines through each point, but there are four. We need now to investigate the other two lines through any vertex, and we choose 12,45. These lines are (e) 123546 and (f) 213456. The two lines which form a quadrilateral with these are (g) 132564 and (h) 625341. We shall call this quad. B. If we investigate quad. B in the same manner as we did quad. A, we find that these two quadrilaterals have only one vertex in common. The similarity of the hexagram now shows that the six vertices of a quadrilateral serve, one each, as the vertices of six other separate quadrilaterals. Each Pascal line is therefore used in only one quadrilateral.

We shall call this group of fifteen quadrilaterals the α collection. The six quadrilaterals which have one vertex each in common with quad. A, we shall call the α_1 collection for quad. A. We shall call the eight which have no common vertices with quad. A the α_2 collection for quad. A. We shall also define a closed set of quadrilaterals to mean a set of three quadrilaterals, any two of which have only one common vertex, and all three of which have a pair of opposite vertices confined to three points.

THEOREM 2. *Any quadrilateral of the α collection forms three closed sets with the quadrilaterals of its α_1 collection, and the α collection contains fifteen closed sets.*

Proof. The four lines (i) 314625, (j) 341652, (k) 651342, and (l) 615324 form a quadrilateral which we shall call quad. C. Quads. A and B have the common point 12,45. Quads. B and C have the common point 14,25, and A and C have 15,24 in common. The first two of these points do not lie on the same Pascal line, hence they are opposite vertices of quad. B. Likewise the second and third

are opposite for quad. C and the first and third for quad. A. Therefore, quads. A, B, and C form a closed set. We may notice that the three points that determine a closed set are formed by the same four digits, and that any two are formed from the third by permuting pairs of digits across the comma. The number of combinations of six things used four at a time is fifteen, which is the number of closed sets possible in the α collection. Since each quadrilateral has three pairs of opposite vertices it is a member of three closed sets.

THEOREM 3. *The 45 diagonals of the 15 quadrilaterals of the α collection are the 45 Brianchon lines which are the duals of the 45 Pascal points.*

Proof. The proof of this theorem follows almost immediately from the theorem from projective geometry on the inscribed quadrilateral. If we choose the quadrilateral whose vertices are the points 1, 2, 4, and 5, then the pairs of tangents through 2 and 5, and 1 and 4 meet on a line through 15,24 and 12,45. But this line is both a Brianchon line and a diagonal of quad. A.

Now we shall define an open set of quadrilaterals to be a set of three, no two of which have common vertices, but the nine diagonals of the set must meet by three's in three points such that one diagonal from each quadrilateral passes through each point.

THEOREM 4. *Each quadrilateral of the α collection is a member of four open sets and there are twenty such sets in the α collection.*

Proof. We shall introduce a notation here that we shall call a dual notation. In this notation the six digits represent the six tangents to the conic at the respective vertices of the hexagon. Then the notation, 123456, gives the three Brianchon lines through a point; these lines are the duals of the three Pascal points on a line determined by the same notation. Choose now a pair of opposite vertices on quad. A, such as 12,45 and 24,51. If we apply the theorem on the inscribed quadrilateral again, we find the diagonal joining these points to be, in dual notation, 25,41. This is exactly the same as the standard notation for the third Pascal point which, with the two above, determines a closed set of quadrilaterals. By reversing this scheme we may find a pair of opposite vertices on a quadrilateral from a given diagonal. Then by writing the Pascal lines through these, and selecting the ones that have common Pascal points, we can find a quadrilateral from a given diagonal.

Starting in dual notation with the diagonal of quad. A, that is, 52,41, we build the permutation 143256 which gives three diagonals that meet in a point. Each diagonal is from a different quadrilateral. The first, 14,25, is from quad. A, the second, 43,56, is from one we shall call quad. D, and 32,61 is from quad. E. In the following table the first three rows give these three quadrilaterals in standard notation. Rows 4, 5, and 6 give their vertices with their opposite vertices paired in parentheses. Rows 7, 8, and 9 give their diagonals in dual notation with each set of diagonals directly below the vertices they connect. Row 10 gives the permutations which show that these diagonals meet in three points.

TABLE I

1.	Quad. A.	123456, 213546,	132465, 624351.	
2.	Quad. D.	451362, 541632,	461352, 641532.	
3.	Quad. E.	314265, 134625,	364215, 634125.	
4.	Quad. A.	(12, 45 − 24, 51),	(23, 56 − 35, 62),	(34, 61 − 13, 46).
5.	Quad. D.	(45, 36 − 64, 53),	(24, 13 − 41, 32),	(16, 25 − 15, 26).
6.	Quad. E.	(31, 26 − 63, 21),	(15, 46 − 14, 56),	(24, 53 − 25, 34).
7.	Quad. A.	14, 25,	36, 25,	14, 36.
8.	Quad. D.	43, 56,	43, 12,	56, 12.
9.	Quad. E.	32, 61,	54, 16,	45, 32.
10.		143256,	634521,	145632.

A comparison of the vertices in rows 4, 5, and 6 of the table shows that quads. A, D, and E have no common vertices. And since their diagonals meet in three points such that one diagonal from each quadrilateral passes through each of the three points, they form an open set. Since the points where the diagonals meet are the Brianchon points, four of these points lie on each diagonal and each quadrilateral is a member of four open sets. Quads. D and E are members of the α_2 collection for quad. A; hence we may state that the α_2 collection is divided into four pairs and that each pair forms an open set with quad. A. Since there are sixty Brianchon points and three are used in each open set, there are twenty open sets in the α collection of quadrilaterals.

We shall now show a correspondence between these sets of quadrilaterals and the results obtained by Steiner, Kirkman, Cayley, and Salmon. Salmon used the notation,

$$\begin{cases} 12.45.36 \\ 65.23.14, \\ 34.16.25 \end{cases}$$

to represent the three Pascal lines, 123654, 163452, and 143256, which intersect in a Steiner g point [4]. We notice that each row and each double column contains all six digits in such a way that each two rows represent a Pascal line. The dual notation for the three Brianchon points in Table I, row 10, is the same as these. Hence the nine diagonals of the quadrilaterals in an open set intersect by three's in three points on a straight line, and this line is the polar of a Steiner g point.

The notation for a Kirkman h point is

$$\begin{cases} 41.35.62 \\ 52.64.31. \\ 63.51.24 \end{cases}$$

Here each row and the first column only contain all six digits without repetition, and the three Pascal lines are 146253, 251364, and 415362. The three h points on a Salmon-Cayley G line are the ones just given and

$$\begin{cases} 65.13.42 \\ 34.26.15 \\ 21.35.46 \end{cases} \text{and} \begin{cases} 16.42.53 \\ 23.51.46. \\ 54.26.31 \end{cases}$$

The notation for the Steiner point on this line is set up by using the first column from each of these. We must also notice that the lines 14,52 and so forth, which make up the first columns, are not duplicated in the notation for any of the h points. Six of the fifteen lines joining the six points on a conic are used in forming any Pascal line. Three Pascal lines are formed without the use of these six. The three thus formed meet in an h point which is said to correspond to the Pascal line formed by the first six.

In the following table the capital letters represent the quadrilaterals of the α_2 collection for quad. A. The first column of six digits is composed of the Pascal lines which make up those quadrilaterals. The second column contains the h points which correspond to the Pascal lines. The small letters with subscripts are used to identify Pascal lines and rows in the notation for a Kirkman h point.

TABLE II

A	i_2	123456	41.35.62 — a_1	A	g_1	132465	41.62.35 — a_3
	h_3		52.64.31 — b_1		g_2		63.45.12 — b_3
	g_4		63.51.24 — c_1		i_4		52.16.43 — c_3
	i_1	213546	52.34.61 — a_2		h_1	624351	36.45.12 — a_4
	g_3		41.65.23 — b_2		h_2		52.13.46 — b_4
	h_4		36.42.15 — c_2		i_3		14.56.23 — c_4
D	f_1 —	451362	43.61.52	E	a_1 —	314265	32.64.15
			65.32.41				61.25.34
			12.46.35				45.36.21
		541632	65.13.24 — g_1		b_1 —	134625	16.24.35
			34.26.15 — h_1				23.65.14
			21.35.46 — i_1				45.12.63
	d_1 —	461352	43.51.62		c_1 —	364215	32.14.65
			56.32.41				16.25.34
			12.45.36				45.31.26
	e_1 —	641532	65.31.42			634125	23.51.46 — e_1
			34.52.61				16.42.35 — d_1
			12.63.54				45.13.62 — f_1

A study of Table II shows a certain correspondence. The digits in the rows a_1, b_1, and c_1 of the h point correspond to Pascal 123456 in quad. A, and they have the same permutations, respectively, as the Pascal lines a_1, b_1, and c_1 in quad. E. The fourth line in quad. E has no such correspondence, but its corresponding h point has the same relationship to Pascal lines d_1, e_1, and f_1 of quad. D. Again, the h point corresponding to the fourth line of quad. D has rows, g_1, h_1, i_1, whose permutations are the same as those of lines g_1, h_1, and i_1 of quad. A. This open set of quadrilaterals forms a complete cycle of these correspondences. The three h points thus chosen lie on the same G line, and the g point formed by their first columns lies on the same line. Also the nine diagonals

of this open set are concurrent by three's on the polar reciprocal of this same g point. We also find that one h point from each quadrilateral of this open set lies on the same Pascal line through this g point, and there are three such groupings in the set. Hence the twelve h points corresponding to the twelve Pascal lines in an open set lie by three's on four lines through a g point. We have thus set up a direct correspondence between an open set of quadrilaterals and a G line and also a g point.

By making a table similar to Table II for the entire α_2 collection, we find that quad. A holds the same relationship to each of the four open sets in its α_2 collection. The four G lines corresponding to these four sets meet in an i point. The four g points, however, do not lie on an I line. (Salmon's proof is sufficient [4].) Now if we examine the twelve Pascal lines that make up the open set A, D, E, we find that the three that are not identified in the Table by a letter with subscript, 1, meet in a g point. These are the three that correspond to three h points on a G line. We also find the other nine lines divided into three collections of three each, such that lines i_1, f_1, and c_1 meet in an h point as do the collections g_1, e_1, a_1, and h_1, b_1, d_1. These four points determine a G line. The corresponding collections of lines determined by the other open sets, identified by subscripts 2, 3, and 4, respectively, each determine a G line in the same way. The four g points so determined lie on an I line but the four G lines do not pass through an i point.

We may now state that the four open sets of an α_2 collection for any quadrilateral determine four g points on an I line and correspond to four G lines through an i point. There are fifteen quadrilaterals, fifteen α_2 collections, fifteen I lines, and fifteen i points. Again, since any open set of three quadrilaterals is a member of three α_2 collections and each collection determines a different I line and corresponds to a different i point, while the set in each case determines the same g point and corresponds to the same G line, there are three I lines through each g point and three i points on each G line. It might be of interest to add that the g point determined by an open set and the one corresponding to an open set are harmonic conjugates with respect to the conic.

In order to make a further study of the α collection we shall designate a quadrilateral by a letter and by the dual notation for one of its diagonals. For the members of the α_2 collection we shall choose the diagonal that passes through one of the four Brianchon points on diagonal 14,25 of quad. A. The α_2 collection, paired as suggested above then becomes: ($D-43,56$; $E-32,61$), ($F-13,56$; $G-32,64$), ($D_1-35,64$; $E_1-13,26$), and ($F_1-35,61$; $G_1-34,26$). We now notice a different grouping of four pairs. The notations for quads. D and D_1 are formed by the same four digits; hence they have a common vertex and the diagonals here represented both come from that vertex. The same is true of the other pairs, E, E_1 and F, F_1 and G, G_1.

If we consider the diagonals 15,42 and 12,45 of quads. B and C, and the Brianchon lines that meet in four points on each of them, we find that the pairs

D, E_1 and D_1, E meet on 15, 42. Hence, the four lines D, E_1, D_1, and E form a quadrilateral, two nonconsecutive vertices of which are Pascal points; two are Brianchon points on 14,25; and two are Brianchon points on 15,42. Similarly we may show that F, G, F_1, and G_1 form a quadrilateral with two vertices in Pascal points, two in Brianchon points on 14,52 and two on 12,54. The other four Brianchon points, two on 15,42 and two on 12,54, also serve as vertices for a similar quadrilateral and the other two vertices are Pascal points. The six Pascal points used in these three quadrilaterals all lie on line 36. Line $\overline{36}$ is the dual of point 36 and the three quadrilaterals above with their properties may be obtained by dualizing the closed set of quads. A, B, and C, and by employing Theorems 5 and 6.

THEOREM 5. *The three diagonals, one from each quadrilateral of a closed set, which passes through the three points which are each common vertices of two members of the set, form a triangle in which each vertex is the pole of the opposite side with respect to the given conic.*

Proof. The vertices of the triangle formed by the diagonals of quads. A, B, and C are the Pascal points 12,45, 24,51, and 25,41. The dual notation for any one of these points gives the Brianchon line which passes through the other two. This follows from the theorem on the inscribed quadrilateral.

THEOREM 6. *The six diagonals, two from each quadrilateral of a closed set, which do not pass through a vertex common to two members of the set, are concurrent.*

Proof. In dual notation, the diagonals of quad. A which are determined by vertices which are not common to a vertex of quads. B or C are 25,36 and 14,36. Likewise those of quad. B, determined by vertices which are not common to a vertex of quads. A or C, are 15,36 and 24,36, and those of quad. C are 12,36 and 45,36. We now notice that these diagonals all pass through the point 36.

The diagonals of quad. A are 14,25, 14,36, and 36,25. If one of these is given we may separate the notation for it at the comma, and use the two digits not used in its notation with each half to form the notation for the other two. Thus if the given diagonal is 14,25, the other two must contain the point 36 in their notations. Since the notation for the diagonals of Theorem 5 are formed by the same four digits, the notation for the other six diagonals in the set must each contain the other two digits in such a way as to represent a point.

This theorem sets up a one to one correspondence between the fifteen closed sets and the fifteen points 36 and 12, and so on. We may therefore designate a closed set by the notation for the point which corresponds to it.

The Kirkman h point,

$$\begin{cases} 15.34.26 \\ 42.56.13, \\ 36.14.25 \end{cases}$$

corresponding to the Pascal line 123456 of quad. A lies on the Pascal line 314625

of quad. C, and, conversely, the h point,

$$\begin{cases} 36.24.15 \\ 12.65.24, \\ 45.32.16 \end{cases}$$

corresponding to the last Pascal line lies on the first. The twelve Pascal lines that make up a closed set of quadrilaterals may be divided into six pairs so related. Since quad. A is a member of three closed sets, each line in quad. A is paired with three other lines, one in each set. It follows that these three lines correspond to the three h points on the given line of quad. A. Conversely, since one Pascal line corresponds to only one h point, the three Pascal lines, one from each set, must pass through that h point. The four lines forming quad. A correspond to four h points, and the twelve Pascal lines concurrent by three's in these are chosen by two's from each of the six quadrilaterals of the α_1 collection for quad. A.

The results that are new in this paper are:

1. The division of the 60 Pascal lines into distinct collections of four each.
2. The relationship between these collections, or quadrilaterals.
3. The correspondence between these quadrilaterals and the results obtained by others.

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NOTES ON COSPHERICAL POINTS

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1. Introduction. Some of the geometrical elements, and some of their related properties, that have been associated with the tetrahedron [1] that is, with four points in space, are here associated with n cospherical points. The analogies derived are so obvious that, to save space, they will not be pointed out.

2. Definitions. Consider n points A_i $(i=1, 2, \cdots, n)$ on a sphere (O), center O, radius R. Let G be the centroid of the n given points and k^2 the sum of the squares of the $n(n-1)/2$ segments determined by the n points A_i. The line OG may be called the *Euler line* of the given points A_i.

The centroids of the n given points taken $(n-1)$ at a time form a new set of n points which may be called the *medial set of points* associated with the n points A_i. The line A_kG_k joining the point A_k to the centroid G_k of the remaining $(n-1)$ given points may be called a *median* of the given points A_i. The given points A_i have n medians A_kG_k.

3. The medial sphere. The points G, A_k, G_k are collinear and we have $GG_k:GA_k=-1:(n-1)$. Thus the set of medial points G_k, $k=1, 2, \cdots, n$, associated with the points A_i correspond to the points A_i in a homothecy $[G, -(n-1)]$ having G for center and $-(n-1)$ for homothetic ratio. Since the points A_i are cospherical, by assumption, we have: *The medial set of points associated with the n given cospherical points A_i lie on a sphere.*

This sphere may be called the *medial sphere* of the given points. It will be denoted by (L).

The point G is the internal center of similitude of the spheres (O), (L). The center L of (L) lies on the Euler line OG and we have

$$GO:GL = -(n-1).$$

The radius of (L) is equal to $R/(n-1)$.

The harmonic conjugate M of G for O and L may be called the *Monge point* of the points A_i; this is the external center of similitude of the spheres (O), (L), and we have

$$ML:MO = 1:(n-1),$$

hence

$$(n-2)MG = 2GO.$$

4. The (Q)-sphere of the given points A_i. The sphere (Q) having M for center and coaxal with (O) and (L), that is, the external sphere of antisimilitude of the latter two spheres will be referred to as the *(Q)-sphere* of the given points A_i. The square of its radius will be denoted by q.

The two spheres (O), (L) are inverse with respect to (Q), hence q^2 is equal to

the product of the powers of M for these two spheres, so that

$$q^2 = (MO^2 - R^2)\frac{ML^2 - R^2}{(n-1)^2}$$
$$= (MO^2 - R^2)\frac{(n-1)^2ML^2 - R^2}{(n-1)^2};$$

hence

$$q = \frac{MO^2 - R^2}{n-1}.$$

Thus: *The square of the radius of the sphere (Q) of the n given points A_i is equal to $1/(n-1)$ part of the power of the Monge point with respect to the circumsphere of the given points.*

5. The (G)-sphere of the points A_i. The sphere (G) having G for center and coaxial with the spheres (O), (L), that is, the internal sphere of antisimilitude of the latter two spheres will be referred to as the *(G)-sphere* of the points A_i. Let g be the square of the radius of (G).

By a calculation analogous to the one which yielded the value of q we obtain

$$g = -\frac{(GO^2 - R^2)}{n-1} = -\frac{g_0}{n-1},$$

where g_0 denotes the power of G for the sphere (O). Now [2]

$$g_0 = -\frac{k^2}{n^2};$$

hence

$$g = \frac{k^2}{n^2(n-1)}.$$

6. The sum of the powers of the given points. Let N_i be the foot of the perpendicular from the point A_i upon the common radical plane σ of the four spheres (O), (L), (G), (Q). The power r_i of A_i for the sphere (Q) is given by the relation [3],

$$r_i = 2N_iA_i\cdot MO;$$

hence

$$\sum r_i = 2MO\cdot\sum N_iA_i.$$

Now if N is the foot of the perpendicular from the centroid G upon the radical

plane σ, we have

$$\sum N_iA_i = n\cdot NG;$$

hence

$$\sum r_i = 2MO\cdot n\cdot NG.$$

The two spheres (G), (Q) being orthogonal, the power of G for the sphere (Q) is equal to g; hence

$$g - g_0 = 2NG\cdot MO.$$

Therefore

$$\sum r_i = n(g - g_0) = - n^2g_0/(n - 1) = k^2/(n - 1).$$

Thus: *The sum of the powers of n cospherical points with respect to their (Q)-sphere is equal to $1/(n-1)$ part of the sum of the squares of the $n(n-1)/2$ segments determined by those points.*

7. Spheres orthogonal to a fixed sphere.

THEOREM. *The spheres having for diameters the medians of n cospherical points are orthogonal to a fixed sphere, namely, the (Q)-sphere of the n given points.*

The points A_i, G_i are homologous on the two spheres (O), (L) with respect to the internal center of similitude G, hence the point G_i and the diametric opposite C of A_i on (O) are homologous points on (L) and (O) with respect to the external center of similitude M. Thus the line MG_iC meets (O) in the antihomologous point D of G_i with respect to M, that is, the points D and G_i are inverse points with respect to the sphere of antisimilitude (Q) of (O) and (L).

The line DA_i is perpendicular to CDM, hence DA_i lies in the polar plane of G_i for the sphere (Q), that is, the points A_i, G_i are conjugate with respect to (Q), and therefore the sphere having A_iG_i for diameter is orthogonal to the sphere (Q).

8. Systems of cospherical circles. From the preceding proposition we derive the following corollaries.

THEOREM 1. *The spheres having for centers n given cospherical points and orthogonal to the (Q)-sphere of these points cut the spheres having for diameters the respective medians along n circles lying on the same sphere, namely, the (G)-sphere of the given points.*

Indeed, let (A_i) be the sphere having A_i for center and orthogonal to (Q). The spheres (A_i), (G), (A_iG_i) are orthogonal to the same sphere (Q) and their centers are collinear. Hence they are coaxal, which proves the proposition.

THEOREM 2. *The polar planes of the median points of n given cospherical points with respect to the (Q)-sphere of the latter points cut the spheres having for diameters the respective medians along n circles lying on the same sphere, namely, the*

circumsphere of the given points.

Indeed, the polar plane of G_i for (Q) passes through the points D and A_i and is perpendicular to the common diametral plane DCA_iGG_iLM of the two spheres (O) and (A_iG_i). Moreover, the points D and A_i belong both to (O) and (A_iG_i); hence the polar plane under consideration cuts the two spheres along the same circle, which proves the proposition.

THEOREM 3. *The polar planes of n given cospherical points with respect to their (Q)-sphere cut the spheres having for diameters the respective medians along n circles lying on the same sphere, namely, the medial sphere of the given points.*

The proof is analogous to the proof of the preceding proposition.

Note. The above propositions are applicable to n concyclic points.

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A PROGRAM FOR IMPROVING THE TEACHING OF SCIENCE AND MATHEMATICS

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1. Science and Public Policy. The suggestions in this paper stem from a recent report that was prepared for the President's Scientific Research Board. The report appears in pages 47 to 149 of the Board's fourth volume on *Science and Public Policy*, and was provided by the Cooperative Committee on Science and Mathematics Teaching. This Committee was created by representatives of several scientific societies in 1941 to work on educational problems which no single scientific group could solve as well working alone as by cooperative action. Another function of the Cooperative Committee has been to serve as a forum in which representatives of the scientific societies have been able to state the views of their own organizations, and to learn the views of other groups on the teaching of science and mathematics at the secondary and elementary levels. The Cooperative Committee is now a standing Committee of the American Association for the Advancement of Science, and is charged with the responsibility of giving attention to the problems involved in the teaching of science and mathematics at all levels.

2. Recent report of the Cooperative Committee.* The Cooperative Committee has made a number of reports to the parent organizations, as, for example, "A Report on High School Science and Mathematics in Relation to Manpower Problems," which was published in 1943 and distributed to more than 12,000 individuals. The most recent report, entitled "The Present Effectiveness of Our Schools in the Training of Scientists," is the one with which this paper deals. Volume Four in the series, *Science and Public Policy*, is devoted to the problem of scientific and technical manpower. In considering the sources from which future scientists may be drawn, the President's Scientific Research Board was naturally driven to consider the effectiveness of science teaching, not only in higher institutions but also in the lower schools. At this point the President's Scientific Research Board turned for aid to the Cooperative Committee.

Although the hundred-page report contributed by the Cooperative Committee is primarily concerned with the early identification of youth with special talent in science and mathematics and with the proper nurture of these gifted youth through school, college, and university, it had to give careful attention to science and mathematics in general education. In fact, the report is unique in that it is the first effort to present a picture of what is going on in a subject area throughout the educational system from the elementary school through graduate work. It indicates the major limitations of the teaching of science and mathematics at each level, and it suggests possible ways for correcting the deficiencies. The Committee spent many hours in study and discussion, and it probably comes closer to unanimous agreement on issues, whether they lie in the elementary school, in the secondary school, in the college, or in the graduate school, than is ordinarily attained by commissions.

The Committee accepted the sobering assignment of making recommendations looking toward the creation of (1) a corps of effective research scientists

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* The present members of the Committee and the societies they represent are:

American Association of Physics Teachers: K. Lark-Horovitz, Purdue University, and Glen W. Warner, Chicago City College.

American Astronomical Society: Oliver J. Lee, Northwestern University.

American Institute of Physics: Lloyd W. Taylor, Oberlin College.

American Society of Zoologists: L. V. Domm, University of Chicago.

Botanical Society of America: Glenn W. Bladyes, Ohio State University.

Central Association of Science and Mathematics Teachers: Arthur O. Baker, Cleveland Board of Education.

Division of Chemistry Education of the American Chemical Society: Lawrence L. Quill, Michigan State College.

Executive Committee of the A.A.A.S.: E. C. Stakman, University of Minnesota.

Geological Society of America: George A. Thiel, University of Minnesota.

Mathematical Association of America: Raleigh Schorling, University of Michigan.

National Association of Biology Teachers: Prevo L. Whitaker, Indiana University.

National Council of Teachers of Mathematics: E. H. C. Hildebrandt, NorthwesternUniversity.

National Science Teachers Association: Morris Meister, Bronx High School of Science.

Chairman: K. Lark-Horovitz;

Secretary: R. W. Lefler

and (2) an organized group of discerning science educators. The report is based on the premise that our people should take such steps as may be necessary to insure (1) enough competent scientists to do whatever job may be ahead, and (2) a voting public that understands and supports the scientists' role in defense and in the design for better living. It should be noted that the second objective involves the crucial task of the education of the general public in the nature and function of science in the world of tomorrow.

This paper represents an attempt to collect the most important recommendations of the report, and to try to piece them together into a program or platform for the teaching of science and mathematics. At this point the reader is entitled to a word of caution. Lifting recommendations out of context may have done violence to the content intended by the Cooperative Committee. In any case, the critical person will wish to check the proposals in this paper with the details of the original report.

3. Program for science and mathematics. In developing a program for science and mathematics, this paper considers twenty major problems, described usually in the form of situations that imply limitations in our offerings in science and mathematics. First, in the treatment of each item, appears a statement of the problem; then a recommendation is given, which seems, at least to the writer, to stem from the Committee's report.

(1). Some of the crucial problems in the teaching of science and mathematics are much the same at all levels—elementary, secondary, and collegiate. For example, efficiency in the teaching of science and mathematics is everywhere cut down by such factors as: (a) inadequate financial support, (b) the lack of materials and space, (c) the shortage of competent teachers, (d) undesirable working conditions of teachers, (e) the unrealistic programs for teacher education, and (f) the lack of good in-service programs. Is it sensible for committees at every level and in every subject-matter area to "rediscover America" by working as if no one had every done anything on the problem?

We recommend the appointment of a National Commission on the Teaching of Science and Mathematics under the auspices of the American Association for the Advancement of Science, representing the scientists, and cooperating with agencies such as the American Council on Education, the National Education Association and the U. S. Office of Education that represent educators, administrators and science teachers (p. 59*). Presumably the suggestion here is that the Cooperative Committee be reconstituted to include representatives from other groups that are also concerned about the effectiveness of the teaching of science and mathematics in our schools. A second aim no doubt is to guarantee continuity of interest in pedagogical and curriculum problems.

* When page number only is given, the reference is to Manpower for Research, Volume Four of Science and Public Policy, a Report to the President, by John R. Steelman, Chairman, the President's Scientific Research Board, 1947. For sale by the Superintendent of Public Documents. U.S. Government Printing Office, Washington 25, D.C., at 35 cents.

(2). The financial support of our schools is wholly inadequate, and suggests that as a nation we are operating with a distorted sense of values. We spend about 35 billions a year on recreation, tobacco, alcohol, soap, chewing gum, beauty aids, crime, and delinquency, but only about one-tenth as much per year for education. We shall not have good teaching of science or anything else throughout the country until our people correct this situation. We are spending about 2 per cent of our annual income for education, whereas England spends a much higher percentage, and persistent rumor has it that Russia spends somewhere between 8 and 10 per cent of her annual income for education.

We should gear the financial support of public education to our economy. It should be a fixed percentage of our national income. The rate should be scientifically determined and the total amount for education should vary with our national income and with the purchasing power of the dollar. It is not sensible for us to pay without much complaint nearly 5 per cent of our national income for education in times of depression, and then recklessly allow our schools to deteriorate by spending less than 2 per cent of our income in prosperous times. Our schools need twice as many teachers as we now have and at least three times as much financial support. The writer has seen many schools, but never a good one that did not cost a lot of money.

(3). The nation faces a devastating shortage of teachers of science and mathematics. The main facts, widely publicized by newspapers, magazines, and radio, are: (a) There is a general shortage of teachers, (b) The situation is just about hopeless in the early grades of the elementary school. (c) Young people are rejecting teaching in the lower schools as a career. (d) Men teachers are disappearing from the high school. (e) The shortage of competent teachers of science and mathematics at the high school level appears to be more critical than in other subjects. (f) Young men who might be good material for teaching in college and graduate school are being lured into industry, business, and government by larger salaries, lighter work, and greater freedom. The report points out that in the school year 1946–47, there were about 200,000 students enrolled in 24 prominent colleges and universities that educate teachers. Of this vast student body, only 600 qualified for the teachers certificate in science and mathematics (p. 96). It is probable that in certain subjects the shortage of teachers may presently be alleviated, but the outlook as regards a sufficient number of competent teachers of science and mathematics for high schools is still dark. The study quoted above suggests that we may consider the University of Michigan as typical of large universities. In the present semester there are about 20,500 full time students at the University of Michigan. The total enrollment in the various sciences is very great. For example, we have 150 graduate students in mathematics, and about the same number in physics. We have over 4,000 elections of undergraduate mathematics courses in the college of Literature, Science, and the Arts alone. At present there are 272 seniors who are qualifying for majors in the physical and biological sciences. Moreover, the number of undergraduate and graduate students majoring in mathematics is a higher per-

centage of the total enrollment than it was 10 years ago. Now comes the shock: In student teaching, which is, of course, the gateway for teaching in high school, we have, in the present semester, the following number of student teachers: 2 in general science, 5 in mathematics, 0 in biology, 3 in chemistry, 0 in physics. Incidentally, the University of Michigan has not had a single student teacher in physics at any time during the last five semesters!

Every national scientific organization should put forth a special effort to make the public aware of the facts regarding the teacher shortage. In addition to higher salaries, we must realize that salaries and social prestige are not the only issues. For example, it is wholly unlikely that we could get or hold enough competent teachers of science merely by doubling the salaries of teachers. There are about a dozen additional things that we need to do, and most of these things relate to the working conditions of teachers. The items were listed and discussed in a document entitled "An Evolving Bill of Rights," first published in 1945. Since this statement has been reproduced frequently, we need do no more here than list the rights as follows: (a) the right to teach classes that are not too large, in general from ten to twenty pupils, (b) the right to have time in the school day for planning, (c) the right to a 45-hour week, (d) the right to adequate compensation for the full year of fifty-two weeks, (e) the right to an adequate amount of helpful and constructive supervision, (f) the right to have good materials and enough of them, (g) the right to work in a room that, with the help of students, can be made pleasant and appropriate to the tasks to be learned, (h) the right to the same personal liberties which other respectable citizens assume for themselves as a matter of course, (i) the right to an internship, (j) the right to a realistic program of in-service education, (k) the right to participate in changing the curriculum and methods, and in formulating school policies, and (l) the right to keep from being lost in the profession.*

(4). The undergraduate curriculum for the education of teachers, in many training institutions, is unrealistic. The report discusses this problem in a number of places. Typical limitations are: (a) Too many courses are, in fact, designed to select and train specialists in science and mathematics (p. 102). (b) Too often prospective teachers of science and mathematics for the lower schools are not trained in broad areas that will enable them to do a good job teaching several subjects in the early years of teaching in small high schools (p. 103). (c) In general, colleges and universities fail to provide a year of professionalized subject matter especially designed to meet the needs of the teachers (p. 105). (d) In science and mathematics, conventional undergraduate courses provide good background for the traditional high school courses, but do not provide adequately for the newer courses that are emerging to meet the general-education needs of pupils or for the close correlation of science and mathematics which is needed in the education of talented science students (p. 106).

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* *Swords into Ploughshares*, p. 37. Superintendent of Public Instruction, Lansing, Michigan, 1946.

The undergraduate training program for teachers of science and mathematics must be a specifically planned professionalized program (p. 103). While the present sequence of science courses may be very satisfactory in the selection and training of future science specialists, and not too bad as a training program for teachers in large high schools, they fail to give breadth of understanding or comprehension of the interrelationships existing between the many specialized fields of science and mathematics. These understandings and relationships are essential for teachers in the secondary school, where they must, at best, teach in broad areas (p. 103). A fifth year of college training for high school teachers is clearly indicated. However, merely adding another year of the same type will not meet the needs of persons who begin their teaching in small high schools.

(5). At all levels, schools and higher institutions have failed to provide strong in-service programs for teachers. Supervision in the lower schools is disappearing and in too many cases has become stigmatized or has shifted from the local schools to remote institutions and to state departments of education. Yet, supervision, when provided in adequate amount and when focused on the growth of the teacher on the job, is obviously badly needed in this day when even a strong high school may each year have as much as a 25 per cent turnover of teachers, and when advances both in subject matter and professional education are great. An important phase of an in-service program is keeping materials up-to-date. School boards and administrators need to realize that curriculum revision is a technical job requiring time and energy. It can not be done by tired teachers after school.

Higher institutions, as well as public school systems, should provide strong in-service programs for teachers. Supervision should be in adequate amounts and of the type that the teacher with professional zeal wants. It should focus on teacher growth. Plans should be developed for curriculum workshops, conferences, institutes, summer school instruction, demonstrations, lectures, field trips, excursions and conferences for the training of secondary school science and mathematics teachers. We need to make provision for science and mathematics counsellors throughout the country in each of the fields of mathematics, life science, and physical science (p. 59). A bright ray in the picture as regards supervision is that an increasing number of colleges and universities, under the pressure of heavy enrollments and the consequent increases in staff, are becoming seriously concerned about training programs for young and inexperienced instructors. Incidentally, some large universities report that even the annoying problem of meager equipment can be solved partially by proper supervision of the use of such equipment as is available.

(6). As regards special talent in science and mathematics, our nation has not been sagacious in the use of human resources. Among the wasteful policies and practices, the report lists the following: the ratio of able youth out of college to those in college has been reported as 1 to 1. Thus we seem to lose precious

talent because many youth in normal times can not afford higher education. Then too, we have no systematic and sure method for the early identification of youngsters with a flair for science and mathematics. Nor can we be confident that the typical teacher of the early grades with little or no science training can spot a youngster with science talent, or that she would be able to design appropriate experiences that would fan the spark of genius. Then, too, high school teachers in general can not provide their students with reliable information about the science and mathematics needed on jobs and in professional careers, for two reasons: (a) the teachers lack knowledge of the applications of science and mathematics made in engineering, agriculture, mining, medicine and the like, and (b) they do not have the necessary guidance materials. The first guidance pamphlet geared to a high school subject was published in the November, 1947, issue of *The Mathematics Teacher*. Our record when identifying superior students who will be highly successful in college is poor, although several investigations suggest that it is possible now to identify them with an accuracy approximating 85 per cent (p. 124). Failure of talented youth to attend college is obviously a costly loss of valuable human resources. In too many cases, however, there is a regrettable but unnecessary waste even when talented youth do go to college, because many of the ablest of our graduate students and our young instructors in colleges and universities waste time and energy in self-support. They find themselves confronted with the dilemma of resigning themselves to a delayed entrance into the profession or struggling with an academic overload, with the almost inevitable consequence of hasty and unsatisfactory graduate work. The road to research is long, and science is a young man's game. A year or two lost can never be regained, not even if the student turns out to be a top-flight scientist.

To meet the needs of especially talented youth the school must have a continuous program of identification and guidance (p. 89). The Cooperative Committee discusses with considerable reservation such practices as rapid promotion and permission to carry extra studies. It seems to have more faith in performance on research projects at the level where the student is operating. It is also asserted that science aptitude tests, in conjunction with vocational interest inventories, will help to locate the student who should be advised of opportunities for success in fields where science is applied or research is utilized. However, the main recommendation aimed to stop the wasteful drains on our human resources is that we develop a more comprehensive record of the individual student and more nearly adequate and more competent guidance service to interpret this record intelligently. As regards the waste of precious time and energy of graduate students and young instructors, the Cooperative Committee recommends the following: "Establish Federal subsistence type scholarships for the scientifically gifted as part of a general program to support able and talented youth in all fields. This will guarantee the utmost utilization of our scientific manpower through collegiate and graduate training" (p. 59). Of special importance is the need for making financial provision that will attract and hold teachers in the

junior instructional staff. The compensation paid in the laboratories of industry and government, particularly to research workers, far exceeds the compensation of staff members even in the large universities. Basic research in the fields of all science, particularly the physical sciences, is carried on in laboratories, in industry, and in government to an extent unknown five years ago. The Committee also recommends that a large number of post-doctorate fellowships be established for: (a) junior staff type such as the National Research Council fellowships, and (b) senior staff type such as the Guggenheim fellowships (p. 59). The two purposes of the post-doctoral fellowships are (1) to utilize the ability of young scientists who have just finished the Ph.D. and are eager to continue with basic research problems, and (2) to give older scientists, who frequently have too many routine duties in their regular jobs, the opportunity to refresh their scientific spirit and to study new methods of science education and research in new environments. Science teachers need to be alerted to the drastic possibility that the proposed Science Foundation may be established without providing fellowships for talented youth. The long delay in passing a bill creating the Science Foundation has provided an opportunity to divert money, that would be needed to provide fellowships, to other projects.

(7). The teaching profession has not shown great resourcefulness or utilized much imagination in the design of curricula appropriate to the nurture of genius. Presumably this is a difficult task for the curriculum builder in any field. It certainly is no oversight that the report of the Cooperative Committee devotes so little space to practical suggestions as to what schools should do with youngsters who are gifted in science and mathematics. We are, therefore, in no position to be critical of the typical teacher in an elementary school or high school with little or no training in science who may not know what experiences to provide for a future scientist.

A scientific study should be undertaken to collect and perhaps to design types of experiences that should be included in a curriculum for our youth with scientific aptitudes (p. 81). It is very clear that we do not now have enough reliable information to arrive at a sensible solution of this problem. The report does suggest that a large city may provide a specialized school of science, as for example, the Bronx High School of Science in New York City. In that set-up, the entire school can be fitted to the needs and purposes of a selected group of students. One might expect that large comprehensive high schools would provide at least a few courses of sufficient rigor to challenge the most able. However, the education of mentally gifted children as a separate group is one of the relatively new programs in public education. Nevertheless, there are a few school systems where groups of children with scientific interests meet with specially qualified teachers who are sympathetic with the idea of developing special talents to a maximum extent, in classrooms equipped to allow the student freedom for individual investigations. It could happen that a comprehensive investigation would challenge some of the materials and methods that long have

been accepted as a matter of course. Thus, the Cooperative Committee raises the question whether the finest algebra and geometry courses of the traditional type are the best mathematics courses that can be provided for the future scientist. In any case, a survey of the type suggested here seems necessary, for certainly neither science nor mathematics teachers have ever come to grips with the design of curricula that are best for youth with special aptitudes for science and mathematics.

(8). There is vagueness concerning the content of science and mathematics courses in general education. To be sure there is also confusion concerning the question, "What is the training in art, music, citizenship, and every other field, that must be considered a part of general education?" The high school has, since the turn of the century, been concerned with general education in that it has had the dual responsibility of providing (a) rigorous training for leadership in science, mathematics, and other learned fields, and (b) good general education for better living and for use in the common affairs of life. The rising tide of students has swept this problem into the higher institutions. The problem of general education is now on the desk of the college teacher, and presumably will be in the graduate school day after tomorrow. The present controversy among college teachers of science and mathematics as regards early specialization is only one phase of a problem that will not be solved until we have a definition of general education that is widely accepted. Since the research worker should first of all be an educated citizen concerned about human welfare, general education is of paramount importance. To many it will seem impossible to put anything more into the curriculum buckets that already seem to be overflowing. Perhaps there will be plenty of time to teach all that needs to be taught of general education—once we identify what it is. Certain it is that no good undergraduate program for the education of teachers can be built until the content of general education is defined.

The content of science and mathematics in general education should be determined in a way that will be widely accepted. It may be that we need two definitions of general education—one as an essential for citizenship and the other appropriate for the well-educated person. Be that as it may, the perplexing question is first to identify this content of general education. The task obviously would involve comprehensive investigation; and perhaps a National Commission on the Teaching of Science and Mathematics, if and when created, will make this one of its first projects. It should be noted that a first step has been taken in the Second Report of the Commission on Post-War Plans of the National Council of Teachers of Mathematics. In that document there are identified 29 specific concepts and principles in mathematics that are considered essential in the education of all citizens. The task is more urgent at the college level, for here the great crowd of hungry guests are already sitting at the table, though the cooks have only started the meal. The Cooperative Committee puts the problem in these cautious words: "There is general agree-

ment among college teachers of science that our general courses are not as well designed in terms of general education as they might be, and that better orientation of teachers toward the place of science in general education is of high importance" (p. 113).

(9). There is a tendency for the general courses in science and mathematics to become stigmatized. It seems difficult to administer general courses in science and mathematics in a way that will make them respectable and desirable. Here the attitude of the teacher no doubt is the determining factor. Far too often he is by training disposed to give a halo of prestige to the older, traditional, sequential courses. This attitude may be due in part to the fact that many of our high school teachers have never had a single general course either in high school or in college. Then too, teachers do not make clear to students and parents that both types of courses—the general and the specialized—are worthwhile for the pupils who need them, and that they offer different experiences for pupils with very different needs. The general courses have not had a good program of publicity. There is still another factor that tends to stigmatize the general courses. It is the practice, in some schools, of attempting to get homogeneous groups by classifying pupils on the basis of intelligence tests alone.

Teachers of science and mathematics need to recognize the importance of the general courses. Two objectives are involved: (a) a public that will vote intelligently on issues involving the role of science in the world today, and (b) a more abundant and meaningful life for the citizen. Teachers need to reject the fallacy that there is a conflict between general education and the rigorous training of the future scientist. We are not dealing with an either/or proposition. Both tasks must be done and done extremely well. Every department of science and of mathematics should include some teachers who see clearly the function of the general courses.

(10). The offering in science and mathematics in the small high school is pathetically meager. Many persons do not realize that more than two-thirds of all high schools are small, with certainly fewer than 200 students and probably fewer than 8 teachers. Such small high schools enroll in all more than a million pupils. In many schools the offering in science and mathematics is limited to the traditional sequential courses designed for college preparation, in spite of the fact that a very small fraction of the graduates ever enter college. On the other hand, some small high schools offer only the general courses in science and mathematics and still others provide no science and mathematics at all in a particular grade.

The Cooperative Committee has made three suggestions. In the first place, the small school can enrich its offerings by teaching simultaneously in small classes of perhaps only a half a dozen students two courses on different levels—one on the level of general education, and one providing a rigorous course for the future science specialist. A second suggestion is to cycle the course in science and mathematics. Physics does not need to be taught every year in order for a

student, with reasonable planning, to pick it up before graduation. Finally, courses in mathematics may be provided for gifted students by correspondence courses that are now serviced by more than 80 higher institutions.

(11). Efficiency of instruction in science and mathematics in grades 1–6 is low. Science instruction at all levels leaves much to be desired. The Cooperative Committee includes some distinguished scientists who are no less critical of advanced courses in their own fields than they are of science in the lower grades. However, it is of the greatest importance that the foundation in science instruction be solid. Therefore, the following indictments are cited: (a) Science instruction is in many schools incidental. (b) It too often neglects the natural phenomena that surround the child. (c) The science and mathematics materials are not geared to each other. (d) In arithmetic there is often not sufficient effort to teach meaning—often pupils do not understand what they are doing even when they get the correct answers. (e) The general level of competence in arithmetic is very low (p. 63). (f) Basic concepts and principles are taught by reading and verbalizing, rather than through the varied experiences with the materials in the immediate environment. (g) Symbols and generalizations too often run ahead of experience. (h) In many schools there is no list of specific objectives for the school year (p. 66). (i) The method of science is not in the spotlight. (j) The science and mathematics work of these grades is not planned with the teachers of the next six higher grades to form a 12-year unit of continuous, systematic instruction.

More assistance to classroom teachers must be provided through the use of supervisors, science consultants, and others, who are able to design or recast curriculum materials and to integrate science with other subject areas. School boards might well employ competent teachers of science and mathematics in the summer months to build source units and otherwise organize the curriculum in science and mathematics. Another suggestion is to allow competent teachers time in the school day to plan the units of work. All of these suggestions are important for the reason that the undergraduate curriculum so seldom teaches prospective teachers how to collect and organize instructional materials. It may be that this task requires a degree of maturity and an amount of experience quite beyond the undergraduate. In any event, some teacher-training institutions should provide for at least one year of graduate work in which ample courses in science and mathematics are offered to meet the needs of the teacher instead of the research scientists.

(12). The main goal of science and mathematics in the junior high school grades has not been recognized. In both science and mathematics the offerings in these grades have been characterized by great variation. A good illustration is to be found in the case of mathematics for the ninth school year. In adjacent towns one may find the following offerings in small high schools: School *A* offers only general mathematics; school *B* provides only algebra; school *C* teaches a diluted algebra; school *D* requires pupils to elect commercial arithmetic; school

E teaches agriculture mathematics only; and school *F* teaches no mathematics at all in order to cut down failures! While this illustration is a bit extreme, the picture of science offered in grades 7 and 8 would also be shocking.

Grades 7, 8, and 9 provide the crucial years for achieving the objectives of general education. Most of the youngsters at this age level are still in school, where they can be taught those things which society deems essential. The general courses have greater flexibility, and therefore may include a greater variety of activities to meet the needs of varied interests and abilities. The foregoing statement does not imply that all ninth grade students should be enrolled in general mathematics and general science. A goodly fraction of the students will have achieved functional competence in mathematics by the end of the eighth school year. Moreover, the future scientist is almost certain, by that time, to want more organization, more rigor, more continuity than can be or should be provided in the general courses. In brief, the beginning of the ninth year is the time for differentiation of courses and classification of students. In the ninth grade we should operate a double track, at least, in mathematics and in science.

(13). As regards senior high school, no one should assume that all is well with the traditional courses in science and mathematics. In most schools they are woefully out of date as regards both content and method. The inertia is largely due, as has been pointed out earlier, to the fact that teachers are not given time in the school day or employed in the summer months to do the technical and arduous task of curriculum revision.

Science and mathematics teachers should check the content and method of the traditional sequential courses with the recommendations that have long been advocated in the reports of our national committees. We will do well to follow the example of the industrialists and meet new needs with improved materials, and with more efficient methods. We should recognize that future scientists are not desirous that we make these courses easy. Moreover, the traditional sequential courses should be reserved for the pupils who can profit by such courses. The unique values of the traditional courses can not be achieved by a constant gearing down. Failure should, for the most part, be avoided by guiding pupils to appropriate general courses.

(14). The science and mathematics courses in the junior college have been allowed to develop without much design. These courses have grown up like Topsy, and, as concerns the sequential courses, largely in imitation of freshman and sophomore courses in the colleges. On the other hand, an inspection of the great variety of new general courses suggests that a systematic evaluation might uncover many courses that are superficial and lacking in specific objectives. The junior college, or perhaps better, the community college, is now undergoing its second great spurt of growth. Witness the fact that more than one-fourth of the institutions of higher learning are junior colleges. In the fall of 1946 there were 119,000 new students enrolled in junior colleges as compared to 163,000 new students in colleges of arts and sciences, and 269,000 in universities and large

institutions of complex organization (p. 111). The evidence that the community college is here to stay and that it will play an important part in public education is completely convincing. It is clear that national organizations can not afford to neglect the science and and mathematics of so large a group of college students.

We need to provide science and mathematics courses for three very different types of students enrolled in the community college. It is assumed that the community college provides educational opportunities which otherwise might be inaccessible to a large number of educatable youth. In that case the three types of courses called for are: (a) courses that attempt to guarantee youngsters competence in mathematics and science as regards the content of general education for all who can possibly achieve it, (b) courses that provide certain science and mathematics competencies needed by students who have a desire to follow specific vocational interests, and (c) rigorous sequential courses in science and mathematics for students who plan a career in science or mathematics. As a minimum it would seem that the community college should offer (1) at least a year of mathematics and of science which is general in appeal, flexible in purpose, challenging in content, and functional in outcome, (2) a one-year pre-vocational course consisting of units that correlate materials from mathematics, physics, and industrial arts, and (3) ample provision for the student with a major interest in science and mathematics.

(15). Too often college courses are geared to the needs of the future specialist, to the neglect of the general student body whose main interest is in the science and mathematics of general education. This, in the judgment of the Cooperative Committee, is one of the main weaknesses of the science and mathematics programs in the colleges. This criticism seems valid also for many other undergraduate courses—even in the social studies, as, for example, sociology, psychology, and economics. Perhaps the freshman courses are the worst offenders, although there have been some sincere efforts, such as those at Columbia, Chicago, and Harvard, to design courses that would provide the science and mathematics of general education for the educated person. However, the typical freshman course in mathematics, when evaluated in terms of the needs of women, for example, suggests that we still have a long way to go.

A department of science and mathematics in a college needs to give careful attention to its functions as a service unit to other departments—medicine, economics, etc. Quoting directly from the report:

> In many of the large State universities and institutions of complex organization, the science departments are largely service departments for the professional schools. In many cases it is not the ability of the student but his membership in a professional group which determines the type of science offering the student has to take. The insistence of many professions that every student, during his college career, be exposed to every branch of the natural sciences, produces an antiscientific attitude on the part of the student who does not understand basic principles but is forced to assimilate superficially an agglomeration of facts and formulas. It produces a frustration on the part of the science teacher who is unable to hold and inspire the majority of his students (p. 118).

In an earlier section, the reader's attention was called to the controversy among college teachers of science and mathematics as regards early specialization. Perhaps a precise identification of the elements of science and mathematics in general education will go a long way in resolving this difficulty. In the meantime, the able and interested student should have the opportunity of starting courses in his special field as early as he knows what his field is, and certainly by the beginning of the second year of college (p. 123). A drastic indictment of science and mathematics in the colleges is that "In a great many institutions departmentalized instruction is so rigorous that scientific progress is impeded by it" (p. 127). To solve this problem, the Cooperative Committee suggests that "Advances can be expected at the borderline of the sciences and special attention should be given to the creation of strong departments in universities to cover overlapping fields of science such as meteorology, geophysics, geochemistry, biophysics, radiobiology, and to develop these newly recognized fundamental areas. These borderline fields have been developed by capable individuals, partly self-trained, but they merit the attention of the curriculum planners and justify specially trained scientific personnel for further fundamental work in these areas" (p. 127).

(16). The rapid expansion of enrollments in graduate work is creating serious problems. The discussion here will treat lightly such aspects of the situation as inadequate material and lack of space. There are certain problems stemming from heavy enrollments that will not be solved easily and probably not very soon. Such problems may be illustrated by the following: (1) With fewer contacts, desirable student-professor relationships are now more difficult to build. (2) There is a lack of competent supervision in the classroom and laboratory. (3) It is even more difficult than in the past to give teaching the full attention which it deserves by placing the most experienced, capable, and inspiring members of the staff in charge of the all-important task of introducing the beginning students to science (p. 117). (4) The value of the master's degree is being deflated. In this paper we shall discuss only the problems concerning the master's degree and the need for supervision.

The vast up-surge in enrollment makes the definition of the master's degree one of the major problems of the graduate school. There are two attacks on the master's degree, and curiously enough, they come from opposite directions. In the first place, some scientists insist that the master's degree is no longer a good screening device for selecting the competent research worker. This is due not only to lack of uniformity in requirements for the master's degree, but to gearing down of standards under the pressure of large enrollments. The other criticism is expressed by those who recognize that in many institutions the master's degree has for some time been, to no small extent, a teacher's degree. This does not imply that the curriculum in such institutions has been especially designed to serve the needs of prospective teachers, but rather that a goodly fraction of the graduates, if they continued their education, turn out to be teachers, espe-

cially in the lower schools. As regards the master's degree, the Cooperative Committee recommends: A master's degree designed as a powerful screening device for the doctoral candidate that would require evidence of research ability, the competence to find and evaluate the literature bearing on a given piece of research, the discerning mind to choose significant problems of investigation, and the ability to devise effective means of solving such problems. It is hoped that if graduate schools define the degree of master of science as proposed in this report, that a truly professional master's degree will also be made available to teachers. Indeed it might be feasible and desirable to challenge a teacher to strive for two master's degrees: (a) A master's degree (M.A. in Education) given upon the completion of a curriculum that represents a conscientious effort to meet the needs of future teachers, especially in elementary and secondary schools. Under sensible administration this degree need not turn out to be a stigmatized "cheap" degree, for in that case the teachers would be selected for high competence on the total college record which obviously would include many courses in which future research workers would also be enrolled. (b) A master's degree, perhaps called Master's Degree in Professional Education, to be awarded to educators who have completed two years of graduate work, and who have convincingly demonstrated that they are master teachers or highly skilled practitioners in other phases of professional education. The argument is that sooner or later we must differentiate between the competent teacher of science and mathematics and the mere transient and the utterly incompetent.

(17). One of the low spots is the training now offered in statistical methods. The Committee on Applied Mathematical Statistics* reports that: "There has been an unprecedented growth of interest in the use of statistical methods during the past 10 years, which has caught the American educational system unprepared. . . . There is a shortage of adequately trained statisticians in both academic and non-academic categories and a more severe shortage is expected. . . . The shortage of training facilities for mathematical and especially for applied statisticians is critical. . . . "

Mathematics departments of colleges should provide the kind of statistical training as a service to other departments that will bridge the gap between theory and applications. Operating as a service unit, the mathematics department of a college should provide statistical training needed for students whose major interest is in other fields. The problem will not be solved by requiring students in these various fields to take one or two courses of theoretical statistics. Rather the department of mathematics needs to provide types of statistical training that will bridge the gap between theory and the applications in these fields. Undergraduate courses even in the social studies are becoming more systematic and therefore more statistical. Undergraduates to an increasing de-

* "Personnel and Training Problems Created by the Recent Growth of Applied Statistics in the United States," a report by the Committee on Applied Mathematical Statistics, National Research Council, Reprint and Circular Series, No. 128, May 1947. For a summary of this report, refer to this MONTHLY, vol. 54, 1947, p. 525.

gree are not prepared to deal with the statistical problems that they encounter. Neither the college nor the high school is doing a good job in meeting this need. Some one should analyze the statistics needed in the undergraduate courses. In all probability a full semester of high school training in statistics—an elective—would be indicated. If this should turn out to be the case, colleges might well encourage the strong high schools to offer a semester in statistics. Finally, colleges and universities should keep an eye on the rapid development in statistics and should realize that the demand for persons well trained in statistics is likely to exceed the supply in the years immediately ahead.

(18). It is inefficient and uneconomical for a teacher of science to spend an excessive amount of time as a technician or as a serviceman for his own apparatus. It is generally recognized that teachers in all subjects waste precious time doing clerical tasks that should go into instruction. However, the fact that time and effort is wasted by science instructors on repairing and rebuilding apparatus is not realized by most administrators. Moreover, in high school courses these tasks should be done by selected students with a bit of special training, and by hired technicians at the college and university level, for the reason that in the learning stage such work may provide valuable experiences. Incidentally, our schools probably do not give students enough of the practice in leadership they would get by appearing before a class or a group with responsibility for such specific tasks.

In high school, college and university, we need to utilize trained assistants and hired technicians to increase the efficiency of science teachers. Prospective teachers of the high school, if they are going to teach students the skills of a technician, must themselves learn to repair, design and rebuild pieces of equipment and this means they must be able to use glass blowing techniques, hand tools, and some machine tools. Quoting from the report:

> It is especially important under the conditions now existing in graduate schools that adequate technician services be available. Otherwise those who are directing the research of graduate students will be immersed in a mass of detail which will materially reduce the number of students whom they can supervise. In addition to machinists, and glass blowers for the physical sciences, and technical assistants for the biological sciences, both now require electronics aids of the type which did much of the development and all of the servicing of radar and communications equipment during the war. The new electrical techniques are as important in the physiology laboratory as in the physics laboratory. Because they are new, the necessity for that type of technician is not yet recognized by many university administrators, who will have to realize their importance (p. 138).

(19). The scientific approach to problems is not given enough emphasis. It is reasonable to assume that the scientific method would be about the last thing to be neglected in courses in science. This, however, does not seem to be true. Throughout the report, whether the discussion deals with science in the early grades of the elementary school, or with the advanced courses in the college, there is the implied criticism that the main objective of science instruction is not being achieved to the extent desired. Let us consider this sample quotation: "The scientific approach to problems of the nature of man should occupy a cen-

tral role both in the courses offered in general education and those offered to science specialists. In the former, insufficient stress is now laid on the objectivity of scientific approach and the method of scientific analysis. An accumulation of facts, no matter how well stated or numbered, fails to provide the student with the tools he so urgently needs, namely those of searching inquiry, critical analysis, and unbiased conclusions" (p. 114).

In the science courses at all levels, both in general education and in the training of the future scientist, there should be an emphasis on the method of science. It is understandable that there might be a neglect of the techniques of scientific method in general courses in science and mathematics. However, the following quotation implies that the scientific method is not even given adequate attention in the traditional sequential courses. "In courses for science specialists the scientific method usually appears by implication and in time some students acquire it as second nature. But frequently a conscious and explicit exposition of the scientific approach and its implication is lacking. More effort should be spent in clarifying the nature of scientific thought and its historic development and the cooperative aspects of investigations, the free nature of exchange of ideas and findings, and the important effects of scientific discoveries on our society" (p. 114). Finally it is recommended that courses be given in which a conscious effort is made to clarify the nature of scientific thought, the cooperative aspects of scientific investigations, the ethical implications of the free exchange of ideas and the important effects of scientific discoveries on society. We also recommend that courses be required in the history of scientific thought, in the philosophy of science and in general philosophy, so that the science specialist may recognize the social implications of scientific endeavor (p. 143). The growth of interest in a college course on the Foundations of Mathematics is to be commended.

(20). As regards efficiency of science instruction, the ceiling is low, for we do not know the answer to many of our problems in science education. A host of systematic studies are needed to provide the data from which sensible solutions may evolve. These studies, now so urgently needed, require funds and personnel far greater than any existing organization can provide. The task is beyond what any higher institution or any single national society can do.

Some national organization should be set up to study the problems in science education in a systematic, a comprehensive, and a reliable manner. There are at least two possibilities. In the first place there is the hope that the National Science Foundation, if and when created, will make available the funds and the personnel to undertake the necessary investigations. In the second place, the Cooperative Committee is recommending that a National Commission on the Teaching of Science and Mathematics be created. Some form of cooperative action on matters relating to science education by the national societies is clearly indicated, and almost certain to prove useful. In the event that the Science Research Foundation is not established, cooperative action would seem to be a necessity.

MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California

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RATIONAL SOLUTIONS OF A DIOPHANTINE EQUATION

P. A. CARIS, University of Pennsylvania

1. Introduction. It is the purpose of this paper to find a general non-trivial rational solution of the diophantine equation

$$u^3 + v^3 + w^3 - 3uvw = x^3 + y^3. \tag{1}$$

The result, though readily obtained, is believed to be new.

First it may be remarked that if $u+v+w=0$, or if $u=v=w$, then $x+y=0$. Conversely, if $x+y=0$, either $u+v+w=0$, or $u=v=w$. The latter result comes from the fact that the quadratic factor of $u^3+v^3+w^3-3uvw$ is a definite form. Such solutions are trivial and will be ignored in this discussion.

The substitutions, $u=r-s$, $v=r+s$, $w=r+t$, transform (1) into

$$(3r + t)(t^2 + 3s^2) = x^3 + y^3.$$

If $t=s=0$, then $u=v=w$. Hence it will be assumed that $ts\neq 0$.

2. Solution. Multiply the above transformed equation by 4 for the sake of convenience. Then (1) may be written thus:

$$4(3r + t)(t^2 + 3s^2) = (x + y)[(2x - y)^2 + 3y^2].$$

This equation will be satisfied in rational numbers if the following system of equations is satisfied in rational numbers:

$$\begin{aligned} p(3r + t) &= q(x + y), \\ 2q(t + s\sqrt{-3}) &= (a + b\sqrt{-3})(2x - y + y\sqrt{-3}), \\ 2(a + b\sqrt{-3})(t - s\sqrt{-3}) &= p(2x - y - y\sqrt{-3}). \end{aligned}$$

This system will be satisfied if

$$3pr + pt = qx + qy, \tag{2}$$

$$2qt = 2ax - ay - 3by, \tag{3}$$

$$2qs = ay + 2bx - by, \tag{4}$$

$$2at + 6bs = 2px - py, \tag{5}$$

$$py = 2as - 2bt. \tag{6}$$

Here x, y, r, s, t are to be found. Assume however, for the moment, that these values are known, i.e., assume that $(3r+t)(t^2+3s^2)=x^3+y^3$. Then the following values,

$$p = 2t^2 + 6s^2,$$

$$q = 2(x^2 - xy + y^2),$$
$$a = 2tx - ty + 3sy,$$
$$b = 2sx - ty - sy,$$

satisfy the system of equations, as can readily be seen by substitution. Hence, to find the general solution of (1), it will suffice to solve the system of equations for x, y, r, s, t.

Now write equation (3) thus:

$$2pqt = a(2px - py) - 3b(py).$$

Substitute in this equation from (5) and (6). Then

$$2pqt = 2a^2t + 6b^2t,$$

whence, since $t \neq 0$, $pq = a^2 + 3b^2$.

Similarly, (4) will be satisfied by (5) and (6) with $pq = a^2 + 3b^2$, since $s \neq 0$.

From (5), making use of (6), we get

$$px = (a - b)t + (a + 3b)s.$$

Now write (2) thus:

$$3p^2r + p^2t = q(px + py).$$

In this equation substitute the values of px and py already known. Then

$$3p^2r + p^2t = q(a - 3b)t + q(3a + 3b)s,$$

whence

$$3p^2r = (aq - 3bq - p^2)t + q(3a + 3b)s.$$

Here let $t = 3cp^2$ and $s = dp^2$. Since $t \neq 0$, we have $p \neq 0$. Then

$$r = acq - 3bcq - cp^2 + adq + bdq.$$

These values for t and s, when substituted in the equation found for px, give

$$x = 3acp - 3bcp + adp + 3bdp,$$

and, similarly, from (6),

$$y = 2adp - 6bcp.$$

Hence,

$$u = r - s = acq - 3bcq - cp^2 - dp^2 + adq + bdq,$$
$$v = r + s = acq - 3bcq - cp^2 + dp^2 + adq + bdq,$$
$$w = r + t = acq - 3bcq + 2cp^2 + adq + bdq,$$
$$x = 3acp - 3bcp + adp + 3bdp,$$
$$y = 2adp - 6bcp,$$

where $pq=a^2+3b^2$.

As a special case, the values $a=1$, $b=-1$, $p=q=2$, give the neat identity:

$$(c-d)^3+(c+d)^3+(4c)^3-12c(c^2-d^2)=(3c-d)^3+(3c+d)^3.$$

ON A CONTACT TRANSFORMATION THEOREM

HUAN-TING KUO, National Wuhan University, Wuchang, China

The aim of this paper is to give a vectorial proof of the following classic contact transformation theorem.

THEOREM. *Given a surface S and a fixed point O; join the point O to any point M of the surface S, and pass a plane OMN through OM and the normal MN to the surface S at the point M. In the plane OMN draw through the point O a perpendicular to the line OM, and lay off on it a length $OP=OM$. The point P describes a surface Σ, which is called the apsidal surface to the given surface S. The transformation is a contact transformation, and the relation between the surfaces S and Σ is a reciprocal one. When the given surface S is an ellipsoid and the point O is its center, the surface Σ is Fresnel's wave surface.*

Proof. Take O as the origin of a system of rectangular coordinates. Let the coordinates of M and P be x, y, z and X, Y, Z respectively. Then obviously we have

$$xX+yY+zZ=0, \tag{1}$$

$$X^2+Y^2+Z^2=x^2+y^2+z^2, \tag{2}$$

$$\begin{vmatrix} x & y & z \\ X & Y & Z \\ p & q & -1 \end{vmatrix}=0. \tag{3}$$

Differentiating (1) and (2), transposing, and dividing by $\sqrt{x^2+y^2+z^2}$ or $\sqrt{X^2+Y^2+Z^2}$, we get

$$\frac{xdX+ydY+zdZ}{\sqrt{x^2+y^2+z^2}}=-\frac{Xdx+Ydy+Zdz}{\sqrt{X^2+Y^2+Z^2}}, \tag{4}$$

$$\frac{XdX+YdY+ZdZ}{\sqrt{X^2+Y^2+Z^2}}=\frac{xdx+ydy+zdz}{\sqrt{x^2+y^2+z^2}}. \tag{5}$$

Let us denote the vectors (dx, dy, dz) and (dX, dY, dZ) by a and A respectively. Using OM and OP as the first and the second axes of a rectangular system, and denoting the components of a and A referred to this system of coordinates by a_1, a_2, a_3 and A_1, A_2, A_3 respectively, we have from (4) and (5)

$$A_1=-a_2, \qquad A_2=a_1.$$

Thus

$$a_1A_1 + a_2A_2 = - a_1a_2 + a_2a_1 = 0.$$

Consequently the projection of (dx, dy, dz) on the plane MOP is perpendicular to the projection of (dX, dY, dZ) on this same plane. But the vector (dx, dy, dz) always lies on the tangent plane at M, hence the projection of (dx, dy, dz) on the plane MOP is along the line of intersection of MOP and the tangent at M. It follows that the tangent plane at P is perpendicular to the tangent plane at M. Thus the transformation is a contact transformation. Obviously the normal at P is on the plane MOP, and we have

$$\begin{vmatrix} x & y & z \\ X & Y & Z \\ P & Q & -1 \end{vmatrix} = 0. \tag{6}$$

By (1), (2), (3) and (6) we know the transformation is a reciprocal one. To find P and Q we use the equation $pP+qQ+1=0$ and the equation (6).

CLASSROOM NOTES

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GENERALIZATION, SPECIALIZATION, ANALOGY*

GEORGE PÓLYA, Stanford University

My personal opinion is that the choice of problems and their discussion in class must be, first and foremost, *instructive*. I shall be in a better position to explain the meaning of the word "instructive" after an example. I take as an example the proof of the best known theorem of elementary geometry, the theorem of Pythagoras. The proof on which I shall comment is not new; it is due to Euclid himself (*Elements* VI, 31).

1. We consider a right triangle with sides a, b and c, of which the first, a, is the hypotenuse. We wish to show that

$$a^2 = b^2 + c^2. \tag{1}$$

This aim suggests that we describe squares on the three sides of our right triangle. And so we arrive at the not unfamiliar part I of our compound figure.

* Presented at the summer meeting of the Mathematical Association of America, New Haven, Conn., September 1, 1947.

(The reader should draw the parts of this figure as they arise, in order to see it in the making.)

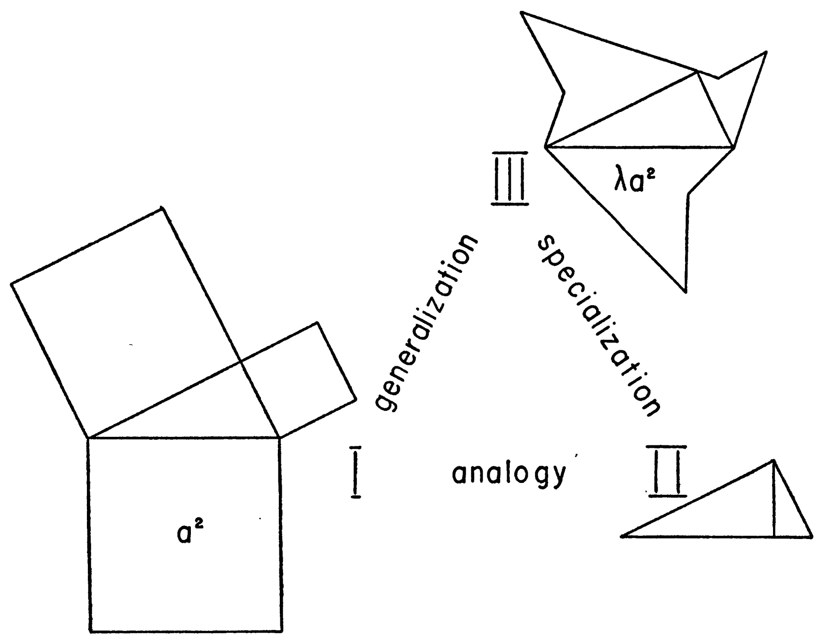

Fig. I, II and III

2. Discoveries, even very modest discoveries, need some remark, the recognition of some relation. We can discover the following proof by observing the *analogy* between the familiar part I of our compound figure and the scarcely less familiar part II: the same right triangle that arises in I is divided in II into two parts by the altitude perpendicular to the hypotenuse.

3. Perhaps, you fail to perceive the analogy between Figures I and II. This analogy, however, can be made quite explicit by a common *generalization* of I and II which is expressed by Figure III. There we find again the same right triangle, and on its three sides three polygons are described which are similar to each other but arbitrary otherwise.

4. The area of the square described on the hypotenuse in Figure I is a^2. The area of the irregular polygon described on the hypotenuse in Figure III can be put equal to λa^2; the factor λ is determined as the ratio of two given areas. Yet then, it follows from the similarity of the three polygons described on the sides a, b and c of the triangle in Figure III that their areas are equal to λa^2, λb^2 and λc^2, respectively.

Now, if the equation (1) should be true (as stated by the theorem that we wish to prove), then also the following would be true:

$$\lambda a^2 = \lambda b^2 + \lambda c^2. \tag{2}$$

In fact, very little algebra is needed to derive (2) from (1). Now, (2) represents a *generalization* of the original theorem of Pythagoras: *If three similar polygons are described on the three sides of a right triangle, the one described on the hypotenuse is equal in area to the sum of the two others.*

It is instructive to observe that this generalization is *equivalent* to the special case from which we started. In fact, we can derive the equations (1) and (2) from each other, by multiplying or dividing by λ (which is, as the ratio of two areas, different from 0).

5. The general theorem expressed by (2) is equivalent not only to the special case (1), but to any other special case. Therefore, if any such special case should turn out to be obvious, the general case would be demonstrated.

Now, trying to *specialize* usefully, we look around for a suitable special case. Indeed Figure II represents such a case. In fact, the right triangle described on its own hypotenuse is similar to the two other triangles described on the two legs, as is well known and easy to see. And, obviously, the area of the whole triangle is equal to the sum of its two parts. And so, the theorem of Pythagoras has been proved.

6. I took the liberty of presenting the foregoing reasoning so broadly because, in almost all its phases, it is so eminently instructive. A case is instructive if we can learn from its something applicable to other cases, and the more instructive the wider the range of possible applications. Now, from the foregoing example we can learn the use of such fundamental mental operations as generalization, specialization and the perception of analogies. There is perhaps no discovery either in elementary or in advanced mathematics or, for that matter, in any other subject that could do without these operations, especially without analogy.

The foregoing example shows how we can ascend by generalization from a special case, as from the one represented by Figure I, to a more general situation as to that of Figure III, and redescend hence by specialization to an analogous case, as to that of Figure II. It shows also the fact, so usual in mathematics and still so surprising to the beginner, or to the philosopher who takes himself for advanced, that the general case can be logically equivalent to a special case. Our example shows, naively and suggestively, how generalization, specialization and analogy are naturally combined in the effort to attain the desired solution. Observe that only a minimum of preliminary knowledge is needed to understand fully the foregoing reasoning. And then we can really regret that mathematics teachers usually do not emphasize such things and neglect such excellent opportunities to teach their students to think.*

* The author's views are presented more fully in his booklet, *How to Solve It* (Princeton, 5th enlarged printing 1948). For more about generalization, specialization and analogy see the sections starting on pp. 97, 164 and 37.

AREAS OF PLANE FIGURES

HOI-CHEUNG PANG, Provincial Canton Technical High School, China

1. Triangles. A well known formula of analytic geometry states that the area of a triangle whose vertices are $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ is given by:

$$\text{(1)} \qquad \text{Area } (P_1P_2P_3P_1) = \tfrac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3).$$

If the passage around the triangle places the interior of the triangle on the left, (1) gives a positive result; the opposite direction of passage gives a negative result.

This area may also be computed by the following rule:

(a) Write down the vertices in two rows in the order $P_1P_2P_3P_1$

$$\begin{matrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{matrix}$$

(b) Multiply each abscissa by the ordinate of the following column and add, giving: $x_1y_2+x_2y_3+x_3y_1$.

(c) Multiply each ordinate by the abscissa of the following column and add, giving: $x_2y_1+x_3y_2+x_1y_2$.

(d) Subtract the result of (c) from that of (b) and divide by 2. This gives the result in formula (1).

2. Other polygons. This rule equally well gives the areas of polygons of all descriptions. The proof consists of dividing the polygon into triangles, and applying the above rule to each triangle dropping out terms which cancel. As an illustration consider quadrilateral $P_1P_2P_3P_4P_1$. Divide the quadrilateral into

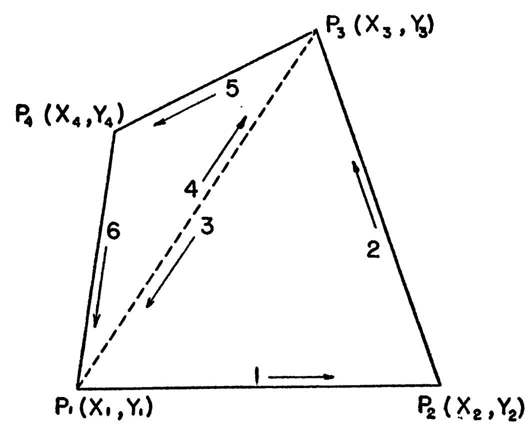

Fig. 1

the triangles $P_1P_2P_3P_1$ and $P_1P_3P_4P_1$ and write down the coordinates of the two triangles in two sets:

$$P_1P_2P_3P_1 \qquad\qquad P_1P_3P_4P_1$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{array} \qquad\qquad \begin{array}{cccc} x_1 & x_3 & x_4 & x_1 \\ y_1 & y_3 & y_4 & y_1 \end{array}$$

Applying the above rule to each triangle we see that the last part of the result obtained from the first triangle, namely $x_3y_1-x_1y_3$ cancels with the first part of the result obtained from the second triangle, namely $y_1y_3-x_3y_1$. So we can combine the two sets of coordinates into a third one, namely:

$$P_1P_2P_3P_4P_1$$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_1 \\ y_1 & y_2 & y_3 & y_4 & .y_1 \end{array}$$

and apply the above rule to find the area of the quadrilateral with the following caution in step (a):

(a′) Write down the coordinates of the vertices in an order agreeing with that established by passing continuously around the perimeter.

This proof generalizes at once to any non-self-intersecting polygon—whether convex or not. When the polygon intersects itself the procedure gives a result whose interpretation is suggested by the example:

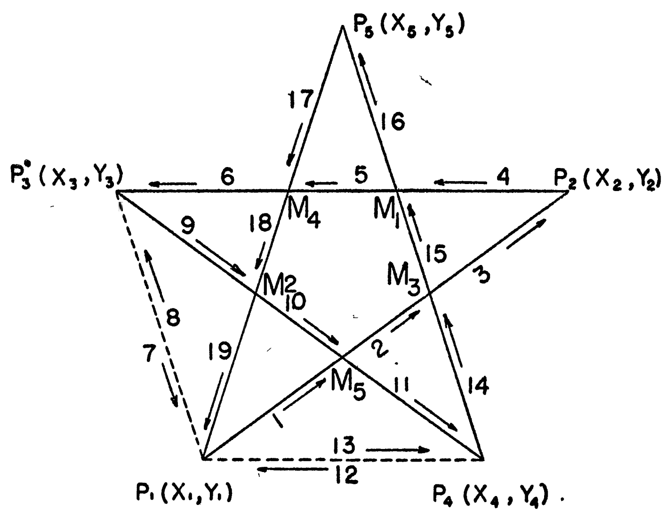

Fig. 2

Following the arrows around the perimeter we note that **7** and **8** and also **12** and 13 cancel with each other. The area given by the rule represents the sums of the areas of:

$$\Delta P_1M_5M_2P_1+\Delta P_4M_3M_5P_4+\Delta P_2M_1M_3P_2+\Delta P_5M_4M_1P_5+\Delta P_3M_2M_4P_3$$
$$+\,2(\text{Pentagon } M_1M_4M_2M_5M_3M_1).$$

3. Closed curves. Through a limiting process the rule gives a formula for the area bounded by a closed curve. For example divide the arc of the first quadrant

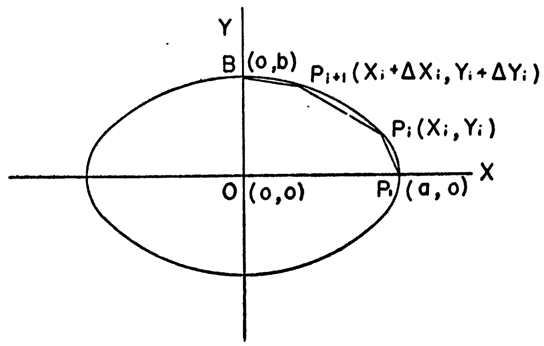

Fig. 3

of an ellipse into n divisions, equal or unequal. Let P_i and P_{i+1} be two consecutive points of division, and apply the rule to the polygon $P_1 \cdots P_iP_{i+1} \cdots BOP_1$. We have the array:

$$\begin{array}{llllll} x_1(=a) \cdots x_i & x_i+\Delta x_i(=x_{i+1}) \cdots & x_{n+1}(=0) & 0 & a \\ y_1(=0) \cdots y_i & y_i+\Delta y_i(=y_{i+1}) \cdots & y_{n+1}(=b) & 0 & 0 \end{array}$$

So Area $P_1 \cdots P_nBOP_1 = \frac{1}{2} \sum_{i=1}^{n} x_i(y_i+\Delta y_i) - y_i(x_i+\Delta x_i)$. The limit of this area gives the area of the quadrant of the ellipse. Hence from the fundamental theorem of calculus:

$$\text{(2)} \qquad \text{Area Quadrant of Ellipse} = \frac{1}{2}\int_{P1}^{B} xdy - ydx.$$

This formula immediately generalizes to give the area of any closed curve as a line integral around its boundary, C:

$$\text{(3)} \qquad \text{Area} = \frac{1}{2}\int_{C} xdy - ydx.$$

This result, of course, is a standard one, but it is usually proved in a different fashion.

4. Polar coördinates. In polar coördinates the area of the triangle OP_1P_2O where P_1 is (ρ_1, θ_1) and P_2 is (ρ_2, θ_2) is given by:

$$\text{(4)} \qquad \text{Area } OP_1P_2O = \tfrac{1}{2}\rho_1\rho_2 \sin(\theta_2 - \theta_1).$$

Hence the area of any polygon of n sides is

$$\text{(5)} \qquad \text{Area } P_1P_2 \cdots P_nP_1 = \frac{1}{2}\sum_{i=1}^{n} \rho_i\rho_{i+1} \sin(\theta_{i+1} - \theta_i)$$

assuming $(\rho_{n+1}, \theta_{n+1})$ is (ρ_1, θ_1).

Passing to the case of a curve, we have that the area between radii whose polar angles are α and β is given by

$$\begin{aligned}\text{Area} &= \frac{1}{2}\lim_{n\to\infty}\sum_{i=1}^{n}\left[\rho_i(\rho_i+\Delta\rho_i)\sin\Delta\theta_i\right]\\ &= \frac{1}{2}\lim_{n\to\infty}\sum_{i=1}^{n}\rho_i^2\Delta\theta_i + \frac{1}{2}\lim_{n\to\infty}\sum_{i=1}^{n}\rho_i\Delta\rho_i\Delta\theta_i\\ &= \frac{1}{2}\int_\alpha^\beta \rho^2 d\theta. \qquad (6)\end{aligned}$$

For $\lim_{x\to 0}(\sin x/x)=1$ and $\Delta\rho_i\Delta\theta_i$ is an infinitesimal of higher order than the first.

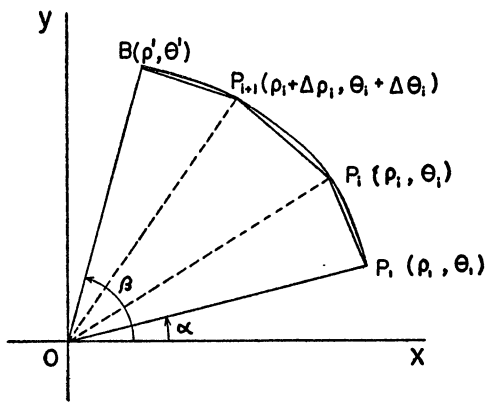

Fig. 4

CORRECTION

In the article, *A Problem of Collinear Points*, by H. S. M. Coxeter, this MONTHLY, vol. 55, p. 27 the first displayed formula should read:

$$x^3 + y^3 + z^3 = xyz = 0.$$

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 811. *Proposed by H. D. Larsen, Albion College, Michigan*

A, B, and C participate in a novel dart game, the targets consisting of three small balloons marked A, B, and C, respectively. At each turn one dart is thrown, the order of the turns being determined in advance by drawing lots. As soon as a balloon is hit and destroyed, the owner of that balloon is eliminated from the game. The balloons are placed in such a manner that there is no danger of destroying a balloon by a dart aimed at another balloon. It is known by all participants that A can hit a balloon 4 out of 5 times, B 3 out of 5 times, and C 2 out of 5 times; this knowledge is used by each player to his best advantage. What is each contestant's chance of winning the game?

E 812. *Proposed by Monte Dernham, San Francisco, California*

Find the shortest perimeter common to two different primitive Pythagorean triangles.

E 813. *Proposed by C. W. Trigg, Los Angeles City College, California*

Let S be the sum of the integer elements of a magic square of order three, and let D be the value of the square considered as a determinant. Show that D/S is an integer.

E 814. *Proposed by Sidney Kravitz, New York, New York*

Given the curve $y=e^x/x$. Consider all areas under the curve, over the x-axis, and between two ordinates one unit apart. Locate the boundary lines of the area which is a minimum.

E 815. *Proposed by J. S. Miller, Dillard University*

A marble rolls in a hemispherical bowl. Find its period.

SOLUTIONS

Another Ball in a Vase Problem

E 781 [1947, 412]. *Proposed by P. D. Thomas, Navy Department, Washington, D.C.*

A heavy ball is gently dropped into a vase full of water. A section through the vertical axis of the vase is a semiellipse, the height being the semimajor axis, the diameter being then the minor axis. The size of the ball is such as to cause

the maximum displacement. (1) Find the radius of the ball. (2) Show that the plane of the circle of tangency bisects the height of the submerged segment of the ball. (See E 687 [1946, 334].)

Solution by the Proposer. Let the semimajor and semiminor axes of the semi-ellipse be a and b respectively. Let the major and minor axes be the x and y axes, and C the origin. Then a point P on the ellipse has coordinates

$$P: \quad (x, b\sqrt{1 - x^2/a^2}).$$

The normal at P meets the x-axis in the point

$$G: \left(\frac{a^2 - b^2}{a^2} x, 0\right).$$

The radius R of the variable ball is equal to GP; that is

$$GP = R = b\sqrt{1 + (b^2 - a^2)x^2/a^4}. \tag{1}$$

The height h of the segment of the ball submerged is $CG+R$, or

$$h = CG + R = (a^2 - b^2)x/a^2 + R, \tag{2}$$

where R is given by (1). The volume of a spherical segment is

$$V = \pi h^2(3R - h)/3,$$

whence

$$dV/dx = \pi h[(2R - h)dh/dx + hdR/dx]. \tag{3}$$

The second factor of (3), with the values of R, h, dR/dx, dh/dx obtained from (1) and (2), becomes

$$a^2b\sqrt{1 + (b^2 - a^2)x^2/a^4} = (a^2 + b^2)x,$$

whence we find

$$x_{\max} = ab/(a^2 + 3b^2)^{1/2},$$

and therefore from (1)

$$R_{\max} = b(a^2 + b^2)/a(a^2 + 3b^2)^{1/2}.$$

Computing h from (2) we find

$$h = 2ab/(a^2 + 3b^2)^{1/2} = 2x_{\max},$$

which shows that the plane of the circle of tangency bisects the height h of the submerged spherical segment.

Also solved by Ragnar Dybvik.

Numbers of the Form $x^2+kxy+y^2$

E 782 [1947, 412]. *Proposed by Joseph Rosenbaum, The Milford School, Connecticut*

Show that the product of two numbers each of the form $x^2+kxy+y^2$, where k, x, y are integers, k fixed, is also of that form. If x and y are relatively prime, are all factors of $x^2+kxy+y^2$ also of the same form?

Solution by Leo Moser, University of Manitoba. The first part is verified by the identity

$$(a^2 + kab + b^2)(c^2 + kcd + d^2) = x^2 + kxy + y^2,$$

where

$$x = ac - bd, \qquad y = ad + bc + kbd.$$

The question is answered in the negative by taking $x=4$, $k=4$, $y=5$. Then

$$x^2 + kxy + y^2 = 121.$$

But we cannot have

$$a^2 + 4ab + b^2 = 11$$

inasmuch as

$$a^2 \equiv 0 \quad \text{or} \quad 1 \pmod 4,$$

and thus

$$a^2 + 4ab + b^2 \equiv 0, 1, \quad \text{or} \quad 2 \pmod 4.$$

Also solved, in part, by the proposer.

Enumeration of Triangles

E 783 [1947, 412.] *Proposed by C. D. Olds, San Jose State College*

Given a parallelogram and its diagonals. Let each side of the parallelogram be divided into n equal parts and let lines be drawn through the points of division, parallel to the sides and to the diagonals of the parallelogram. Find the total number of triangles in the resulting figure.

Editorial Note. L. S. Shively pointed out that this essentially is the same as problem 3264 (of this MONTHLY), which he proposed, and whose solution can be found on p. 212 of Vol. 35 (1928). Leo Moser stated that the problem has been treated (at least for special values of n) by several puzzlers. Thus, on p. 86 of *Mathematics Clubs and Recreations* by S. I. Jones, is found the case where $n=3$, with the incorrect solution 150. The number of triangles for the general case is

$$\{12n^3 + 18n^2 + 4n - 1 + (-1)^n\}/4.$$

Salt Solutions

E 785 [1947, 412]. *Proposed by R. J. Walker, Cornell University*

Each of $n-1$ tanks, $T_1, \cdots, T_{n-1}$, holds V gallons of water, and an nth tank, T_n, holds V gallons of a salt solution containing M pounds of salt. Liquid is circulated at the rate of g gallons per minute from T_n to T_{n-1}, T_{n-1} to $T_{n-2}, \cdots, T_2$ to T_1, T_1 to T_n. How much salt is in T_n after t minutes?

Solution by Norman Miller, Queen's University, Ontario. Suppose that tank T_i contains x_i pounds of salt t minutes after the beginning of the operation. In Δt minutes thereafter T_i receives from T_{i+1} $gx_{i+1}\Delta t/V$ pounds and loses to T_{i-1} $gx_i\Delta t/V$ pounds, correct to infinitesimals of higher order. This leads to the differential equation $dx_i/dt=g(x_{i+1}-x_i)/V$, or $(D+k)x_i=kx_{i+1}$, where $k=g/V$. The complete set of equations is

$$\begin{aligned}(D+k)x_1 &= kx_2\\ (D+k)x_2 &= kx_3\\ &\cdots\cdots\\ (D+k)x_n &= kx_1\end{aligned} \tag{1}$$

From (1), by differentiation and substitution, we obtain the following equations, each in one dependent variable:

$$(D+k)^n x_i = k^n x_i, \qquad i=1, 2, \cdots, n. \tag{2}$$

The set (2) is to be solved subject to the initial conditions: $x_1=x_2=\cdots=x_{n-1}=0$, $x_n=M$, when $t=0$.

The auxiliary equation of (2) is $(D+K)^n=k^n$, whose roots are 0, $k(\omega-1)$, $k(\omega^2-1), \cdots, k(\omega^{n-1}-1)$, where $1, \omega, \omega^2, \cdots, \omega^{n-1}$ are the n nth roots of unity. It follows that

$$x_1=\sum_{i=1}^{n} c_i e^{k(\omega^{i-1}-1)t}.$$

The first equation of (1) gives x_2 in terms of the same constants:

$$x_2=\sum_{i=1}^{n} \omega^{i-1}c_i e^{k(\omega^{i-1}-1)t},$$

and similarly for $x_3, \cdots, x_n$. The last equation is

$$x_n=\sum_{i=1}^{n} \omega^{(i-1)(n-1)}c_i e^{k(\omega^{i-1}-1)t}.$$

To find the c's we have when $t=0$,

$$\sum_{i=1}^{n} c_i = 0$$

$$\sum_{i=1}^{n} \omega^{i-1} c_i = 0 \tag{3}$$

$$\cdots\cdots\cdots\cdots$$

$$\sum_{i=1}^{n} \omega^{(i-1)(n-1)} c_i = M$$

By using the properties $1+\omega+\omega^2+\cdots+\omega^{n-1}=0$, $\omega^n=1$, we get from (3), by addition, $c_1=M/n$. By multiplying the equations of (3) in order by $1, \omega, \omega^2, \cdots, \omega^{n-1}$ and adding, we get $c_n=\omega^{n-1}M/n$. By using multipliers $1, \omega^2, \omega^4, \cdots, \omega^{2(n-1)}$, we get $c_{n-1}=\omega^{n-2}M/n$, etc. Finally, $c_2=\omega M/n$. It follows that

$$x_n = (M/n)\sum_{i=1}^{n} e^{k(\omega^{i-1}-1)t}$$

$$= (M/n)\left[1 + e^{-kt}\sum_{i=1}^{n-1} e^{k\omega^i t}\right].$$

Of the terms in the summation, those equidistant from the ends are either real or conjugate imaginaries. Hence the sum is real and can be expressed without the use of imaginaries. Thus, for $n=2, 3, 4$, the values of x_n reduce to

$$x_2 = (M/2)[1 + e^{-2gt/V}],$$
$$x_3 = (M/3)[1 + 2e^{-3gt/2v}\cos(\sqrt{3}\,gt/2V)],$$
$$x_4 = (M/2)e^{-gt/V}(\cos gt/V + \cosh gt/V).$$

Also solved by Joseph Rosenbaum (partially), Dagnar Dybvik, Hugh Stelson, and the proposer.

The proposer showed that

$$x_n = (M/n)(1 + 2Q) \qquad \text{if } n = 2m+1,$$
$$x_n = (M/n)(1 + e^{-2gt/V} + 2Q) \qquad \text{if } n = 2m+2,$$

where

$$Q = \sum_{j=1}^{m} e^{(-1+\cos 2\pi j/n)gt/V}\cos[(\sin 2\pi j/n)gt/V].$$

In either case x_n (and also $x_1, x_2, \cdots, x_{n-1}$) approaches M/n as t increases.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4290. *Proposed by P. T. Bateman, Yale University*

The function $-\log |2 \sin \frac{1}{2}x|$ has the Fourier series

$$\cos x + \tfrac{1}{2}\cos 2x + \tfrac{1}{3}\cos 3x + \cdots .$$

Prove that no partial sum of the series is ever less than -1.

4291. *Proposed by J. P. Ballantine, University of Washington, Seattle*

$$\begin{matrix} 1 & 2 & 3\cdots & n \\ 0 & 1 & 2\cdots & n-1 \\ 0 & 0 & 1\cdots & n-2 \\ \cdot & \cdot & \cdot\cdots & \ddots \\ 0 & 0 & 0\cdots & 1 \end{matrix}$$

Consider the square array with 1's down the principal diagonal, increasing consecutive integers above the diagonal, and zeros below. (a) How many different paths of $i+j-2$ steps are possible from the 1 in the upper left corner to the element in the ith row and jth column, $j \geqq i$, without passing through any zero element?

(b) Define the value of each path as the product of the elements through which the path passes, not counting the terminal elements. Show that the sum of the values of all the paths in (a) is the coefficient of $(\tan x)^{j-i+1}$ in the $(j+i-2)$nd derivative of $\tan x$ with respect to x, expressed in terms of $\tan x$.

4292. *Proposed by R. Goormaghtigh, Bruges, Belgium*

If s and ρ are the arc length and the radius of curvature of a plane curve Γ at a variable point M, and if it be required that s have a constant ratio to the distance of M from a fixed point, then Γ must be a cycloidal curve and

$$\lambda^2 s^2 - \rho^2 = a^2,$$

λ and a being constants.

Prove that, in the case of a twisted curve, the condition is

$$\lambda^2 s^2 - \rho^2 = \left(\int \frac{\rho}{\tau} ds + a\right)^2,$$

τ being the radius of torsion of Γ at M.

4293. *Proposed by H. F. Sandham, Trinity College, Ireland*

Evaluate

$$\sum_{1}^{\infty} \frac{\left(\frac{3-\sqrt{5}}{2}\right)^n}{n^3}.$$

4294. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The lines joining the orthocenter of a triangle to the points of intersection of the medians with the nine-point circle (other than the mid-points of the sides), pass through the vertices of parabolas tangent to two sides of the triangle and having the third side for chord of contact.

SOLUTIONS

Feuerbach Hyperbola

4169 [1946, 103]. *Corrected. Proposed by Victor Thébault, Tennie, Sarthe, France*

The tangents at the vertices A, B, C of a given triangle to its Feuerbach hyperbola form a triangle whose conjugate circle is tangent to the nine-point circle of ABC at its Feuerbach point.

*Solution by the Proposer.**

I. LEMMA: *The polar circle* [1] *of a triangle $A_1B_1C_1$ circumscribed about an equilateral hyperbola (H) and the nine-point circle of the triangle ABC, having as its vertices the points of contact of B_1C_1, C_1A_1, A_1B_1, are tangent at the center ω of (H).*†

Since the equilateral hyperbola (H) is inscribed in triangle $A_1B_1C_1$ and circumscribed about triangle ABC, its center ω is on the polar circle (h) of $A_1B_1C_1$ and on the nine-point circle (O_9) of triangle ABC. The two circles (h) and (O_9) are tangent at ω, for if they had another point ω' in common, there would exist a second equilateral hyperbola (H') passing through A, B, C and inscribed in $A_1B_1C_1$, which is impossible.

Note. If (H) varies, remaining inscribed in triangle $A_1B_1C_1$, the polar circle (h) of $A_1B_1C_1$ is the envelope of the circle (O_9) of the triangle ABC of the points of contact. If (H) varies, remaining circumscribed about ABC, the circle (O_9) of ABC is the envelope of the polar circle (h) of the triangle formed by the tangent lines at A, B, C.

* Translated and checked by W. E. Byrne, Virginia Military Institute.

† V. Thébault, Mathesis, t. LIV, Supplément, p. 38.

II. THEOREM: *The cevians* [2] *of an arbitrary point* Q *of the equilateral hyperbola* (H) *circumscribed about a triangle* ABC *meet the sides* BC, CA, AB *at* D, E, F. *If the sides* $B'C'$, $C'A'$, $A'B'$ *of the orthic triangle* $A'B'C'$ *of* ABC *cut the sides* EF, FD, DE *of triangle* DEF *at* M, N, P, *the triangle* MNP *is circumscribed about the hyperbola* (H) *touching it at* A, B, C, *and its polar circle* (h) *is tangent to the nine-point circle* (O_9) *of triangle* ABC *at the center* ω *of* (H).

If the point Q varies on the equilateral hyperbola (H), to a new position Q_1 there corresponds a triangle $D_1E_1F_1$, the sides of which pass through the points M, N, P [3]. The tangents of (H) at A, B, C, which correspond to three particular positions of the point Q, pass through M, N, P. Hence the triangle MNP is circumscribed about (H) at A, B, C. From the preceding lemma we conclude that the polar circle (h) of triangle MNP is tangent to the nine-point circle (O_9) of triangle ABC at the center ω of (H). Thus the theorem is proved.

Note. Triangle MNP, the vertices of which are at finite distance, is obtuse-angled.* Hence circle (h) is always real.

III. The Feuerbach hyperbola (H) [4], corresponding to the incenter or to one of the three excenters of triangle ABC, contains not only the orthocenter but the corresponding Gergonne point as well. For each Feuerbach hyperbola of triangle ABC the triangle MNP has as its vertices the points of intersection of the sides $B'C'$, $C'A'$, $A'B'$ of the orthic triangle with the sides EF, FD, DE of the pedal triangle of the incenter (or excenter) with respect to ABC. Triangle MNP is circumscribed about (H) at A, B, C, and its polar circle (h) is tangent to the nine-point circle (O_9) of ABC at the center ω of (H), that is, at the Feuerbach point associated with the incircle (or excircle). It is also known that the incircle (or excircle) is the polar circle (h) of MNP.† Our question 4169 leads therefore to a possibly new demonstration of Feuerbach's theorem: *In every triangle the incircle and the excircles are tangent to the nine-point circle.*

We may extend our results by proving the following:

IV. THEOREM: *The polar circle of a triangle* $A_1B_1C_1$ *circumscribed about an equilateral hyperbola* (H), *of center* ω, *is orthogonal to the circle circumscribed about the triangle* $\alpha\beta\gamma$, *whose vertices are symmetric to* ω *with respect to* B_1C_1, C_1A_1, A_1B_1.

If we consider a circle (C), the center of which is the orthocenter H_1 of triangle $A_1B_1C_1$, the locus of points P in its plane such that the circle passing through the points symmetric to P with respect to B_1C_1, C_1A_1, A_1B_1 is orthogonal to (C) consists of the circumcircle of $A_1B_1C_1$ and of another circle of center H_1. The radius of this latter circle (H_1, σ), which is the proper locus of P, may be determined in the following manner. (H_1, σ) cuts B_1C_1 at the points S, S' of intersection of B_1C_1 and the circle of center A_1 orthogonal to (C). Let r designate the radius of (C). If A_1' is the foot of the altitude A_1H_1 on B_1C_1, we have

* Ad. Mineur, Mathesis, 1937, p. 434.

† C. Servais, Mathesis, 1915, p. 134.

$$\overline{SA_1'}^2 = \overline{A_1H_1}^2 - r^2 - \overline{A_1A_1'}^2$$
$$\sigma^2 = \overline{H_1S}^2 = \overline{A_1H_1}^2 - \overline{A_1A_1'}^2 + \overline{H_1A_1}^2 - r^2$$
$$= 2\overline{H_1A_1}\cdot\overline{H_1A_1'} - r^2 = 2\rho^2 - r^2$$

where ρ is the radius of the polar circle of $A_1B_1C_1$. Hence, if we consider a point P of a circle (C) having as its center the orthocenter H_1 of a triangle $A_1B_1C_1$ and as its radius r, the circle passing through the points α, β, γ symmetric to P with respect to B_1C_1, C_1A_1, A_1B_1 is orthogonal to the circle (H_1, σ). As a special case we conclude that if P is a point of the polar circle (H_1, ρ) of $A_1B_1C_1$, the circle $\alpha\beta\gamma$ is orthogonal to (H_1, ρ). In other words: the circumcircle of triangle $\alpha\beta\gamma$, where α, β, γ are the symmetric points of the center ω of the equilateral hyperbola inscribed in triangle $A_1B_1C_1$ with respect to B_1C_1, C_1A_1, A_1B_1, is orthogonal to the polar circle of $A_1B_1C_1$.

By combining Theorems 2 and 4, we obtain the following:

THEOREM: *The tangents at the vertices A, B, C of a triangle ABC to any of its four Feuerbach hyperbolas form a triangle $A_1B_1C_1$ whose polar circle is tangent to the nine-point circle of ABC at the corresponding Feuerbach point ϕ and is orthogonal to the circumcircle of triangle $\alpha\beta\gamma$, the vertices of which are the points symmetric to ϕ with respect to B_1C_1, C_1A_1, A_1B_1.*

Note. In general, a triangle $A_1B_1C_1$ being given, the locus of the points P in its plane such that the circle passing through the points symmetric to P with respect to B_1C_1, C_1A_1, A_1B_1 is orthogonal to a fixed circle (C) is a bicircular quartic (K) circumscribed about $A_1B_1C_1$. (K) intersects the circumcircle of $A_1B_1C_1$ at the focus P of the parabola of directrix PH_1 inscribed in $A_1B_1C_1$. (K) intersects B_1C_1 at the points in which the circle of center A_1, orthogonal to (C), cuts B_1C_1. If (C) is orthogonal to the circumcircle of $A_1B_1C_1$, (K) passes through the orthocenter H_1 of $A_1B_1C_1$. If the center of (C) is the orthocenter H_1, the quartic degenerates as was stated above.

Notes by the translator.

1. Court, College Geometry, p. 155.
2. Court, College Geometry, p. 128.
3. The point M is the pole of BC with respect to (H). If a conic Γ is circumscribed about a triangle ABC the necessary and sufficient condition that the pole of BC with respect to Γ lie on the side $B'C'$ of the orthic triangle is that the conic Γ be an equilateral hyperbola.
4. The equilateral hyperbola determined by the three vertices of a triangle ABC and the incenter (or one of the excenters) is called the Feuerbach hyperbola of ABC. There are four of them. The Gergonne point is the point common to the lines drawn from the vertices A, B, C to the points of contact of the incircle (or of the excircle) with the opposite sides of ABC. Usually, in American texts, only the Gergonne point associated with the incircle is known by that name.

Arc with a Maximum Property

4232 [1947, 49]. *Proposed by H. D. Ruderman, New York City*

$A:(0, 0)$ and $B:(0, u)$ are joined by the straight line segment AB and a curve

Π to enclose a region of area s. Let k represent the length of Π from A to B. Find the equation of Π such that the ratio s/k^2 is a maximum.

Solution by Abraham Miller, Washington, D. C. Let Π' be the reflection of Π with respect to AB, and consider the resulting closed curve $\Pi+\Pi'$. Its area, $2s$, cannot exceed the area of a circle of circumference $2k$ (radius k/π). That is, $2s\leq k^2/\pi$ or $s/k^2 \leq \frac{1}{2}\pi$. The equality sign holds if and only if $\Pi+\Pi'$ is a circle, and in this case s/k^2 is a maximum. Hence Π is a semicircle with AB as diameter, the equation being

$$x^2 + y^2 - uy = 0.$$

Also solved by R. C. Buck, R. Lessard, H. J. Zimmerberg, and the Proposer.

Determinant Divisible by a Power of $x-1$

4235 [1947, 112]. *Proposed by Irving Kaplansky, University of Chicago, and D. C. Lewis, University of New Hampshire.*

Show that the determinant

$$\begin{vmatrix} (x-1)/1 & (x^2-1)/2 & \cdots & (x^n-1)/n \\ (x^2-1)/2 & (x^3-1)/3 & \cdots & (x^{n+1}-1)/(n+1) \\ \cdots & \cdots & \cdots & \cdots \\ (x^n-1)/n & (x^{n+1}-1)/(n+1) & \cdots & (x^{2n-1}-1)/(2n-1) \end{vmatrix}$$

is a constant times $(x-1)^{n^2}$.

Solution by Fritz Herzog, Michigan State College. We shall first prove the following:

THEOREM. *Let $f_1(x), f_2(x), \cdots, f_n(x), g_1(x), g_2(x), \cdots, g_n(x)$ be $2n$ polynomials and let $\phi_{ij}(x)=\int_a^x f_i(u)g_j(u)du$. Then the determinant $|\phi_{ij}(x)|$ vanishes at $x=a$ with a multiplicity of at least n^2.*

We have to show that, if $F(x)$ denotes the given determinant, $F^{(k)}(a)=0$ for $0\leq k<n^2$. Using the well-known method of differentiating a determinant (by rows, say), we obtain

$$F^{(k)}(x) = \sum [k!/(k_1!k_2!\cdots k_n!)]\,|\phi_{ij}^{(k_i)}(x)|, \tag{1}$$

where $\sum$ is taken over all sets of non-negative integers $k_1, k_2, \cdots, k_n$ with $k_1+k_2+\cdots+k_n=k$. Let $F^{(k)}(x)=G_k(x)+H_k(x)$, where $G_k(x)$ consists of those terms in (1) in which at least one of the k_i is zero and $H_k(x)$ of those in which all k_i are positive. Since $\phi_{ij}(a)=0$ we have at once $G_k(a)=0$ (which completes the proof when $0\leq k<n$). We shall now show that $H_k(x)=0$ identically in x and may henceforth omit the variable x in our notation.

Since $\phi_{ij}'=f_ig_j$ we have for $i,j=1, 2, \cdots, n$ and $k_i\geq 1$

$$\phi_{ij}^{(k_i)} = \sum [(s_i+t_i)!/(s_i!t_i!)]f_i^{(s_i)}g_j^{(t_i)}, \tag{2}$$

where $\sum$ is taken over all non-negative values of s_i and t_i with $s_i+t_i=k_i-1$. Substituting (2) for the elements of $|\phi_{ij}^{(k_i)}|$ in each term of (1) that enters into H_k, we obtain a determinant each element of whose ith row is written as a sum of k_i terms. Thus we can write this determinant as a sum of $k_1k_2\cdots k_n$ determinants and arrive, after further simplifications, at the following expression for H_k:

$$(3)\qquad \sum \frac{k!}{(s_1+t_1+1)\cdots(s_n+t_n+1)}\cdot\frac{f_1^{(s_1)}\cdots f_n^{(s_n)}}{s_1!\cdots s_n!t_1!\cdots t_n!}\cdot\left|g_j^{(t_i)}\right|,$$

where $\sum$ is taken over all sets of non-negative integers $s_1, s_2, \cdots, s_n, t_1, t_2, \cdots, t_n$ with $\sum_i(s_i+t_i)=k-n$.

Those terms in (3) in which at least two t_i are equal vanish. In the remaining terms (in which the t_i are distinct) at least two s_i are equal to one another, since otherwise $k-n=\sum_i(s_i+t_i)\geq n(n-1)$, which contradicts $k<n^2$. These remaining terms of (3) can now be paired according to the following principle: In any such term of (3), let m be the smallest index for which s_m equals one of the other s_i and let $r(>m)$ be the smallest index for which $s_r=s_m$. Then we associate with this term the one with the same values of the s_i and the t_i, except that the values of t_m and t_r are interchanged. It is easily seen that this constitutes actually a pairing of the remaining terms of (3) and that the sum of the members of each such pair vanishes. This completes the proof of the Theorem.

We now put $f_i(x)=g_i(x)=x^{i-1}$, $i=1, 2, \cdots, n$, and $a=1$; we obtain $\phi_{ij}(x)=(x^{i+j-1}-1)/(i+j-1)$ and hence, by the Theorem, the determinant given in the problem is divisible by $(x-1)^{n^2}$. It is easily verified that each term in the expansion of this determinant is a polynomial of degree n^2 and thus we obtain $|(x^{i+j-1}-1)/(i+j-1)|=A_n(x-1)^{n^2}$, where $A_n=|1/(i+j-1)|$ depends on n only.

Remarks: 1. The Theorem, proved in the above solution, is a generalization of Problem 2 of Part I of the Putnam Mathematical Competition for 1946 (see this MONTHLY, vol. 53 (1946), p. 484).

2. The value of the constant A_n is given by

$$A_1=1;\qquad A_n=\prod_{r=1}^{n-1}(2r+1)^{-1}(C_{2r,r})^{-2},\qquad n=2,3,\cdots.$$

Also solved by R. P. Brady, H. E. Bray, Robert Breusch, William Gustin, W. J. Harrington, Ivan Niven, J. W. Popow, J. T. Tate, Jr., F. Underwood, John Williamson, and the Proposer.

Editorial Note. Most of the other solvers found transformations to establish the equality of the given determinant with the determinant $|(x-1)^{i+j-1}/(i+j-1)|$, whence the proof follows immediately.

Stern's Diatomic Series

4236 [1947, 112]. *Proposed by H. D. Grossman, New York City*

Write down two 1's, then a 2 between them, then a 3 between any two numbers whose sum is 3, then a 4 between any two numbers whose sum is 4, and so forth. Prove that the number of n's written down is $\phi(n)$, ($n>1$, ϕ is Euler's totient).

Solution by Leo Moser and R. Steinberg, University of Toronto. The sequence obtained after the n's have been inserted is precisely the sequence of denominators in the Farey series F_n. Since F_n consists of all irreducible fractions from 0 to 1, inclusive, with denominators not exceeding n, it will contain $\phi(n)$ irreducible fractions with denominator n, so that this will be sufficient to prove the theorem.

If h/k, h''/k'', h'/k' are three successive terms of F_n, and if h''/k'' occurs in F_n but not in F_{n-1}, then $h+h'=h''$ and $k+k'=k''$. (See Hardy and Wright, *The Theory of Numbers*, second edition, p. 25.) Thus in going from F_{n-1} to F_n we insert terms with denominator n exactly between those terms whose denominators sum to n. Hence, by induction, our statement is proved.

Also solved by P. T. Bateman, Howard Eves, William Gustin, L. M. Kelly, Y. S. Luan, and Ivan Niven.

Editorial Note. Howard Eves remarks that the sequences here obtained were investigated by Stern in the *Journal für Mathematik*, v. 55, p. 193. The present result is one of many interesting facts proved by Stern. Further study of these sequences was made by D. H. Lehmer, who obtained a curious tie-up with the famous Fibonacci series. Some of Lehmer's results may be found in this MONTHLY [1929, 59–67].

These sequences are also closely related to those treated by Williams and Browne, *A Family of Integers and a Theorem on Circles* (this MONTHLY, 1947, 534–536). Theorem 7 of this article applies, with almost no change, as a proof of the present problem.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Compléments de Géométrie Plane. By R. Deaux. Bruxelles, Maison D'Edition A. de Boeck, 1945. 7+150 pages. 45 Fr.

This small book of 156 pages is devoted to materials comprising that which is generally called College Geometry in America. Its avowed purpose is to help prepare the participants in the competitive examinations for admission to the Polytechnical Faculties of Mons. The three chapters are entitled: I Properties of segments of lines, II Properties of circles, and III Two geometric transformations. Among the topics treated in Chapter I are the usual theorems of Möbius, of Menelaus and Ceva, and those of Papus, Pascal and Brianchon (with respect to the circle). Signed segments, areas of triangles, and angles are used consistently. A rather complete discussion of cross-ratio is given and applications are made to problems of concurrence of lines and collinearity of points and to proofs of Desargues' theorem and its dual. In Chapter II are considered the theorems concerning poles and polars with respect to a circle, radical and antiradical axes, orthogonality of circles and pencils of circles. In Chapter III are treated the similarity transformations and inversion in quite complete detail.

Probably the most important part of the text is that devoted to problems, of which there are 383, grouped according to the sections in the text proper. These vary from quite easy examples of the theory to rather difficult problems.

In general one may say that the text covers considerable, but not all, of the usual topics in College Geometry, but at the same time includes topics often relegated to a chapter on metric applications in our Synthetic Projective Geometries. Definitions of terms are carefully made, and theorems accurately demonstrated.

V. G. GROVE

Introduction a la Géométrie des Nombres Complexes. By R. Deaux. Bruxelles, Maison D'Édition A. de Boeck, 1947. 163 pages. 100 Fr.

The chief purpose of this book, as the author explains, is to exploit the use of complex numbers as a tool for geometric investigations. The material in the first part of the book has been used by the author in a course for electrical engineers, and this flavor can be observed in some of the choice of material. However, the writer has chosen to keep the book definitely in the realm of pure geometry rather than to delve into applications, and only a fraction of the material would be of great interest to the engineer.

The first chapter lays the basic groundwork for the discussion, including the usual geometric interpretation of algebraic operations on complex numbers,

and the relation of complex numbers to vectors in the plane. After a brief discussion of the algebraic forms of such simple transformations as reflection, rotation, translation and inversion, the chapter concludes with a consideration of cross ratio. This includes a study of the geometric meaning of harmonic division in the complex plane, and construction problems related to harmonic division and harmonic quadrangles.

In chapter two the techniques developed are applied to problems in plane analytic geometry, and two main topics are selected for consideration. The first is a study of curves that can be generated as a sum of two vectors from a common point, each rotating at constant speed. These are shown to coincide with the cycloidal curves which are described by a fixed point of a moving circular disc as it rolls without slipping on a fixed circle. In particular, the ellipse is shown to be such a curve, and the relation of the geometric properties of the ellipse to the generating vectors developed. The second main topic is that of unicursal curves, that is, curves whose complex equation may be written parametrically as a quotient of polynomials in the real variable t. Special emphasis is laid on the cases when the polynomials are linear or quadratic. These curves include the straight line, the circle, all the conics, the unicursal bicircular quartics, and the unicursal circular cubics. Various geometric properties of the curves, such as center and foci of the conics, and double points of the cubics and quartics are neatly obtained in terms of the parametric representations.

Chapter three is devoted to a detailed discussion of the direct circular transformations or homographies and a somewhat less complete consideration of the indirect circular transformations of antihomographies.

It is assumed in the book that the reader has a working knowledge of projective geometry, since the author draws on this freely in places. A more explicit reference to some of the results thus assumed from projective geometry in a few places would add clarity for the casual reader. The book as a whole seems clearly written and typographical errors surprisingly few. While naturally the material considered is not new, the development is interesting, and the book should be a welcome addition to other literature in the field.

S. B. JACKSON

Elements of Symbolic Logic. By Hans Reichenbach. The Macmillan Company, 1947. 13+444 pages. $5.00.

This book is intended as a college textbook for a first course in logic. Since the emphasis is more on general ideas, than on the techniques of manipulating symbolic formulas, the work would probably be more suited for a first course in a philosophy, than in a mathematics, department. But it might reasonably be put on a reading list for advanced mathematical students of logic, along with the more recent works of Carnap: mathematicians approach philosophical ideas diffidently, and late in life.

Roughly the first half of the book is taken up with an exposition of the propositional calculus, the lower calculi of predicates and of classes, and the

higher calculus of predicates. This part follows essentially the usual treatment of these topics. A beginning mathematical student (if he knew German) might find it more agreeable to become acquainted with this material through the elegant deductive presentation of Hilbert and Ackermann's *Grundzüge der Theoretischen Logik*. But even here the author sometimes illuminates his topics by flashes of original philosophical insight—so that the mature student will find at least something to learn from these pages.

The remainder of the book is taken up with the analysis of conversational language, and with modalities. In connection with the analysis of conversational language, the author makes the point that the analysis offered by traditional grammar is artificial and inadequate. This inadequacy of traditional grammar rests in turn on the fact that the first grammarians were of a generation trained in the inadequate logic of Aristotle: for there is an intimate connection between logic and grammar. The author attempts to reform traditional grammar in the spirit of modern logic. While much of his discussion here is admittedly exploratory and tentative, he makes some contributions which seem to be of permanent value, particularly in his application to conversational language of Russell's theory of descriptions, and in his treatment of the tenses of verbs (which is an extension of an analysis given by Jespersen). On the other hand, in some cases, the author introduces notation which slavishly imitates terms used in ordinary language, but without giving adequate rules for manipulating such symbols: he sometimes, so to speak, merely translates difficult words of English into symbolic language. An example of this is the notation "Θ^*," which is introduced on page 287 to symbolize the reflexive "this" of English (as when we say, "This sentence consists of six words"). It is clear, that the symbol introduced in this way cannot be governed by the same rules that are assumed for ordinary substantives; for example, from the two true sentences

"This sentence is written in roman type, and contains at least twelve words"

and

"This sentence contains at most seven words"

we cannot deduce the false sentence

"This sentence contains at most seven words, and at least twelve words."

If any light is to be thrown on the use of the reflexive "this" of conversational language, exact rules must be laid down which govern its usage; it is not sufficient merely to replace "this" by another symbol "Θ^*."

The final chapter, which is concerned with modalities, presents an original view of this subject—but a view which is rather close to the views of Carnap, in that, in order to define "necessity," use is made of the notion of provability. A specific criticism of the ideas of this chapter will be found in the last paragraph of the review.

The book is written from the point of view of logical empiricism, a philosophical position which holds that a proposition is meaningful only if it is veri-

fiable as true or false, and that two propositions have the same meaning if they obtain the same verification, as true or false, for all possible observations. The author applies the general point of view of empiricism in a forceful manner: as in the passage on page 356, for example, "we do not wish to say that physical necessity is due to invisible forces tying things together, or that it is a law of reason projected into nature, or whatever else has been subtly devised by certain metaphysicians." Or, to take another example, the brilliant analysis, in terms of grammar, on page 273, of some of the grand philosophical systems of the past: "Disregarding the arbitrariness in the choice of the argument-object has led to an unfortunate absolutism in certain philosophical systems. Thus materialism seems to be guilty of an absolutism of thing-arguments; other philosophical systems are on the search for absolute argument-things in the construction of substances beyond material things. On the other hand, insight into the arbitrariness of the argument has led to the mistake of denying the existence of things. . . . We therefore do not agree with philosophical systems that want to abolish things; instead, we consider the definition of the argument-object a matter of volitional decision."

It is unfortunate that a work so remarkable for philosophical common sense, should suffer from grave technical deficiencies. At many points statements are made, which can hardly stand up under rational examination. In some places the text is almost incomprehensible because of a lack of precision in the introduction of defined terms.

These technical inadequacies are perhaps best illustrated by consideration of the definition, on page 369, of an original nomological statement: "An original nomological statement is an all-statement that is demonstrably true, fully exhaustive, and universal." We notice, in the first place, that the condition that the statement be an all-statement has little force, since, if A is any statement without free variables, then A is equivalent to $(x)A$. Moreover, if we turn back to page 264, to consult the definition of "universal statement," we find it asserted that a statement is not a universal statement if it can be written in a tautologically equivalent form which contains an individual-term. Since for every two statements A and B, however, A is tautologically equivalent to $A\cdot(B\supset B)$, we conclude that no statement A is universal (since we can always choose B so that it contains an individual-term)—from which it would follow that original nomological statements themselves do not exist!

J. C. C. McKinsey

A Chapter in the Theory of Numbers. By L. J. Mordell. Cambridge, at the University Press; New York, The Macmillan Company, 1947, 31 pages. $.40.

This little book contains Professor Mordell's inaugural lecture at the University of Cambridge, on the equation $y^2=x^3+k$. The solution of this equation, in integers or rationals x and y, is an example, perhaps on a par with Fermat's $x^n+y^n=z^n$, of a problem which, at first merely difficult or barren, has ultimately led to interesting and significant developments.

A survey is first made of contributions to the equation by Bachet (1621, $k=-2$ and 17, in rational x, y), Fermat (1657, $k=-2$ and 4, in integers), Euler (1738), and others. Fermat's claim to have discovered a "beautiful and subtle method which enabled him to solve such questions in integers," is to be doubted as regards the general equation. Thus, Mordell suggests Fermat could hardly have obtained Nagell's result (1930) that $y^2-17=x^3$ has integral solutions only for $y=3, 4, 5, 9, 23, 282, 375$, and 378661. The first of several examples proved to have no integral solutions, by properties of particular binary quadratic forms and elementary congruences, was that of Lebesgue (1869) who wrote the equation for $k=7$ as $y^2+1=(x+2)(x^2-2x+4)$, and observed that $x+2$ must be positive and of the form $4n+3$. Euler solved $ay^2-kf^2=x^3$ by assuming that $y\sqrt{a}+f\sqrt{k}=(p\sqrt{a}+q\sqrt{k})^3$, whence $y=ap^3+3kpq^2$, $f=3ap^2q+fq^3$; and recognized (without knowing why) that in some cases this assumption fails to give all solutions in integers. The first rigorous proofs of results of some generality were given in 1875, by Pepin, using the Gaussian theory of binary quadratic forms.

A lucid account is given of the bearing on the problem of unique factorization in the quadratic field $R(\sqrt{k})$, and of the connection (suggested above in Euler's example) with the representation of 1 by a binary cubic form. One of the illustrations chosen is $R(\sqrt{6})$, where it is stated there is unique factorization, since the class-number $h=1$. This worried the reviewer for a while, since there are two classes of binary quadratic forms (x^2-6y^2 and $2x^2-3y^2$) of this determinant, but serves to illustrate the no longer one-one correspondence between classes of forms and of ideals when -1 is not a norm. Then there are twice as many classes of forms as of ideals.

The problem of solving in integers the slightly more general equation $Ey^2=Ax^3+Bx^2+Cx+D$ is reduced to that of solving a finite number of equations of the form $ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey^4=1$. Applying Thue's Theorem (1908) it follows that $Ey^2=Ax^3+Bx^2+Cx+D$ has (unless the right member has a repeated factor) only a finite number of integral solutions. Mordell's attempt to extend this to $y^2=Ax^4+\cdots+E$ led him to discover his "basis theorem" for the rational points of a general cubic curve, previously conjectured by Poincaré. Weil's generalization to curves of genus p and Siegel's generalization of Thue's Theorem soon followed. The work of Delaunay and Nagell giving bounds to the number of integral solutions, and Fueter's examples of equations with no rational solutions, are mentioned.

This makes an interesting, well-written chapter in the Theory of Numbers, which, in view of the author's concluding remarks, is not yet closed.

GORDON PALL

NEW BOOKS RECEIVED

Analytic Geometry. By R. S. Underwood and F. W. Sparks, Boston, Houghton Mifflin Co., 1947. 10+225 pages. $2.75.

Newtonian Society, Lehigh University

The Newtonian Society, freshman honorary mathematics society at Lehigh University, held five meetings during the school year 1946–47.

Speakers and topics were:

Curve fitting, by Professor V. V. Latshaw

Euler's formula and the regular polyhedra, by Professor R. R. Stoll

Mapping, by Professor J. O. Chellevold

What is a number?, by Professor T. Hailperin

Officers for the year were: President, D. N. Love; Vice-President-Treasurer, J. M. Christie; Secretary, H. Dowling.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

A.A.A.S. CENTENNIAL CELEBRATION

Dr. Harlow Shapley, Chairman of the Centennial Policy Committee, has announced that the Centennial Celebration of the American Association for the Advancement of Science will be held on September 13–17, 1948 in Washington, D. C. The keynote of the Centenary will be "One World of Science."

SEVENTH INTERNATIONAL CONGRESS OF APPLIED MECHANICS

The Seventh International Congress of Applied Mechanics will be held at the Imperial College of Science and Technology, South Kensington, London, England, on September 5–11, 1948.

Applications and other material concerning the Congress may be secured by writing to the Secretary of the American Mathematical Society, University of Pennsylvania, Philadelphia 4, Pennsylvania.

BACK COPIES OF THE "MONTHLY"

During the past two years, the membership of the Association has grown at such a rapid rate that it has been difficult to estimate in advance the proper number of copies of the MONTHLY which were to be printed each month.

As a result, the office of the Association finds that it is lacking sufficient copies of the MONTHLY for 1946 (especially January, May, June–July and August–September), 1947 (especially January and December) and January 1948.

The Association will be glad to refund postage to any member who wishes to dispose of the above-mentioned numbers of the MONTHLY by contributing them

to the Association. Mail to: Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

PERSONAL ITEMS

The Duodecimal Society of America has conferred the 1948 Annual Award upon Mr. H. C. Robert of Atlanta, Georgia.

The Institute of Mathematical Statistics has announced the election of the following officers: Professor Abraham Wald, head of the Department of Mathematical Statistics of Columbia University, as president; Dr. Churchill Eisenhart, chief of the Statistical Engineering Laboratory of the National Bureau of Standards, as a vice-president; Professor Henry Scheffé of the University of California at Los Angeles, as a vice-president.

The Royal Society of London has awarded the Copley Medal to the late Professor Emeritus G. H. Hardy of the University of Cambridge for his part in the development of mathematical analysis in Britain during the last thirty years.

Associate Director Joseph Slepian of Westinghouse Research Laboratories has received the Edison medal for 1947 from the American Institute of Electrical Engineers.

Professor Hassler Whitney of Harvard University received from Yale University in 1947 an honorary doctorate of science.

Professor R. L. Wilder of the University of Michigan has been elected vice-president of Section A of the American Association for the Advancement of Science.

Associate Professor J. W. T. Youngs of Indiana University has received a Guggenheim fellowship.

Associate Professor E. W. Anderson of Iowa State College of Agriculture and Mechanic Arts has been promoted to a research professorship in mechanical engineering and a professorship in mathematics.

Assistant Professor H. G. Apostle of Amherst College has been appointed to an assistant professorship in the Department of Philosophy of the University of Chicago.

Mr. L. C. Bagby of the Jam Handy Corporation has been appointed to a professorship at the Lawrence Institute of Technology, Detroit, Michigan.

Mr. J. W. Beach, Iowa State College of Agriculture and Mechanic Arts, has been promoted to an assistant professorship.

Professor Arne Beurling will be a visiting lecturer at Harvard University for the coming academic year.

Mr. F. E. Bortle, Iowa State College of Agriculture and Mechanic Arts, has been promoted to an assistant professorship.

Associate Professor A. T. Brauer of the University of North Carolina has been promoted to a professorship.

Assistant Professor A. B. Brown of Queens College has been promoted to an associate professorship.

Professor C. L. Buxton is serving as Director of the Malone Branch of Clarkson College of Technology.

Professor Henri Cartan of the University of Paris is a visiting lecturer at Harvard University this term.

Mr. P. T. Copp is now a research engineer with the Hays Corporation, Michigan City, Indiana.

Dr. C. L. Dolph, University of Michigan, has been promoted to an assistant professorship.

Professor Philip Franklin of the Massachusetts Institute of Technology is also serving as visiting lecturer on mathematical physics at Harvard University during the current spring term.

Assistant Professor G. N. Garrison, Lehigh University, has been promoted to an associate professorship.

Professor V. G. Grove of Michigan State College will be visiting professor at the University of Puerto Rico during the present term.

Associate Professor E. R. Heineman of Texas Technological College has been promoted to a professorship.

Assistant Professor Rufus Isaacs of the University of Notre Dame has accepted a position as research engineer with the North American Aviation Corporation, Los Angeles, California.

Mr. P. B. Johnson of the University of Illinois has been appointed to an assistant professorship at Occidental College.

Mr. C. H. Lindahl, Iowa State College of Agriculture and Mechanic Arts, has been promoted to an assistant professorship.

Professor C. E. Love of the University of Michigan has retired.

Mr. C. R. Morris of Indiana University has accepted a position as mathematician with the National Union Radio Corporation, Orange, New Jersey.

Mr. Bob Parker, Texas Technological College, has been promoted to an assistant professorship.

Assistant Professor Sallie Pence of the University of Kentucky has been promoted to an associate professorship.

Associate Professor D. E. South, University of Kentucky, has been promoted to a professorship.

Assistant Professor R. R. Stoll of Lehigh University has been promoted to an associate professorship.

Professor Alessandro Terracini of the University of Tucumân has returned to the University of Turin.

Professor Gerhard Tintner of Iowa State College of Agriculture and Mechanic Arts will be on leave of absence during the coming academic year and will attend the Institute of Economics at Cambridge University.

Dr. C. A. Truesdell, Research Department, Naval Ordnance Laboratory, White Oak, Maryland, has been promoted to the position of chief of the theoretical mechanics section.

Assistant Professor F. E. Ulrich, Rice Institute, has been promoted to an associate professorship.

Professor J. L. Walsh of Harvard University will be on leave of absence during this term.

Professor D. V. Widder of Harvard University will be on leave of absence during the coming academic year.

The following appointments to instructorships are announced:

New York State College for Teachers, Buffalo: Mr. Rudolph Cherkauer.

Texas Technological College: Mrs. Susie Kammerdiener, Miss Lillian McGlothlin, Mr. E. H. Thomas, Mrs. Naomi Thompson.

University of Illinois, Chicago Undergraduate Division: Mr. Furio Alberti.

University of Kentucky: Miss Elsie T. Church, Mrs. L. C. Cooper, Mr. D. L. Daly, Mr. D. C. Rose, Miss Genevieve Snider.

University of Pennsylvania: Miss Jean B. Walton.

Dr. Mellen W. Haskell, formerly head of the Department of Mathematics of the University of California, died January 15, 1948 at the age of eighty-four years.

Professor Emeritus Vladimir Karapetoff of Cornell University died January 11, 1948.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

REPORT OF THE TREASURER FOR THE YEAR 1947

The following report of Professor W. B. Carver as Treasurer for the year 1947 has been approved by the Finance Committee and accepted by vote of the Board of Governors.

I. Total Funds of the Association on December 31, 1946

(See Treasurer's report, pp. 249–252 of the Monthly for April, 1947)

Current Fund (checking account)		\$ 6,348.92
Savings Account (Ithaca Savings Bank)		1,022.66
Invested Funds (Cleveland Trust Company)		
Carus Fund	\$10,513.65	
Chace Fund	10,000.06	
Houck Fund	9,382.63	
Chauvenet Fund	665.58	
Life Membership Fund	648.60	
General Fund	25,659.78	56,870.30
		\$64,241.88

II. Current Fund Account

Receipts		Expenditures	
Balance, Jan. 1, 1947	$ 6,348.92	Monthly	
Dues	11,617.05	Publication	$ 7,828.50
Initiation fees	656.00	Reprints	223.92
Subscriptions to the Monthly	3,685.14	Editor-in-Chief's Office	528.39
Sale of back numbers Monthly	915.50	Secretary-Treasurer's Office	
Advertisements	1,109.50	Clerical help	3,650.16
Interest on General Fund	919.02	Postage	402.81
Interest on Carus Fund	376.07	Printing	434.53
Interest on Chace Fund	358.17	Office supplies	282.97
Interest on Houck Fund	336.67	Bank fee	100.00
Interest on Chauvenet Fund	23.64	Moving office	210.82
Interest on Life Membership Fund	23.28	Exec. and Finance Committees	799.27
Interest from Hardy Fund	120.00	Coordinating Committee	19.66
Sale of Archibald's Outline	210.13	Coop. Committee on Teaching	108.30
Sales of Monographs (Carus)	2,146.43	Sections and Regions	387.75
Sale of Papyrus (Chace)	230.00	Subventions	
Sale of Slaught Memorial Paper I		American Math. Society	100.00
(Chace)	14.00	Mathematical Reviews	350.00
Sales of exchange periodicals	28.60	Amer. Council on Educ.	180.76
Miscellaneous sources	3.56	Back numbers Monthly	225.12
Transferred from Chace Fund	494.54	Bank exchange charges	6.41
Transferred from General Fund	901.92	Reprinting Monos. 6 & 7 (Carus)	1,509.63
		Printing Slaught Mem. Paper I (Chace)	1,096.71
		Handling exchanges of periodicals	76.00
		Com. on Aid to Devastated Libraries	250.00
		Mrs. B. F. Finkel (Hardy Fund)	120.00
		Half cost of "Smith list"	120.49
		Addition to Trust Fund	6,500.00
		Transferred to Carus Fund	1,012.87
		Transferred to Houck Fund	336.67
		Transferred to Chauvenet Fund	23.64
		Transferred to Life Membership Fund	23.28
		Balance, Dec. 31, 1947	3,609.48
	$30,518.14		$30,518.14

III. Savings Account, Ithaca Savings Bank

Balance, Jan. 1, 1947	$ 1,022.66		
Interest	23.16	Balance, Dec. 31, 1947	$ 1,045.82
	$ 1,045.82		$ 1,045.82

IV. Invested Funds, Cleveland Trust Company

Cash Balance, Jan. 1, 1947	$ 174.30	Decrease in values of securities	$ 2,536.00
Values of Securities, Dec. 31, 1946	56,696.00	Values of securities, Dec. 31, 1947	60,635.00
From Current Fund	6,500.00	Cash bal., Dec. 31, 1947	199.30
	$63,370.30		$63,370.30

List of Securities

	Par Value	Market Value Dec. 31, 1947
U.S. Treasury Bonds, 2%, 1950	$3,000.00	$ 3,000.00
U.S. Treasury Bonds, 2%, 1954	2,000.00	2,020.00
U.S. Treasury Bonds, $2\frac{1}{2}$%, 1969	2,000.00	2,020.00
U.S. Treasury Bonds, $2\frac{1}{2}$%, 1972	1,000.00	1,000.00
U.S. Treasury Bonds, $2\frac{1}{4}$, 1962	5,000.00	5,000.00
U.S. Treasury Bonds, $1\frac{3}{4}$%, 1948	2,000.00	2,000.00
U.S. Savings Bonds, Ser. G, $2\frac{1}{2}$%, 1953	3,000.00	2,850.00
U.S. Savings Bonds, Ser. G, $2\frac{1}{2}$%, 1954	8,200.00	7,790.00
U.S. Savings Bonds, Ser. G, $2\frac{1}{2}$%, 1958	3,000.00	2,910.00
U.S. Savings Bonds, Ser. G, $2\frac{1}{2}$%, 1959	6,500.00	6,435.00
Canadian Nat. Ry. Co. Bonds, $4\frac{1}{2}$%, 1956	2,000.00	2,220.00
Amer. Tel & Tel Co. Conv. Deb. Bonds, $2\frac{3}{4}$%, 1957	500.00	540.00
C. and O. Ry. Co. Ref. Bonds, Ser. D, $3\frac{1}{2}$%, 1996	3,000.00	3,030.00
Columbus and So. Ohio Elec. Co. Bonds, $3\frac{1}{4}$%, 1970	2,000.00	2,100.00
New York Steam Corp. 1st Mort. Bonds, $3\frac{1}{2}$%, 1963	1,000.00	1,040.00
Amer. Tobacco Co. Bonds, 3%, 1969	4,000.00	4,000.00
C. & O. Ry. Co. common stock, 25 sh.		1,075.00
Amer. Tel. & Tel. Co. common stock 30 sh.		4,530.00
Standard Oil Co. of New Jersey com. stock, 20 sh.		1,580.00
Atch. Top. and Santa Fe. R. R. pfd stock, 15 sh.		1,515.00
Commonwealth Edison Co. common stock, 80 sh.		2,160.00
Dana Corp. cum. pfd. stock, 20 sh.		1,820.00
		$60,635.00

V. Carus Fund

Balance, Jan. 1, 1947	$10,513.65	Reprinting Monographs	$ 1,509.63
Sale of Monographs	2,146.43	Decrease in value of securities	468.22
Interest	376.07	Balance, Dec. 31, 1947	11,058.30
	$13,036.15		$13,036.15

VI. Chace Fund

Balance Jan. 1, 1947	$10,000.06	Printing Slaught Mem. Paper I.	$ 1,096.71
Sale of Papyrus	230.00	Decrease in values of securities	445.93
Sales of Slaught Mem. Paper I.	14.00	Balance, Dec. 31, 1947	9.059,59
Interest	358.17		
	$10,602.23		$10,602.23

VII. Houck Fund

Balance, Jan. 1, 1947	$ 9,382.63	Decrease in value of securities	$ 419.18
Interest	336.67	Balance, Dec. 31, 1947	9,300.12
	$9,719.30		$9,719.30

VIII. Chauvenet Fund

Balance, Jan. 1, 1947	$ 665.58	Decrease in value of securities	$ 29.43
Interest	23.64	Balance, Dec. 31, 1947	659.79
	$ 689.22		$ 689.22

IX. Life Membership Fund

Balance, Jan 1, 1947	$ 648.60	To General Fund	$ 44.30
Interest	23.28	Decrease in value of securities	28.98
		Balance, Dec. 31, 1947	$ 598.60
	$ 671.88		$ 671.88

X. General Fund

Balance, Jan. 1, 1947	$25,659.78	Decrease in value of securities	$ 1,144.26
From Current Fund	6,500.00	Transferred to Current Fund	901.92
From Life Membership Fund	44.30	Balance, Dec. 31, 1947	$30,157.90
	$32,204.08		$32,204.08

XI. Total Funds of the Association, December 31, 1947

Current Fund (checking account)		$ 3,609.48
Savings Account (Ithaca Savings Bank)		1,045.82
Invested Funds (Cleveland Trust Company)		
Carus Fund	$11,058.30	
Chase Fund	9,059.59	
Houck Fund	9,300.12	
Chauvenet Fund	659.79	
Life Membership Fund	598.60	
General Fund	30,157.90	60,834.30
		$65,489.60

DECEMBER MEETING OF THE TEXAS SECTION

A special meeting of the Texas Section of the Mathematical Association of America was held at Texas Christian University, Fort Worth, Texas, on Friday and Saturday, December 6–7, 1946. Vice-Chairman R. S. Underwood presided.

The following members of the Association were present: H. E. Bray, H. F. Bright, Myrtle Brown, J. E. Burnam, J. V. Cooke, Alice Dean, Elizabeth Davis, H. J. Ettlinger, R. E. Greenwood, E. R. Heineman, J. E. Howard, Harlan Miller, J. W. Querry, W. A. Rees, D. W. Starr, E. J. Stulken, F. E. Ulrich, R. S. Underwood, W. M. Whyburn, Mabel Williams, H. E. Woodward.

At the business meting it was decided that the present officers should continue until the annual meeting in the spring. The annual meeting will be held at Texas Technological College. The officers are: Chairman, E. H. Hanson, North Texas State Teachers College; Vice-Chairman, R. S. Underwood, Texas Technological College; Acting-Secretary, C. R. Sherer, Texas Christian University.

An informal dinner was held at the Worth Hotel Friday evening. At this time the Section heard an address entitled *The Role of Mathematics in the Post World War II Educational Programs*, by W. M. Whyburn, President, Texas Technological College.

The following papers were given at the meetings of the Section:

1. *A plane version of solid analytical geometry*, by R. S. Underwood, Texas Technological College.

This paper adds to the results obtained in the article entitled *An Analytic Geometry for n Variables*, this MONTHLY, vol. 52, 1945, pp. 253–262. In particular it deals with the alternative to solid analytic geometry provided by the three-axes plane, suggesting briefly the relative advantages of the new system for certain special purposes. Subjects discussed included generalized coördinates, the straight line, point-to-point and point-to-line formulas, and applications to algebra and calculus.

2. *Note on the zeros of* $P_n^m(\cos\theta)=0$ and $dP_n^m(\cos\theta)/d\theta=0$, *considered as functions of n*, by C. W. Horton, Defense Research Laboratory Staff, University of Texas, introduced by H. J. Ettlinger.

In many physical problems in which the boundary conditions are specified over the surface of a cone, it is necessary to know the roots of the equations

$$P_n^m(\cos\theta) = 0 \tag{1}$$

and

$$\frac{d}{d\theta}P_n^m(\cos\theta) = 0 \tag{2}$$

considered as functions of n. This problem has been solved by Bholanath Pal. He develops infinite series for the roots which converge rapidly and are very suitable for numerical computation. In deriving his solution Pal introduced a parameter k which takes on successive integral values and thereby yields successive roots of the equations.

It is the purpose of this note to point out that the value $k=1$ with which Pal commenced the series does not always give the first root of the equation, and sometimes it gives a number which is not a root of the equation. A corrected table of roots based on the calculations of Pal is given.

3. *Recent high speed methods of computation developed for war research*, by H. J. Ettlinger, University of Texas.

The tremendous program of technical development of the past six or seven years has resulted in an accelerated growth in the use of mechanical and electronic devices for carrying out extensive and complicated computations arising in scientific fields. The nature of the problems treated extends from that of solving linear algebraic equations in a substantial number of unknowns to obtaining solutions of systems of ordinary differential equations, partial differential equations, integral equations, and integro-differential equations. The results obtained are not limited to a narrow practical field of obtaining numerical answers to a desired number of decimal places, but it is envisaged that these devices will be helpful in theoretical problems, particularly in relation to non-linear systems.

4. *Riemann surfaces, aspects of the type problem*, by F. E. Ulrich, Rice Institute.

5. *On the roots of the derivatives of a complex polynomial*, by H. E. Bray, Rice Institute.

Let

$$P(z) = z(z - z_1)(z - z_2) \cdots (z - z_n)$$
$$Q(x) = x(x - |z_1|)(x - |z_2|) \cdots (x - |z_k|)(x + |z_{k+1}|)$$
$$P'(z) = \frac{dP}{dz}, \qquad Q'(x) = \frac{dQ}{dx}$$

where $z_1 < z_2 < \cdots < z_n$. The real polynomial $Q'(x)$ has k positive roots $\xi_1 < \xi_2 < \cdots < \xi_k$, and $n-k$ negative roots, $\xi_n < \xi_{n-1} < \cdots < \xi_{k+1}$. It is proved that, if $|\xi_{k+1}| > \xi_k$, then $P'(z)$ has exactly k roots in the circle $|z| \leqq \xi_k$ and $n-k$ roots in the region $|z| \geqq \xi_{k+1}$.

6. *Long range navigation*, by R. E. Greenwood, University of Texas.

7. *Some pages from the history of trigonometric analysis*, by R. E. Langer, University of Texas.

This was an extended address delivered at the invitation of the Section.

8. *Round table discussion of teaching problems in mathematics*, by Elizabeth Dice, Mable Williams, and W. L. Porter of Texas A. and M., all introduced by the Secretary.

C. R. SHERER, *Acting Secretary*

APRIL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the Mathematical Association of America was held at the Texas Technological College, Lubbock, Texas, on Friday and Saturday, April 25–26, 1947. Professor E. H. Hanson, Chairman of the Section, presided.

The following members of the Association were present: Ina Mae Bramblett, H. E. Bray, Myrtle Brown, H. D. Brunk, J. E. Burnam, J. V. Cooke, H. J. Ettlinger, E. H. Hanson, E. R. Heineman, P. E. Lerret, J. N. Michie, Elva Miller, Harlan Miller, C. R. Sherer, F. W. Sparks, F. E. Ulrich, R. S. Underwood, H. S. Wall, G. T. Whyburn.

At the business meeting the following officers were elected for next year: Chairman, R. S. Underwood, Texas Technological College; Vice-Chairman, F. E. Ulrich, Rice Institute; Secretary-Treasurer, C. R. Sherer, Texas Christian University. The location of the next meeting was not determined.

An informal dinner was held at the Hilton Hotel on Friday, April 25, after which an address was given by Professor H. J. Ettlinger.

The following papers were presented on Saturday.

1. *Regions in which an analytic function is flat*, by F. E. Ulrich, Rice Institute.
2. *The strong law of large numbers*, by H. D. Brunk, Rice Institute.
3. *Extended analytic geometry*, by R. S. Underwood, Texas Technological College.
4. *A general criterion for convergence of Fourier series*, by H. E. Bray, Rice Institute.
5. *Application of continued fractions to the location of roots of polynomials*, by H. S. Wall, University of Texas.

This was an invited address of an hour's duration.

6. *Round table discussion on examination papers*, by E. R. Heineman, Texas Technological College, and C. R. Sherer, Texas Christian University.

C. R. SHERER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirtieth Summer Meeting, Madison, Wisconsin, September 6–7, 1948.

Thirty-second Annual Meeting, Columbus, Ohio, December 31, 1948.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pennsylvania State College, Pennsylvania, May 8, 1948

ILLINOIS, Illinois Institute of Technology, Chicago, May 14–15, 1948

INDIANA, Purdue University, West Lafayette, May 8, 1948

IOWA, Fairfield, April 16–17, 1948

KANSAS, Atchison, April 10, 1948

KENTUCKY, Berea, May, 1948

LOUISIANA-MISSISSIPPI, Southwestern Louisiana Institute, Lafayette, Louisiana, April 23–24, 1948

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, United States Naval Academy, Annapolis, Maryland, May 8, 1948

METROPOLITAN NEW YORK, Washington Irving High School, April 24, 1948

MICHIGAN, University of Michigan, Ann Arbor, April 3, 1948

MINNESOTA, College of St. Thomas, St. Paul, May 8, 1948

MISSOURI, University of Kansas City, Kansas City, April 23, 1948

NEBRASKA, University of Nebraska, Lincoln, May 1, 1948

NORTHERN CALIFORNIA, San Francisco, January 29, 1949

OHIO, Ohio State University, Columbus, April 3, 1948

OKLAHOMA

PACIFIC NORTHWEST

PHILADELPHIA, Philadelphia, Pa., November 27, 1948

ROCKY MOUNTAIN, April 23–24, 1948

SOUTHEASTERN

SOUTHERN CALIFORNIA

SOUTHWESTERN, New Mexico Highlands University, Las Vegas, New Mexico, May 3–6, 1948

TEXAS, Rice Institute, Houston, April 23–24, 1948

UPPER NEW YORK STATE, Union College, Schenectady, May 1, 1948

WISCONSIN, Beloit, May 8, 1948

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CONTENTS

MAY 1948

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. V. NEWSOM, Oberlin College, Oberlin, Ohio.

ADVERTISING CORRESPONDENCE should be addressed to J. F. RANDOLPH, Oberlin College, Oberlin, Ohio.

NOTICE OF CHANGE OF ADDRESS by members of the Association, as well as correspondence regarding subscriptions to the MONTHLY, should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y. Change of address must reach the Secretary-Treasurer about six weeks before the change can become effective.

THIS IS THE OFFICIAL JOURNAL OF THE

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(Devoted to the Interests of Collegiate Mathematics)

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THE FORCE OF MORTALITY FUNCTION*

F. C. SMITH, College of St. Thomas

1. Introduction. In this paper we shall discuss a function which plays an important role in the theory of life contingencies, especially when limiting processes are involved. The paper will be entirely expository in nature, and no knowledge of actuarial theory will be assumed.

For most of the practical work in the actuarial field, we require a table showing the number living at each integral age n (usually denoted by l_n), the number of deaths in the year of age n to $n+1$ (usually denoted by d_n), and the average death rate over this same year of age (usually denoted by $q_n=d_n/l_n$). Now from a purely mathematical point of view and often for practical reasons as well, we should like to regard these tabulated numbers as the values of continuous functions l_x, d_x, and q_x for integral values n of the variable x. There is nothing about the nature of q_x which would prevent us from assuming that it is a continuous function, but since l_x and d_x denote numbers of persons, they cannot be continuous but, on the contrary, are always discrete. We can, however, assume the existence of a continuous function, l_x, whose values *for integral values of* x are equal to the number living of exact age x and whose intermediate values furnish a close approximation to the number living at such intermediate ages; similarly, we can assume the existence of a continuous function, d_x, whose values *for integral values of* x are equal to the number of deaths in the year of age x to $x+1$, and whose intermediate values furnish a close approximation to the number dying within a year from the attainment of such non-integral ages. We do not have any *a priori* reasons for making these assumptions about the functions l_x, d_x, and q_x, but are forced to rely on the weight of statistical evidence and to point to the well-known philosophical principle, *Natura non agit per saltus.*

When we pass to the continuous picture, we need to have an instantaneous death rate as well as the average death rate q_x. The minute an instantaneous rate is mentioned, one immediately thinks of a derivative, and we shall find that the instantaneous death rate takes the form of a derivative. The number of deaths in a small age interval Δx following age x is $l_x - l_{x+\Delta x}$, and the average death rate during this age interval may be defined as

$$\frac{l_x - l_{x+\Delta x}}{l_x}. \tag{1}$$

But since a year of age is the basic unit of time, we wish to have the average death rate during a year of age following age x. This can be obtained by multiplying (1) by $1/\Delta x$. As usual, we define the instantaneous rate to be the limiting value of the average rate, namely,

$$\lim_{\Delta x\to 0}\frac{l_x - l_{x+\Delta x}}{l_x\cdot \Delta x}. \tag{2}$$

* This paper is a combination and extension of two papers, one presented to the Indiana Section on October 17, 1946 and the other presented to the Minnesota Section on May 10, 1947.

This instantaneous rate, which is a function of x, is known as the Force of Mortality Function, and in the actuarial literature it is denoted by the symbol μ_x. It is of fundamental importance in actuarial theory as we shall show presently [see also **2**, Chapter 1].

If one compares the limit contained in (2) with the usual definition of a derivative, he finds the numerator turned around and the extra term l_x in the denominator. Thus, we have [see **3**, p. 11].

$$\mu_x = -\frac{1}{l_x}\frac{dl_x}{dx} = -\frac{d(\ln l_x)}{dx}. \tag{3}$$

2. Approximate calculation of μ_x. Unless we have a formula expressing l_x in terms of well-known mathematical functions (a situation which will be discussed in §4), it is impossible to determine the form of μ_x from the definition (3). As we have mentioned, all we usually have at our disposal are the values of l_x for integral values of x. Therefore, all we usually can do is approximate the values of μ_x by means of various formulas for numerical differentiation. Thus, if we make the assumption that l_x can be approximated by a polynomial of the second degree (which is a very restrictive assumption), then by Gauss' forward formula [see **1**, p. 64]

$$l_{x+t} = l_x + t\cdot\Delta l_x + \frac{t(t-1)}{2!}\Delta^2 l_{x-1} \tag{4}$$

where the symbol Δ denotes a difference with unit interval. By definition (3), then, we have

$$\mu_x = -\frac{1}{l_x}\left[\frac{dl_{x+t}}{dt}\right]_{t=0} = -\frac{1}{l_x}\left[\Delta l_x - \tfrac{1}{2}\Delta^2 l_{x-1}\right] = \frac{l_{x-1} - l_{x+1}}{2\cdot l_x}. \tag{5}$$

If we make the less restrictive assumption that l_x can be approximated by a polynomial of the fourth degree and again use Gauss' forward formula, then

$$\begin{aligned} l_{x+t} = l_x + t\cdot\Delta l_x &+ \frac{t(t-1)}{2!}\Delta^2 l_{x-1} + \frac{(t+1)t(t-1)}{3!}\Delta^3 l_{x-1} \\ &+ \frac{(t+1)t(t-1)(t-2)}{4!}\Delta^4 l_{x-2} \end{aligned} \tag{6}$$

and

$$\begin{aligned} \mu_x &= -\frac{1}{l_x}\left[\frac{dl_{x+t}}{dt}\right]_{t=0} = -\frac{1}{l_x}\left[\Delta l_x - \frac{1}{2}\Delta^2 l_{x-1} - \frac{1}{6}\Delta^3 l_{x-1} + \frac{1}{12}\Delta^4 l_{x-2}\right] \\ &= \frac{8(l_{x-1} - l_{x+1}) - (l_{x-2} - l_{x+2})}{12\cdot l_x}. \end{aligned} \tag{7}$$

Formula (7) has been used extensively [see, for example, **3**, p. 397].

Other formulas can be obtained from the relation connecting derivatives and differences, namely, [see **1**, p. 155]

$$\frac{dy_x}{dx} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\Delta^k y_x}{k}. \tag{8}$$

If (8) is used in (3), we obtain either

$$\mu_x = \frac{1}{l_x} \sum_{k=1}^{\infty} (-1)^k \frac{\Delta^k l_x}{k} \tag{9}$$

or

$$\mu_x = \sum_{k=1}^{\infty} (-1)^k \frac{\Delta^k (\ln l_x)}{k} \tag{10}$$

in which we may stop at any order of differences indicated by the data at hand [see **3**, p. 14].

3. Use of μ_x in actuarial theory. The function μ_x, as stated above, is of fundamental importance in the field of life contingencies, especially when limiting processes are involved. Let us examine a few illustrations which show the role of this function in actuarial theory.

Consider any age interval x to $x+k$. Following a familiar procedure, let us divide this interval into n subdivisions of length $\Delta t_1, \Delta t_2, \cdots, \Delta t_n$, and let us choose a point in each interval $x+t_1, x+t_2, \cdots, x+t_n$. Since μ_x is the instantaneous death rate at age x, the product

$$\mu_{x+t_j} l_{x+t_j} \Delta t_j \tag{11}$$

is approximately equal to the number of deaths in the jth age interval, an approximation which becomes better as Δt_j is decreased. Then the total number of deaths between ages x and $x+k$ is approximately equal to the sum of all products of the form (11) for $j=1, 2, 3, \cdots, n$; and the total number of deaths is equal to the limiting value of this sum as $\Delta t = \max \{\Delta t_j\}$ shrinks, namely,

$$\lim_{\Delta t \to 0} \sum_{j=1}^{n} \mu_{x+t_j} l_{x+t_j} \Delta t_j = \int_0^k \mu_{x+t} l_{x+t} dt. \tag{12}$$

If in (12) we replace μ_{x+t} by the form (3), we obtain

$$\int_0^k \mu_{x+t} l_{x+t} dt = \int_0^k \left[-\frac{1}{l_{x+t}} \frac{dl_{x+t}}{dt} \right] l_{x+t} dt = -\int_0^k dl_{x+t} = l_x - l_{x+k}, \tag{13}$$

a result which was anticipated since the number of deaths between ages x and $x+k$ is naturally equal to the difference between the number living at age x and those who survive k years later.

When we recall the fact that the integral $\int_a^b f(x)dx$ gives the area under the curve $y=f(x)$ between the ordinates erected at $x=a$ and $x=b$, we see that the

integral (12) gives the area under the curve $y=\mu_x l_x$ between the ordinates erected at $x=x$ and $x=x+k$. Since this area is thus numerically equal to the number of deaths between ages x and $x+k$, the curve whose equation is $y=\mu_x l_x$ is called the Curve of Deaths [see **4**; also see **3**, p. 196].

As a second illustration of the role of the function μ_x in actuarial theory, let us consider the problem of determining a formula for the net single premium for a whole life annuity of $1 per year payable continuously throughout the remaining lifetime of a person whose present age is x; such a net single premium is usually denoted by $\bar{a}_x$. To prepare for the discussion of this problem, let us first recall the formula for a similar annuity payable annually at the end of each year; such a net single premium is usually denoted by a_x. Assume that l_x persons pay this premium and thus produce a fund of $l_x \cdot a_x$. At the end of t years, since l_{x+t} persons survive, l_{x+t} dollars are paid out. If the fund earns interest at the rate i, compounded annually, the present value of this payment is $v^t l_{x+t}$ where v is the discount factor $(1+i)^{-1}$. The present value of all payments to be made until the last survivor dies is therefore $\sum_{t=1}^{\omega-x-1} v^t l_{x+t}$ where ω is the smallest age for which $l_x=0$. Since the present value of all payments to be made must equal the fund originally set up, we obtain after dividing by l_x

$$a_x = \frac{1}{l_x} \sum_{t=1}^{\omega-x-1} v^t l_{x+t}. \tag{14}$$

Returning to the annuity payable continuously, we note that the fund originally set up in this case is $l_x \cdot \bar{a}_x$. Let us divide the interval from age x to age ω as described in the second paragraph of this section. Since $1 is paid out to each survivor in a year's time, Δt_j dollars are paid to each survivor in the interval of time Δt_j. Thus, the present value of the total amount paid out in the jth age interval is approximately equal to

$$v^{t_j} l_{x+t_j} \Delta t_j$$

which means that the usual reasoning will lead us to the result

$$l_x \cdot \bar{a}_x = \lim_{\Delta t \to 0} \sum_{j=1}^{n} v^{t_j} l_{x+t_j} \Delta t_j = \int_0^{\omega-x} v^t l_{x+t} dt$$

or

$$\bar{a}_x = \frac{1}{l_x} \int_0^{\omega-x} v^t l_{x+t} dt. \tag{15}$$

Unless l_x can be expressed as a simple function, we must resort to approximate integration formulas in order to evaluate the integral on the right in (15). For example, if the Euler-Maclaurin formula is used as far as the term involving the first derivative [see **1**, pp. 187 ff.], we obtain

$$\bar{a}_x = \frac{1}{l_x} \left\{ \sum_{t=0}^{\omega-x-1} v^t l_{x+t} + \frac{1}{2} (v^{\omega-x} l_\omega - l_x) - \frac{1}{12} \left[\frac{d}{dt} (v^t l_{x+t}) \right]_{t=0}^{t=\omega-x} \right\}.$$

$$(16)\quad \bar{a}_x = \frac{1}{l_x}\sum_{t=1}^{\omega-x-1} v^t l_{x+t} + \frac{1}{2} - \frac{1}{12\cdot l_x}\left[v^t\frac{dl_{x+t}}{dt} + v^t l_{x+t}\ln v\right]_{t=0}^{t=\omega-x}$$

$$= a_x + \frac{1}{2} - \frac{1}{12}(\mu_x + \delta)$$

where $\delta = -\ln v$ is the so-called force of interest, the nominal rate which compounded continuously produces the given effective rate i [see 3, p. 133].

We might digress for a moment at this point to note the analogy between the function μ_x and the quantity δ mentioned above. If we have an effective annual interest rate i, the amount of interest earned in a small interval of time Δx is

$$(1 + i)^{\Delta x} - 1.$$

If we multiply this by $1/\Delta x$, we obtain the total interest which would be earned in one year under the assumption that like amounts of interest are earned in each interval of length Δx; that is, we obtain the nominal annual rate which if converted $1/\Delta x$ times per year gives the effective rate i. We define the nominal annual rate which if converted continuously gives the effective rate i to be the limiting value of this expression, namely,

$$\lim_{\Delta x\to 0}\frac{(1+i)^{\Delta x} - 1}{\Delta x} = \lim_{\Delta x\to 0}(1+i)^{\Delta x}\cdot\ln(1+i) = \ln(1+i) = \delta.$$

If we compare this discussion with that preceding formula (2), we see that δ plays a role in the theory of compound interest similar to that played by the function μ_x in the theory of life contingencies.

As a final illustration of the role of the function μ_x in actuarial theory, let us consider the problem of determining the net single premium for a whole life insurance of $1 payable at the moment of death of a person whose present age is x; this is usually denoted by $\bar{A}_x$. In this case, the fund originally set up is $l_x\cdot\bar{A}_x$. Let us again divide the interval from age x to age ω as described above. As before, the product (11) gives us approximately the number of deaths in the jth age interval, and since the amount of each insurance is $1, this product also gives us approximately the total number of dollars paid out in claims during the jth age interval. The present value of this amount is

$$v^{t_j}\mu_{x+t_j}l_{x+t_j}\Delta t_j$$

which means that the usual reasoning will lead us to the result

$$l_x\cdot\bar{A}_x = \lim_{\Delta t\to 0}\sum_{j=1}^{n} v^{t_j}\mu_{x+t_j}l_{x+t_j}\Delta t_j = \int_0^{\omega-x} v^t\mu_{x+t}l_{x+t}dt$$

or

$$(17)\quad \bar{A}_x = \frac{1}{l_x}\int_0^{\omega-x} v^t\mu_{x+t}l_{x+t}dt.$$

When we use (3) in the integral of (17), we get

$$\overline{A}_x = -\frac{1}{l_x}\int_0^{\omega-x} v^t \frac{dl_{x+t}}{dt}\,dt = -\frac{1}{l_x}\int_0^{\omega-x} v^t dl_{x+t}.$$

The last integral suggests an integration by parts which leads to the result

$$\begin{aligned}\overline{A}_x &= -\frac{1}{l_x}\Big[v^t l_{x+t}\Big]_{t=0}^{t=\omega-x} + \frac{\ln v}{l_x}\int_0^{\omega-x} v^t l_{x+t}\,dt \\ &= -\frac{1}{l_x}(v^{\omega-x}l_\omega - l_x) - \delta\bar{a}_x \\ &= 1 - \delta\bar{a}_x \qquad \text{[see } \mathbf{3}\text{, p. 152].}\end{aligned} \tag{18}$$

4. The law of mortality. Those who work in the empirical sciences feel the greatest satisfaction when the data they obtain follow the pattern of some familiar mathematical function. In §1 we assumed the existence of a continuous function l_x and from it defined the derived function μ_x. A number of attempts have been made to represent l_x and μ_x by means of familiar mathematical functions, and although a perfect fit has never been obtained, three of these attempts are of particular interest.

The first noteworthy proposal was made in 1825 [7] by Benjamin Gompertz (1779–1865), a British actuary who, we might mention, also made some contributions to astronomy and pure mathematics [5]. He stated his basic assumption as follows: "A person's resistance to death decreases as his age increases in such a way that at the end of equally infinitely small intervals of time he loses equally infinitely small proportions of his remaining power to oppose destruction." Mathematically, this statement is equivalent to the assumption that the force of mortality increases in geometrical progression as x increases, or that

$$\mu_x = Bc^x \tag{19}$$

where B and c are constants. Formula (19) is known as Gompertz's Law.

In 1860, an important modification of this proposal [8] was made by William Matthew Makeham (18?–1893), another British actuary, who over a period of thirty-five years contributed many brilliant papers to the *Journal of the Institute of Actuaries* [6]. He assumed that the force of mortality consists not only of a part which increases in geometrical progression but also contains a part which remains constant throughout life; that is,

$$\mu_x = A + Bc^x \tag{20}$$

where A, B, and c are constants. This modification of Gompertz's Law is known as Makeham's Law.

Makeham later proposed a second modification [9] in which the force of mortality was assumed to contain a third part which increases in arithmetical progression as x increases; that is,

$$\mu_x = A + Bc^x + Hx. \tag{21}$$

Of these three proposals, formula (20) or Makeham's Law is of greatest importance. It fits many important tables at ages 20 and over. Furthermore, if a table follows Makeham's Law, much of the practical work with the table is simplified materially. In fact, the saving in labor is so great that some sets of mortality statistics have been graduated by means of the Makeham formula even though the formula did not fit the original data too closely.

It should be noted at this point that Gompertz's Law is the special case of Makeham's Law in which $A=0$. Therefore, everything we have said and shall say about the latter is also true about the former. If a table follows Gompertz's Law, the simplification mentioned above is even greater. While Makeham's Law does have wider application, an annuity mortality table now used by many insurance companies, the 1937 Standard Annuity Table, follows Gompertz's Law at most ages.

5. The law of uniform seniority. We shall show the basic reason for the great simplification mentioned above. If in equation (20) we substitute (3) for μ_x, we obtain

$$-\frac{d(\ln l_x)}{dx} = A + Bc^x \tag{22}$$

whence

$$\ln l_x = -\int (A + Bc^x)dx = -Ax - \frac{Bc^x}{\ln c} + C \tag{23}$$

where C is the constant of integration. If in (23) we replace $-A$ by $\ln s$, $-B/(\ln c)$ by $\ln g$, and C by $\ln k$, then l_x takes the form

$$l_x = ks^x g^{c^x} \qquad \text{[see 3, p. 192].} \tag{24}$$

In (24), it should be noted, s, g, and c are constants which depend upon the mortality statistics themselves while k merely plays the role of the constant of integration.

Now of the l_x lives of age x, l_{x+t} will survive to age $x+t$. Thus, under Makeham's assumption, the probability that a life of age x will survive t years (which is usually denoted by ${}_tp_x$) is

$${}_tp_x = \frac{l_{x+t}}{l_x} = \frac{ks^{x+t}g^{c^{x+t}}}{ks^x g^{c^x}} = s^t g^{c^x(c^t-1)}. \tag{25}$$

Likewise the probability that a life of age y will survive t years is

$${}_tp_y = s^t g^{c^y(c^t-1)}. \tag{26}$$

If we assume that the survival of the one life is independent of the survival of the other, the probability that both will survive t years (which is usually denoted by ${}_tp_{xy}$) is the product of (25) and (26), namely,

$${}_tp_{xy} = s^{2t} g^{(c^x+c^y)(c^t-1)}. \tag{27}$$

Now, it is possible to determine an age w such that

$$2c^{w} = c^{x} + c^{y}. \tag{28}$$

In fact, if we let x denote the younger of the two ages, factor c^x from the right side of (28), and then take the logarithm of both sides. We obtain for w

$$w = x + \frac{\ln (1 + c^{y-x}) - \ln 2}{\ln c}. \tag{29}$$

Our result (29) shows not only that it is possible to obtain the desired age w but also that the difference $w-x$ is a function only of the difference $y-x$ and does not depend upon the individual values of x and y themselves. This fact is called the Law of Uniform Seniority [see **3**, p. 258]. Thus, in order to determine w easily in practice, all we need is a table of the values of the difference $w-x$ corresponding to the commonly-occurring values of the difference $y-x$.

With w determined, we have upon using (28) in (27)

$$_tp_{xy} = s^{2t}g^{2c^{w}(c^{t}-1)} = s^{2t}g^{(c^{w}+c^{w})(c^{t}-1)} = {_tp_{ww}} \tag{30}$$

which shows that we need to calculate the values of this function of two age variables only for equal values of the variables.

The same type of result is obtained when one studies the other actuarial functions of two age variables. The results, moreover, are not confined to functions of only two age variables. One can make the general statement that whenever Makeham's Law holds, any problem involving several lives can be reduced to one in which the ages of those lives are all equal. Thus, if a table contains 100 ages and we have a problem involving n lives, we need to prepare a table of 100 values of each actuarial function required instead of a table of 100^n values which would have to be calculated if all combinations of ages had to be used. This fact illustrates very forcibly the great saving in labor which results whenever Makeham's Law holds.

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SOME CONVEXITY PROPERTIES OF SURFACES OF NEGATIVE CURVATURE

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1. Introduction. A very simple formula occurring in the first semester of a course in differential geometry is the expression (2), below, for the Gaussian, or total, curvature K of a surface S. The purpose of the present note is the analysis, by means of elementary calculus, of the implications of (2) in the two cases $K \leqq 0$ and $K \geqq 0$.

In the neighborhood of a point where $K < 0$, the surface lies on both sides of the tangent plane, as in the depressions between one's knuckles; accordingly, a surface S with $K \leqq 0$ at each of its points is called a *saddle surface*. In the neighborhood of a point where $K > 0$, the surface lies entirely on one side of the tangent plane, as on the tops of one's knuckles. A visualization of the surfaces involved should make our result intuitively more plausible.

2. Geodesic parameters. If an analytic surface S is given in parametric representation,

$$S: \quad x_j = x_j(u, v), \qquad (j = 1, 2, 3),$$

(u, v) in a domain D, then the element of length ds and the element of area da are given by

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

and

$$da = (EG - F^2)^{1/2}dudv,$$

respectively, where E, F, and G are the coefficients of the first fundamental differential form for S:

$$E = \sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial u}\right)^2, \qquad F = \sum_{j=1}^{3} \frac{\partial x_j}{\partial u}\,\frac{\partial x_j}{\partial v}, \qquad G = \sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial v}\right)^2.$$

Given a smooth curve C_0 on S, there exists a unique family of geodesics on S intersecting C_0 orthogonally; if segments of equal length s be measured along the geodesics from C_0, then the locus of their end-points is an orthogonal trajectory C_s of the geodesics. Parameters u, v, called *geodesic parameters*, can then be chosen such that the coefficients of the first fundamental form satisfy

$$(1) \qquad E = 1, \qquad F = 0, \qquad G = [\mu(u, v)]^2, \qquad (\mu \geqq 0),$$

so that we have

$$ds^2 = du^2 + \mu^2 dv^2$$

and

$$da = \mu dudv.$$

The curves v = a constant $=v_0$ are the geodesics, of length u_2-u_1, between the points (u_1, v_0) and (u_2, v_0) on S, while the curves u = a constant are the *geodesic parallels* C_s [2, p. 206]. The surface is said to be given in *geodesic representation.*

The geodesic representation holds for the part S^* of S which is physically covered by the family of geodesics. It is to such a part S^*, which we shall henceforth denote S, that our results apply. The mapping is one-to-one between the points of the (u, v)-domain of definition and the points of S insofar as the geodesics of the family do not intersect; but we do not necessarily restrict our attention to such a part of the surface.

Singular points of the surface, or of the geodesic family, are points where $\mu=0$; the remaining points, where $\mu>0$, are *regular points.*

The Gaussian curvature K of S exists at all regular points of S.

By a classical theorem of Gauss [2, p. 155], K is a function of E, F, and G, and their partial derivatives of the first and second orders. If (1) is satisfied, the formula reduces [3] to

$$K = -\frac{1}{\mu}\frac{\partial^2\mu}{\partial u^2}. \tag{2}$$

3. Convex functions. A function $f(u)$, defined in an interval, is said to be *convex* there provided we have

$$f[ta + (1-t)b] \leqq tf(a) + (1-t)f(b)$$

for all a, b in the interval and for all t satisfying $0<t<1$; that is, provided each chord of the curve lies nowhere below the curve.

It follows that if $f(u)$ is convex in an open interval, then $f(u)$ is continous there.

If a function $f(u)$ is continuous in an interval, then $f(u)$ is convex there if any and only if, for all u and h such that $u+h$ and $u-h$ are in the interval, we have

$$f(u) \leqq \tfrac{1}{2}[f(u+h) + f(u-h)]. \tag{3}$$

If $f''(u)$ exists throughout an interval, then $f(u)$ is convex there if and only if we have $f''(u)\geqq 0$ at each point of the interval.

4. Lemma. We shall say that S is a *surface of non-positive Gaussian curvature* provided we have $K\leqq 0$ at all regular points of S.

LEMMA. *If an analytic surface S is given in geodesic representation, then a necessary and sufficient condition that S be a surface of non-positive Gaussian curvature is that the function $\mu(u, v_0)$ be a convex function of u for each line-segment $u_1<u<u_2$, $v=v_0$, in the (u, v)-domain of definition D.*

Proof. The function $\mu(u, v)$ has continuous second partial derivatives at all regular points of S.

If $\mu(u, v_0)$ is a convex function of u, then we have $\partial^2\mu/\partial n^2\geqq 0$ at all regular

points; therefore, by (2), we have $K \leqq 0$ at these points.

Conversely, if $K \leqq 0$ at all regular points of S, then, by (2) μ satisfies the convexity property $\partial^2\mu/\partial u^2 \geqq 0$ at these points. Further, since the non-negative function μ satisfies $\mu = 0$ at all singular points, it follows that for any singular point (u_0, v_0), μ satisfies the convexity inequality

$$\mu(u_0, v_0) \leqq \tfrac{1}{2}[\mu(u_0 + h, v_0) + \mu(u_0 - h, v_0)].$$

Therefore, by (3), since μ is continuous, $\mu(u, v_0)$ is a convex function of u.

We note parenthetically that if S is given in isothermic representation,

$$E = G = \lambda(u, v), \qquad F = 0,$$

instead of in geodesic representation, then a somewhat analogous situation holds. For now K is given by

$$K = \frac{-1}{2\lambda}\left(\frac{\partial^2 \log \lambda}{\partial u^2} + \frac{\partial^2 \log \lambda}{\partial v^2}\right),$$

so that S is a surface of non-positive Gaussian curvature if and only if $\log \lambda$ satisfies

$$\frac{\partial^2 \log \lambda}{\partial u^2} + \frac{\partial^2 \log \lambda}{\partial v^2} \geqq 0$$

at all regular points; that is, if and only if $\log \lambda$ is a subharmonic function of (u, v) [1, 4]. Subharmonic functions furnish a natural generalization of convex functions to functions of more than one variable. A consideration of subharmonic functions leads to a number of results concerning surfaces of non-positive Gaussian curvature; for example, it can be shown by this means that the isoperimetric inequality

$$a \leqq \frac{1}{4\pi} l^2 \tag{4}$$

characterizes surfaces of non-positive Gaussian curvature.

5. On geodesic parallels. We shall prove the following theorem.

THEOREM 1. *Let the arcs $C(u)$, $u_0 \leqq u \leqq u_1$, of length $l(u)$, be arcs of geodesic parallels between geodesics $v = v_0$ and $v = v_1$, $v_0 < v_1$, on a surface S of non-positive Gaussian curvature. Then the length $l(u)$ is a convex function of u (that is, of the geodesic length $u - u_0$); $l(u)$ is strictly convex if S is not a developable surface, and is linear if S is developable.*

Proof. By Lemma 1, $\mu(u, c)$ is a convex function of u, for $v_0 < c < v_1$. Hence for a, b, c, and t satisfying $u_0 < a < b < u_1$, $v_0 < c < v_1$, and $0 < t < 1$, we have

$$\mu[(ta + (1 - t)b), c] \leqq t\mu(a, c) + (1 - t)\mu(b, c),$$

the sign of equality holding if and only if $\partial^2\mu/\partial u^2 \equiv 0$ on the line-segment $u_0 < u < u_1$, $v = c$; that is, by (2), if and only if $K \equiv 0$ there. Hence

$$l(ta + (1-t)b) = \int_{v_0}^{v_1} \mu[(ta + (1-t)b), v]dv$$
$$\leqq \int_{v_0}^{v_1} [t\mu(a, v) + (1-t)\mu(b, v)]dv = tl(a) + (1-t)l(b),$$

the sign of equality holding if and only if $K \equiv 0$ for $u_0 < u < u_1$, $v_0 < v < v_1$, and therefore, by analytic continuation, $K \equiv 0$ on S; that is, if and only if S is a developable surface.

COROLLARY. *If an arc C of length l on an analytic surface S of non-positive Gaussian curvature is translated geodesically arbitrary distances in the two directions on S from C, then at least one of the resulting arcs is of length not less than l; the function $l(u)$ can have an interior maximum only if $l(u)$ is identically constant.*

We note, regarding the corollary, that if $l(u)$ is identically constant then S must be developable; but the hypothesis that S is developable does not imply that $l(u)$ is constant.

THEOREM 2. *Let the arcs $C(u)$, $u_0 - W < u < u_0 + W$, of length $l(u)$, be arcs of geodesic parallels between geodesics $v = v_0$ and $v = v_1$, $v_0 < v_1$, on an analytic surface S of non-positive Gaussian curvature, and let $a(w)$ denote the area of the part of S enclosed by $v = v_0$, $C(u_0 + w)$, $v = v_1$, and $C(u_0 - w)$, $0 \leqq w < W$. Then $a(w)$ is a convex function of w; $a(w)$ is strictly convex if S is not a developable surface, and is linear if S is developable.*

Proof. We have

$$a(w) = \int_{u_0 - w}^{u_0 + w} \int_{v_0}^{v_1} \mu(u, v)dudv = \int_{u_0 - w}^{u_0 + w} l(u)du,$$

whence

$$a'(w) = l(u_0 + w) + l(u_0 - w),$$
$$a''(w) = l'(u_0 + w) - l'(u_0 - w).$$

Since, by Theorem 1, $l(u)$ is convex, it follows that $l'(u)$ is monotonic non-decreasing, so that we have $a''(w) \geqq 0$, the sign of equality holding if and only if $l'(u) \equiv$ a constant; that is, since

$$l''(u) = \int_{v_0}^{v_1} \frac{\partial^2\mu}{\partial u^2} dv,$$

if and only if S is developable.

6. Geodesic polar coördinates. If equal lengths are laid off from a point

P_0 of S along the geodesics in all directions through P_0 on S, then the locus of end-points is an orthogonal trajectory of the geodesics. We take the geodesics for the curves v = a constant, and let u denote distances measured along the geodesics from P_0. The coördinate v can be adjusted to satisfy $v_1=\theta_1$, for all θ_1, where θ_1 is the angle between the tangents at P_0 to the geodesics $v=0$ and $v=v_1$.

We shall write r, θ for u, v, respectively.

Necessary and sufficient conditions for this coördinate system are (1) and

$$\mu(0, \theta) = 0, \qquad \left[\frac{\partial \mu}{\partial r}\right]_{r=0} = 1. \tag{5}$$

Now r, θ are called *geodesic polar coördinates*, with *pole* at P_0; the curve $r=r_0$ is a *geodesic circle* with *center* at P_0 and *geodesic radius* r_0 [2, p. 208].

7. Definitions. Several functions of geometrical significance involving the geodesic radius have certain properties in common. It is convenient to collect these properties in the following definitions.

Condition A. For a given surface S of non-positive Gaussian curvature, and for a given pole P_0 of geodesic polar coordinates on S, a function $\phi(r)$ of the geodesic radius r satisfies Condition A provided: $\phi(0)=0$; for $r\geqq 0$ on S, $\phi(r)$ is a continuous monotonic non-decreasing convex function of r; $\phi(r)\equiv 0$ if S is a developable surface, but otherwise $\phi(r)$ is monotonic increasing and strictly convex.

Condition B. For a given surface S of non-positive Gaussian curvature, and for a given pole P_0 of geodesic polar coordinates on S, a function $\psi(r)$ of the geodesic radius r satisfies Condition B provided $\psi(0)=1$; for $r\geqq 0$ on S, $\psi(r)$ is a continuous monotonic non-decreasing function of r; $\psi(r)\equiv 1$ if S is a developable surface, but otherwise $\psi(r)$ is monotonic increasing.

Thus if $\phi(r)$ satisfies Condition A, then $\phi(r)+1$ satisfies Condition B; but $\psi(r)$ might satisfy Condition B without $\phi(r)-1$ satisfying Condition A.

We note that if a function $\phi(r)$ satisfies Condition A, then $[\phi(r)]^\alpha$, $\alpha>1$, also satisfies Condition A. If $\psi(r)$ satisfies Condition B, then $[\psi(r)]^\beta$, $\beta>0$, also satisfies Condition B.

8. On geodesic circles. We shall establish the following result concerning the length

$$l(r) \equiv \int_0^{2\pi} \mu(r, \theta)d\theta$$

of the circumference of geodesic circles on surfaces of non-positive Gaussian curvature.

Theorem 3. *Let S be an analytic surface of non-positive Gaussian curvature, and let $l(r)$ denote the length of the circumference of the geodesic circle on S with fixed*

center P_0 and geodesic radius r. Then the function

$$\phi_1(r) \equiv l(r) - 2\pi r, \qquad (r \geqq 0 \text{ on } S),$$

satisfies Condition A, and the function

$$\psi_1(r) \equiv \frac{l(r)}{2\pi r}, \qquad (r > 0 \text{ on } S),$$

$$\psi_1(0) \equiv 1$$

satisfies Condition B.

Proof. Since, by Lemma 1, for each θ_0, $0 \leqq \theta_0 < 2\pi$, $\mu(r, \theta_0)$ is a convex function of r and satisfies (5), we have

$$\left[\frac{\partial \mu}{\partial r}\right]_{r>0} \geqq \frac{\mu}{r} \geqq 1, \tag{6}$$

the signs of equality holding if and only if $\partial\mu/\partial r \equiv 1$ on the geodesic $\theta = \theta_0$; that is, by (2), if and only if $K \equiv 0$ on $\theta = \theta_0$.

From (6) it follows that the function

$$\phi_0(r, \theta_0) \equiv \mu(r, \theta_0) - r, \qquad (r \geqq 0 \text{ on } S),$$

which vanishes at $r = 0$, satisfies $\partial\phi_0/\partial r \geqq 0$, so that $\phi_0(r, \theta_0)$ is a continuous monotonic non-decreasing convex function of r. And $\phi_0(r, \theta_0)$ is monotonic increasing and strictly convex unless $K \equiv 0$ on $\theta = \theta_0$.

It follows (see the proof of Theorem 1) that the function

$$\phi_1(r) \equiv l(r) - 2\pi r = \int_0^{2\pi} \phi_0(r, \theta) d\theta$$

satisfies Condition A.

By (5) and l'Hospital's rule, the function

$$\psi_0(r, \theta_0) \equiv \frac{\mu(r, \theta_0)}{r}, \qquad (r > 0 \text{ on } S),$$

$$\psi_0(0, \theta_0) \equiv 1$$

is continuous at $r = 0$. Differentiating $\psi_0(r, \theta_0)$ with respect to r, we see by (6) that $\psi_0(r, \theta_0)$ is monotonic non-decreasing,

$$\psi_0(r_1, \theta_0) \leqq \psi_0(r_2, \theta_0), \qquad (0 \leqq r_1 < r_2 \text{ on } S), \tag{7}$$

the sign of equality holding if and only if $K \equiv 0$ on $\theta = \theta_0$.

By (5) and l'Hospital's rule we have

$$\lim_{r \to 0} \psi_1(r) = \lim_{r \to 0} \frac{\int_0^{2\pi} \mu(r, \theta) d\theta}{2\pi r} = \lim_{r \to 0} \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \mu(r, \theta)}{\partial r} d\theta = 1,$$

so that $\psi_1(r)$ is continuous for $r \geqq 0$ on S. Also, $\psi_1(r)$ is monotonic non-decreasing, since by (7) we have

$$
\begin{aligned}
\psi_1(r_1) &= \frac{\int_0^{2\pi} \mu(r_1, \theta)d\theta}{2\pi r_1} = \frac{1}{2\pi}\int_0^{2\pi} \psi_0(r_1, \theta)d\theta \\
&\leqq \frac{1}{2\pi}\int_0^{2\pi} \psi_0(r_2, \theta)d\theta = \psi_1(r_2), \qquad (0 < r_1 < r_2 \text{ on } S),
\end{aligned} \tag{8}
$$

the sign of equality holding if and only if $K \equiv 0$ on S. Thus $\psi_1(r)$ satisfies Condition B.

We note that (8) could have been obtained equally well by differentiation. By (5) and the convexity of $\mu(r, \theta_0)$, we have $\mu(r, \theta) > 0$ for $r > 0$ on S, so that $\partial\mu/\partial r$ exists for $r > 0$. By (6), we have

$$
2\pi r^2 \frac{d\psi_1(r)}{dr} = r\frac{dl(r)}{dr} - l(r) = \int_0^{2\pi} \left[r\frac{\partial \mu(r, \theta)}{\partial r} - \mu(r, \theta)\right] d\theta \geqq 0, \tag{9}
$$

the sign of equality holding for $r > 0$ on S if and only if $K \equiv 0$ on S.

COROLLARY. *If S is an analytic surface of non-positive Gaussian curvature, then $l(r)$ is a monotonic increasing convex function of r and satisfies the inequality*

$$
l(r) \geqq 2\pi r; \tag{10}
$$

$l(r)$ is strictly convex and satisfies the strict inequality for $r > 0$ on S if S is not a developable surface, and is linear and satisfies the equality if S is developable.

The corollary follows from Theorem 3 and the identity

$$
l(r) \equiv 2\pi r + \phi_1(r).
$$

9. Area theorems. On surfaces of non-positive Gaussian curvature, the area function,

$$
a(r) \equiv \int_0^r \int_0^{2\pi} \mu(\rho, \theta)d\rho d\theta,
$$

has properties similar to those given for $l(r)$.

THEOREM 4. *Let S be an analytic surface of non-positive Gaussian curvature, and let $a(r)$ denote the area of the geodesic circle on S with fixed center P_0 and geodesic radius r. Then the function*

$$
\phi_2(r) \equiv a(r) - \pi r^2, \qquad (r \geqq 0 \text{ on } S),
$$

satisfies Condition A, and the function

$$
\psi_2(r) \equiv \frac{a(r)}{\pi r^2}, \qquad (r > 0 \text{ on } S),
$$

$$\psi_2(0) \equiv 1$$

satisfies Condition B.

Proof. We have

$$\phi_2(r) = \int_0^r \phi_1(\rho)d\rho,$$

whence

$$\phi_2'(r) = \phi_1(r) \geqq 0, \qquad \phi_2''(r) = \phi_1'(r) \geqq 0,$$

the signs of equality holding for $r>0$ on S if and only if $K\equiv 0$ on S. Since $\phi_2(0)=0$, the results indicated for $\phi_2(r)$ follow.

The continuity of $\psi_2(r)$ at $r=0$ follows from (5) and l'Hospital's rule. Since $l(r)$ is convex and satisfies $l(0)=0$, we have

$$l(\rho_1) \leqq \frac{\rho_1}{\rho_2} l(\rho_2), \qquad (0 < \rho_1 < \rho_2 \text{ on } S);$$

the sign of equality holds if and only if $K\equiv 0$ on S. Hence

$$\begin{aligned} a(r_1) &= \int_0^{r_1} l(\sigma)d\sigma = \int_0^{r_2} l\left(\frac{r_1}{r_2}\,\rho\right)\frac{r_1}{r_2}\,d\rho \\ &\leqq \int_0^{r_2} l(\rho)\,\frac{r_1^2}{r_2^2}\,d\rho = a(r_2)\frac{r_1^2}{r_2^2}, \qquad (0 < r_1 < r_2 \text{ on } S), \end{aligned}$$

whence

$$\psi_2(r_1) \leqq \psi_2(r_2), \qquad (0 \leqq r_1 < r_2 \text{ on } S), \tag{11}$$

the sign of equality holding if and only if $K\equiv 0$ on S. This completes the proof of the theorem.

We note that (11) could have been obtained equally well by differentiation. We have

$$\begin{aligned} \pi r^3 \frac{d\psi_2(r)}{dr} &= rl(r) - 2a(r) = rl(r) - 2\int_0^r l(\rho)d\rho \\ &\geqq rl(r) - 2\int_0^r \frac{\rho}{r}\, l(r)d\rho = 0, \qquad (r > 0 \text{ on } S), \end{aligned} \tag{12}$$

the sign of equality holding if and only if $K\equiv 0$ on S.

COROLLARY. *If S is an analytic surface of non-positive Gaussian curvature, then $a(r)$ is a monotonic increasing strictly convex function of r and satisfies the inequality*

$$a(r) \geqq \pi r^2,$$

the sign of equality holding for $r>0$ on S if and only if S is a developable surface.

The corollary follows from Theorem 4 and the identity

$$a(r) \equiv \pi r^2 + \phi_2(r).$$

Actually, $a(r)$ and the function $\phi_2(r) \equiv a(r) - \pi r^2$, which we have shown to be non-negative, satisfy even stronger convexity conditions than those given in Theorem 4 and its corollary, as we now shall show. For a non-negative function $g(r)$, the convexity of $[g(r)]^{1/2}$ implies the convexity of $g(r)$; but the converse does not hold. The proof of Theorem 5 will be shortened by use of the following lemma showing that certain conditions which clearly imply the convexity of $g(r)$ also imply the convexity of $[g(r)]^{1/2}$.

LEMMA 2. *If $g(r)$ is a non-negative function for which $g'''(r)$ exists in the interval $\alpha \leqq r < \beta$, satisfying*

$$h(\alpha) \equiv 2g(\alpha)g''(\alpha) - [g'(\alpha)]^2 \geqq 0$$

and

$$g'''(r) \geqq 0, \qquad (\alpha \leqq r < \beta),$$

then $[g(r)]^{1/2}$ is convex in $\alpha \leqq r < \beta$, and is strictly convex there provided

$$g'''(r) > 0, \qquad (\alpha < r < \beta).$$

Proof. If we let

$$f(r) \equiv [g(r)]^{1/2},$$

then at points where $f(r) \neq 0$ we have

$$f'' = \tfrac{1}{4}g^{-3/2}(2gg'' - g'^2) = \tfrac{1}{4}g^{-3/2}h.$$

Now

$$h' = 2gg''', \tag{13}$$

so that from the hypotheses we get $h(\alpha) \geqq 0$, $h'(r) \geqq 0$, $\alpha \leqq r < \beta$, whence $h(r) \geqq 0$. Then $f''(r) \geqq 0$ at points where $f(r) \neq 0$. Since, further, the non-negative function $f(r)$ satisfies the convexity inequality (3) for points where $f(r) = 0$, it follows that the continuous function $f(r)$ is convex for $\alpha \leqq r < \beta$.

If $g'''(t) > 0$, $\alpha < r < \beta$, then the non-negative convex function $g(r)$ can vanish at no more than one point of $\alpha \leqq r < \beta$, whence, by (13), we have $h(r) > 0$, $\alpha < r < \beta$. It follows that we have $f''(r) > 0$ except at most at one point of $\alpha < r < \beta$, so that $f(r)$ is strictly convex for $\alpha \leqq r < \beta$.

THEOREM 5. *Let S be a surface of non-positive Gaussian curvature, and let $a(r)$ denote the area of the geodesic circle on S with fixed center P and geodesic radius r. Then*

$$f(r) \equiv [a(r)]^{1/2}$$

is a convex function of r for $r \geqq 0$ on S; $f(r)$ is strictly convex there if S is not a developable surface, and is linear if S is developable. Further, the function

$$[\phi_2(r)]^{1/2} \equiv [a(r) - \pi r^2]^{1/2}$$

satisfies Condition A.

Proof. We have

$$a'(r) = l(r) = \int_0^{2\pi} \mu(r, \theta) d\theta, \qquad a''(r) = \int_0^{2\pi} \frac{\partial \mu(r, \theta)}{\partial r} d\theta,$$

$$a'''(r) = \int_0^{2\pi} \frac{\partial^2 \mu(r, \theta)}{\partial r^2} d\theta,$$

whence, in addition to $a(0) = 0$, we have

$$a'(0) = 0, \qquad a'''(r) \geqq 0, \qquad (r \geqq 0 \text{ on } S),$$

with $a'''(r) > 0$ for $r > 0$ on S unless $K \equiv 0$ on S. Then for $0 \leqq r$ on S, $a(r)$ satisfies the hypotheses concerning $g(r)$ in Lemma 2, so that $[a(r)]^{1/2}$ is convex for $r \geqq 0$ on S, and is strictly convex there if S is not a developable surface. If S is developable, then $[a(r)]^{1/2} = \pi^{1/2} r$, so that $[a(r)]^{1/2}$ is linear.

Similarly we have

$$\phi_2'(r) = l(r) - 2\pi r = \phi_1(r), \qquad \phi_2''(r) = \phi_1'(r), \qquad \phi_2'''(r) = \phi_1''(r),$$

whence, in addition to $\phi_2(0) = 0$, we have

$$\phi_2'(0) = 0, \qquad \phi_2'''(r) \geqq 0, \qquad (r \geqq 0 \text{ on } S),$$

with $\phi_2'''(r) > 0$ for $r > 0$ on S unless $K \equiv 0$ on S. It follows from Lemma 2 and Theorem 4 that $[\phi_2(r)]^{1/2}$ satisfies Condition A.

10. The isoperimetric inequality. As was pointed out in §4, the isoperimetric inequality (4) holds for all Jordan regions on surfaces S of non-positive Gaussian curvature, and characterizes these surfaces, so that in particular we have

$$a(r) \leqq \frac{1}{4\pi} [l(r)]^2 \tag{14}$$

for all geodesic circles on these surfaces. In this section we shall include independent proofs of (14).

THEOREM 6. *Let S be an analytic surface of non-positive Gaussian curvature, and $l(r)$ denote the length of the circumference, and $a(r)$ the area, of the geodesic circle on S with fixed center P_0 and geodesic radius r. Then the functions*

$$\phi_3(r) \equiv \frac{1}{4\pi} (2\pi r)[l(r)] - a(r) \equiv \tfrac{1}{2} r l(r) - a(r)$$

and

$$\phi_4(r) \equiv \frac{1}{4\pi}[l(r)]^2 - a(r), \qquad (r \geqq 0 \textit{ on } S),$$

satisfy Condition A, and the function

$$\psi_4(r) \equiv \frac{\frac{1}{4\pi}[l(r)]^2}{a(r)}, \qquad (r > 0 \textit{ on } S),$$
$$\psi_4(0) \equiv 1$$

satisfies Condition B.

Proof. We have

$$\phi_3'(r) = \tfrac{1}{2}[rl'(r) - l(r)] \geqq 0$$

by (9), and

$$\phi_3''(r) = \tfrac{1}{2}rl''(r) \geqq 0$$

by the convexity of $l(r)$; the signs of equality holds for $r>0$ on S if and only if $K\equiv 0$ on S. Since $\phi_3(0)=0$, it follows that $\phi_3(r)$ satisfies Condition A.

Again, we have

$$(15) \qquad \phi_4'(r) = l(r)\left[\frac{1}{2\pi}l'(r) - 1\right] = l(r)\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial\mu(r,\theta)}{\partial r}\,d\theta - 1\right] \geqq 0$$

by (6); and

$$\phi_4''(r) = \frac{1}{2\pi}l(r)l''(r) + l'(r)\left[\frac{1}{2\pi}l'(r) - 1\right] \geqq 0;$$

the signs of equality hold for $r>0$ on S if and only if $K\equiv 0$ on S. Since $\phi_4(0)=0$, it follows that $\phi_4(r)$ satisfies Condition A.

Now $\psi_4(r)$ is continuous at $r=0$ by (5) and l'Hospital's rule. We have

$$[a(r)]^2\psi_4'(r) = \frac{1}{4\pi}l(r)p(r),$$

where

$$p(r) = 2a(r)l'(r) - [l(r)]^2.$$

Since $p(0)=0$ and

$$p'(r) = 2a(r)l''(r) \geqq 0, \qquad (r > 0 \text{ on } S),$$

we have $p(r)\geqq 0$, whence $\psi_4'(r)\geqq 0$, $r>0$ on S; the sign of equality holds if and only if $K\equiv 0$ on S. Hence $\psi_4(r)$ satisfies Condition B.

COROLLARY. *If S is an analytic surface of non-positive Gaussian curvature, then*

$$a(r) \leqq \frac{1}{4\pi}(2\pi r)[l(r)],$$

and a fortiori

$$a(r) \leqq \frac{1}{4\pi}[l(r)]^2, \qquad (r \geqq 0 \text{ on } S),$$

the signs of equality holding for $r>0$ on S if and only if S is a developable surface.

The corollary follows from Theorem 6 and the identities

$$a(r) \equiv \frac{1}{4\pi}(2\pi r)[l(r)] - \phi_3(r),$$

$$a(r) \equiv \frac{1}{4\pi}[l(r)]^2 - \phi_4(r).$$

The corollary follows equally well from (10) and (12).

11. Geodesic circular sectors. The convexity relations derived in §§8, 9, and 10 for geodesic circles depend on convexity relations which hold on each geodesic radius. The integration with respect to θ was taken between limits 0 and 2π for geometric reasons only. We might confine our attention equally well to a *geodesic circular sector* $0 \leqq r < r_0$, $\theta_1 \leqq \theta \leqq \theta_2$, with pole $r=0$ at P_0, and fixed angle (θ_1, θ_2), on S. Equations (1) and (5) still hold, but we do not assume that there necessarily is an entire geodesic circle $0 \leqq r < r_0$, $0 \leqq \theta < 2\pi$, with center at P_0, on S.

For a given surface S, a given pole P_0 of geodesic polar coördinates on S, and a given angle (θ_1, θ_2), we let $l(r; \theta_1, \theta_2)$ denote the length of the bounding arc of geodesic circular sector,

$$l(r; \theta_1, \theta_2) = \int_{\theta_1}^{\theta_2} \mu(r, \theta)d\theta,$$

and $a(r; \theta_1, \theta_2)$ the area,

$$a(r; \theta_1, \theta_2) = \int_0^r \int_{\theta_1}^{\theta_2} \mu(\rho, \theta)d\rho d\theta = \int_0^r l(\rho; \theta_1, \theta_2)d\rho,$$

of the geodesic circular sector of geodesic radius r, and angle from θ_1 to θ_2.

We have the following results.

THEOREM 7. *Let S be an analytic surface of non-positive Gaussian curvature, and let $l(r; \theta_1, \theta_2)$ denote the length of the bounding arc of geodesic circular sector, and $a(r; \theta_1, \theta_2)$ the area, of the geodesic circular sector on S with fixed pole P_0, fixed angle from θ_1 to θ_2, $\theta_1 < \theta_2$, and geodesic radius r. Then the functions*

$$\phi_5(r) \equiv l(r;\theta_1,\theta_2) - (\theta_2-\theta_1)r,$$

$$\phi_6(r) \equiv a(r;\theta_1,\theta_2) - \left(\frac{\theta_2-\theta_1}{2}\right)r^2,$$

$$\phi_7(r) \equiv \frac{[(\theta_2-\theta_1)r][l(r;\theta_1,\theta_2)]}{2(\theta_2-\theta_1)} - a(r;\theta_1,\theta_2) \equiv \tfrac{1}{2}rl(r;\theta_1,\theta_2) - a(r;\theta_1,\theta_2),$$

and

$$\phi_8(r) \equiv \frac{[l(r;\theta_1,\theta_2)]^2}{2(\theta_2-\theta_1)} - a(r;\theta_1,\theta_2), \qquad (r \geqq 0,\ \theta_1 \leqq \theta \leqq \theta_2,\ \text{on } S),$$

satisfy Condition A; and the functions

$$\psi_5(r) \equiv \frac{l(r;\theta_1,\theta_2)}{(\theta_2-\theta_1)r},$$

$$\psi_6(r) \equiv \frac{a(r;\theta_1,\theta_2)}{\left(\frac{\theta_2-\theta_1}{2}\right)r^2},$$

$$\psi_8(r) \equiv \frac{[l(r;\theta_1,\theta_2)]^2}{2(\theta_2-\theta_1)[a(r;\theta_1,\theta_2)]}, \qquad (r > 0,\ \theta_1 \leqq \theta \leqq \theta_2,\ \text{on } S),$$

$$\psi_j(0) \equiv 1, \qquad (j = 5, 6, 8),$$

satisfy Condition B.

The proofs are similar to the proofs of analogous results given in §§8, 9, and 10, and will not be given here.

It follows as a corollary to Theorem 7 that if S is a surface of non-positive Gaussian curvature, then $l(r;\theta_1,\theta_2)$ and $a(r;\theta_1,\theta_2)$ are convex functions which satisfy the inequalities

$$l(r;\theta_1,\theta_2) \geqq (\theta_2-\theta_1)r,$$

$$a(r;\theta_1,\theta_2) \geqq \left(\frac{\theta_2-\theta_1}{2}\right)r^2,$$

and

$$a(r;\theta_1,\theta_2) \leqq \tfrac{1}{2}rl(r;\theta_1,\theta_2) \leqq \frac{[l(r;\theta_1,\theta_2)]^2}{2(\theta_2-\theta_1)}, \qquad (\theta_1 < \theta_2);$$

the signs of equality hold for $r>0$ on S if and only if S is a developable surface.

Similarly, Theorem 5 also generalizes to geodesic circular sectors on surfaces of non-positive Gaussian curvature, with $[a(r;\theta_1,\theta_2)]^{1/2}$ and $[\phi_6(r)]^{1/2}$ in place of $[a(r)]^{1/2}$ and $[\phi_2(r)]^{1/2}$, respectively.

In place of the sector $\theta_1 \leqq \theta \leqq \theta_2$, we might consider an arbitrary measurable set of values of θ, but shall not pursue this further.

12. On regular surfaces of non-negative Gaussian curvature. The above results concerning surfaces of non-positive Gaussian curvature hold in the large and are unaffected by singular points. The following rather analogous results for surfaces of non-negative curvature hold in general only on parts of S where there are no singular points of the surface, or of the family of geodesics, other than at the pole of geodesic polar coördinates; and some of the results hold only in the small even where there are no singular points.

A function $f(x)$ is said to be *concave* provided $-f(x)$ is convex.

A surface S given in terms of geodesic coördinates, or in terms of geodesic polar coördinates, will be said to be *regular* provided there are no singular points on S except, in the case of geodesic polar coördinates, at the pole P_0.

Lemma 1 holds if we add the restriction that S is regular, and replace "non-positive" by "non-negative," and "convex" by "concave." Theorems 1 and 2 hold with the same alterations, and the corollary of Theorem 1 follows *mutatis mutandis.*

We alter Conditions A and B to apply to a "regular surface S of non-negative Gaussian curvature," changing "non-decreasing," "convex," and "increasing" to "non-increasing," "concave," and "decreasing," respectively. The altered conditions will be denoted Conditions A^* and B^*, respectively.

Then *the functions* $\phi_j(r)$, $j=1, 2, 3$, *satisfy Condition* A^*, *and the functions* $\psi_j(r)$, $j=1, 2, 4$, *satisfy Condition* B^*, *on regular analytic surfaces of non-negative Gaussian curvature.*

By the expression for $\phi_4'(r)$ in (15), $\phi_4(r)$ *is monotonic non-increasing on any regular analytic surface* S *of non-negative Gaussian curvature, and is monotonic decreasing if* S *is not developable.* Actually, since $\phi_4(r)$ is analytic, it follows from $\phi_4(0)=\phi_4'(0)=0$ and $\phi_4(r)\leqq 0$ that there is an $r_0=r_0(S, P_0)$ such that $\phi_4(r)$ is concave for $0\leqq r\leqq r_0$. A consideration of the sphere shows, however, that $\phi_4(r)$ is not necessarily concave for all $r>0$ on S.

The properties of the function $\phi_j(r)$ yield the following results concerning $l(r)$ and $a(r)$.

We have

$$l(r) = 2\pi r + \phi_1(r),$$
$$l'(r) = 2\pi + \phi_1'(r),$$

and

$$l''(r) = \phi_1''(r).$$

Since on regular surfaces of non-negative Gaussian curvature the function $\phi_1(r)$ satisfies Condition A^*, on these surfaces we have

$$\phi_1(r) \leqq 0, \qquad \phi_1''(r) \leqq 0, \qquad\qquad (r \geqq 0 \text{ on } S).$$

It follows that *on regular analytic surfaces of non-negative Gaussian curvature the function* $l(r)$ *is concave and satisfies*

$$l(r) \leqq 2\pi r;$$

$l(r)$ is strictly concave and satisfies the strict inequality for $r>0$ on S if S is not a developable surface. Also, on these surfaces we have $\phi_1'(0)=0$, so that, since $\phi_1(r)$ is concave, *for a given regular analytic surface S of non-negative Gaussian curvature and for a given pole P_0 on S, either $l(r)$ is monotonic increasing on S or there is an $r_0=r_0(S, P_0)>0$ such that $l(r)$ is monotonic increasing for $0\leqq r\leqq r_0$ and monotonic decreasing for $r\geqq r_0$ on S.*

Again, we have

$$a(r) = \pi r^2 + \phi_2(r),$$
$$a'(r) = 2\pi r + \phi_2'(r) = 2\pi r + \phi_1(r) = l(r),$$

and

$$a''(r) = 2\pi + \phi_2''(r) = 2\pi + \phi_1'(r) = l'(r).$$

On regular analytic surfaces of non-negative Gaussian curvature we have

$$\phi_2(r) \leqq 0, \qquad \phi_2''(0) = 0, \qquad \phi_2''(r) \leqq 0, \qquad (r \geqq 0 \text{ on } S).$$

Hence *on regular analytic surfaces of non-negative Gaussian curvature, $a(r)$ satisfies*

$$a(r) \leqq \pi r^2;$$

the strict inequality holds for $r>0$ on S if S is not a developable surface. Further, for a given regular analytic surface S of non-negative Gaussian curvature, and for a given pole P_0 on S, either $a(r)$ is strictly convex, or there is an $r_0=r_0(S, P_0)>0$ such that $a(r)$ is strictly convex for $0\leqq r\leqq r_0$ and strictly concave for $r\geqq r_0$ on S.

The interval $0\leqq r\leqq r_0$ on which $a(r)$ is convex coincides with the interval on which $l(r)$ is increasing.

On a sphere, for example, $l(r)$ is increasing and concave, and $a(r)$ is convex, until a hemisphere has been covered. Thereafter, $l(r)$ is decreasing but still concave, and $a(r)$ becomes concave, until the other hemisphere has been covered. At the point diametrically opposite the pole we encounter a singular point of the geodesic representation.

If K_0 is the Gaussian curvature of a surface S at a regular point P_0, then the area and the length of circumference of the geodesic circle with center P_0 and radius r on S are given [2, p. 209] by

$$a(r) = \pi r^2 - \tfrac{1}{12}\pi K_0 r^4 + \cdots,$$

and

$$l(r) = 2\pi r - \tfrac{1}{3}\pi K_0 r^4 + \cdots,$$

respectively, so that

$$a(r) - \frac{1}{4\pi}[l(r)]^2 = \tfrac{1}{4}\pi K_0 r^4 + \cdots.$$

It follows that if $K_0>0$, then *for sufficiently small $r>0$* we have the isoperimetric

inequality with sign reversed:

$$a(r) > \frac{1}{4\pi}[l(r)]^2.$$

We now improve the above result, as follows.

Since $\phi_3(0)=\phi_4(0)=0$, it follows from the properties of $\phi_3(r)$ and $\phi_4(r)$ given earlier in this section that *on regular surfaces of non-negative Gaussian curvature we have*

$$a(r) \geqq \frac{1}{4\pi}(2\pi r)[l(r)] \geqq \frac{1}{4\pi}[l(r)]^2,$$

the signs of equality holding for $r>0$ on S if and only if S is a developable surface.

The functions related to geodesic circular sectors, which we introduced in §11, have, on regular surfaces of non-negative Gaussian curvature, properties analogous to those of the corresponding functions related to geodesic circles on surfaces of the same class.

13. Converse questions. In all the above analyses we have assumed that the Gaussian curvature of S either is non-positive or is non-negative. In most instances we have obtained, in the two cases, conclusions which are distinct except for the dividing class of developable surfaces. Thus by logical exclusion we obtain several characterizations of the above classes of surfaces.

For instance, a regular analytic surface S is a surface of non-negative Gaussian curvature, but not a developable surface, if and only if for each pole P_0 on S we have

$$l(r) < 2\pi r \tag{16}$$

for all $r>0$ on S.

We have shown that the condition $K\geqq 0$, $K\not\equiv 0$, on S implies (16). Conversely, if we should have $K_0<0$ at some P_0 on S, then we would have $K<0$ through some neighborhood of P_0, and therefore, in the neighborhood, we would have

$$l(r) > 2\pi r;$$

also, if we should have $K\equiv 0$ on S, then we would have

$$l(r) \equiv 2\pi r;$$

hence (16) implies $K\geqq 0$, $K\not\equiv 0$, on S.

We might replace (16) by

$$a(r) < \pi r^2,$$

and might give similar characterizations of surfaces with $K\leqq 0$, $K\not\equiv 0$, and of surfaces with $K\equiv 0$.

In the same way we have the following result.

THEOREM 8. *The regular analytic surface S is, i) a surface of non-positive Gaussian curvature, but not developable, ii) a surface of non-negative Gaussian curvature, but not developable, or, iii) a developable surface, if and only if we have*

$$(17) \quad i)\ \phi_j(r) > 0, \quad ii)\ \phi_j(r) < 0, \quad or \quad iii)\ \phi_j(r) \equiv 0, \quad (j = 1, 2, \cdots, 7, or\, 8),$$

respectively, for all poles P_0 and all $r > 0$ on S.

We might replace (17) by the convexity conditions

$$i)\ \phi_j''(r) > 0, \quad ii)\ \phi_j''(r) < 0, \quad or \quad iii)\ \phi_j''(r) \equiv 0, \quad (j = 1, 2, 3, 5, 6, or\ 7),$$

and so on.

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SOME THEOREMS ON CYCLIC POLYGONS INSCRIBED IN A CIRCLE*

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1. Introduction. The Lemoine and Brocard points, as well as their diverse properties, are well known to whoever has studied the geometry of the triangle.

In the present paper we shall show that these points, with their inherent properties, are encountered not only in the triangle but, on the contrary, in all figures which are obtained by joining in pairs the corresponding points of a cyclic projectivity, or more generally of any projectivity on a circle.

The case of the triangle will appear as a particular case if one takes care to consider it as the figure which is obtained by means of the same process, since this gives rise on a circle to a cyclic projectivity of the third order. In this case the center and the axis of collineation are respectively the Lemoine point and line. It goes without saying that in the slightly more general case where the figure is no longer a proper triangle, but a set of any three chords of a circle, the same remarks may be made.

2. Definitions and preliminaries. Let us take, on a conic C, a set of points

* Translated from the French by Howard Eves.

$A_1, A_2, \cdots, A_n$ belonging to a cyclic projectivity of order n. The polygon which is obtained by joining by lines each of these points with its corresponding point (continuing the process until one arrives at the initial point), will be called a *cyclic n-gon.*

The cyclic n-gons will be of the *first, second, third,* $\cdots$ *kind* if each of their sides joins points between which there are no, one, two, $\cdots$ points of the set.

It is evident that if n is odd each set of n points defines $(n-1)/2$ n-gons, while if n is even each set of n points defines $n/2$ n-gons. [1].

On the other hand since the two point sets $(A_1, A_2, \cdots, A_n)$, $(A_2, A_3, \cdots, A_1)$ are projective, the lines $A_1A_2, A_2A_3, \cdots, A_nA_1$ envelop a conic C_1 which has double contact with the conic C. Similarly, the two point sets $(A_1, A_2, A_3, \cdots, A_n)$, $(A_3, A_4, A_5, \cdots, A_2)$ are also projective and consequently the lines $A_1A_3, A_2A_4, \cdots, A_nA_2$ envelop another conic C_2 which has double contact with the conic C, and so on. Thus:

The cyclic n-gons of the first, second, third, $\cdots$ *kind which are inscribed in the same conic C, envelop conics* C_1, C_2, C_3, $\cdots$ *which have double contact with the conic C.*

But since two conics C and C_1, which have double contact, can be projected into two concentric circles, it follows that:

All cyclic n-gons inscribed in a conic may be considered as projections of a regular polygon.

But the circles inscribed in the regular polygons of the first, second, third, $\cdots$ kind are concentric, and thus are tangent to each other and to the circumcircle at the circular points at infinity. Consequently:

The conics C_1, C_2, $\cdots$ *are tangent to C at the same points.*

3. Lemoine point and line. Concerning cyclic n-gons derived from the same cyclic projectivity we shall now establish

THEOREM 1. *All the cyclic n-gons of all kinds, which are inscribed in the same circle, have the same Lemoine point and line.* [2]

First of all we notice that all the cyclic n-gons of all kinds have the same center and axis of collineation.

We shall show that the center and axis of collineation are respectively the Lemoine point and line.

Consider on the circle any n-gon of the first kind, $A_1A_2 \cdots A_n$. The two point sets $(A_1, A_2, A_3, \cdots, A_{n-2}, A_{n-1}, A_n)$, $(A_2, A_3, A_4, \cdots, A_{n-1}, A_n, A_1)$ are projective, and consequently the lines $A_1A_1, A_2A_n, A_3A_{n-1}, \cdots$ pass through a common point M, situated on the axis of collineation of the projectivity. The polar, then, of point M, which will pass through the vertex A_1, will also pass through the center of collineation of the projectivity. But this polar forms a harmonic set with the lines A_1A_1, A_1A_2, A_1A_n. It is then a symmedian of triangle $A_nA_1A_2$ and it follows that for each of its points, the ratio of the distances to the sides A_nA_1, A_1A_2 is equal to the ratio of the segments (A_nA_1), (A_1A_2).

Similarly, the polar of the point of intersection of the lines A_2A_2, A_1A_3, A, $\cdot_4A_n \cdots$ passes through the point A_2 and also through the center of collineation, and is a symmedian of triangle $A_1A_2A_3$, whence each of its points has the ratio of its distances from the sides A_1A_2, A_2A_3 equal to that of the segments (A_1A_2), (A_2A_3). By a similar treatment of the other vertices A_3, A_4, $\cdots$, A_n, the theorem readily follows for cyclic n-gons of the first kind.

The same argument holds for cyclic polygons of other kinds.

4. Brocard points. In regard to Brocard points we shall prove

THEOREM 2. *All the cyclic n-gons of the same kind, inscribed in the same circle, have the same Brocard points.* [3]

Let $A_1A_2A_3 \cdots A_n$ be any cyclic n-gon of the first kind inscribed in a circle, and let K be the Lemoine point. From the center O of the circumcircle let us drop perpendiculars onto the sides A_1A_2, A_2A_3, A_3A_4, $\cdots$, A_nA_1 of the n-gon. These cut the parallels to the sides of the polygon drawn through point K in the points L_1, L_2, L_3, $\cdots$, L_n respectively. The isosceles triangles $A_1L_1A_2$, $A_2L_2A_3$, $\cdots$ are similar. The circle having the segment OK as diameter obviously passes through the points L_1, L_2, $\cdots$, L_n. Let Ω_1 be the point of intersection of the line A_1L_1 with the circle on OK as diameter. We have

$$\begin{aligned} \measuredangle\, KO\Omega_1 &= 180^\circ - \measuredangle\, KL_1\Omega_1 = \measuredangle\, KL_1A_1 \\ &= \measuredangle\, A_2A_1L_1 = \omega = \measuredangle\, A_3A_2L_2 = \measuredangle\, KL_2A_2. \end{aligned}$$

From this we conclude that the line A_2L_2 passes through Ω_1. Similarly we would find that the lines A_3L_3, A_4L_4, $\cdots$, A_nL_n all pass through the same point Ω_1. The lines A_1L_n, A_2L_1, A_3L_2, $\cdots$, in their turn, all pass through a point Ω_2 of the circle on OK as diameter.

The points Ω_1 and Ω_2 are evidently symmetric with respect to the line OK.

COROLLARY 1. *All the cyclic n-gons of all kinds inscribed in the same circle have the same Brocard circle.*

COROLLARY 2. *The Brocard points and the vertices of the isosceles triangles of all the cyclic n-gons of all kinds inscribed in the same circle, lie on the Brocard circle. Also the circles of the two groups of n adjoint circles* [4] *for each n-gon intersect on the Brocard circle.*

It is known that if a side of an angle, of constant magnitude ω, passes through a fixed point Ω, while the vertex of the angle lies on a circle, the other side will envelop a conic which will have double contact with the circle in question, and for which one of the foci is the point Ω.*

COROLLARY 3. *The foci of the conic which envelops all the cyclic n-gons of the same kind, inscribed in the same circle, coincide with the Brocard points of those n-gons.*

* Poncelet, Applications d'analyse et de géométrie (1817), t. II, p. 462.

COROLLARY 4. *The projections of the Brocard points on the sides of all the cyclic n-gons of the same kind, incribed in the same circle, are found on the principal circle of the conic which envelops the n-gons in question.*

5. Calculation of the angle ω. Let $x_1, x_2, \cdots x_n$ be the distances of the point K from the sides $A_1A_2, A_2A_3, \cdots, A_nA_1$, which have lengths $a_1, a_2, \cdots, a_n$ respectively, of a cyclic n-gon of the first kind. We have

$$x_1/a_1 = x_2/a_2 = \cdots = x_n/a_n = \tfrac{1}{2}\tan\omega,$$

or

$$a_1x_1/a_1^2 = a_2x_2/a_2^2 = \cdots = a_nx_n/a_n^2 = \tfrac{1}{2}\tan\omega = 2S/\sum a_i^2,$$

where S is the area of the n-gon. Therefore

$$\tan\omega = 4S/\sum a_i^2 = \sum a_i(4R^2 - a_i^2)^{1/2}/\sum a_i^2,$$

where R is the radius of the circumcircle.

It follows that the quantity $S/\sum a_i^2$ is the same for all the cyclic n-gons of the same kind which are inscribed in the same circle.

6. First Lemoine circle. The generalization to cyclic n-gons of the first Lemoine circle of a triangle is given by the following:

THEOREM 3. *If through the Lemoine point K of an inscriptible cyclic n-gon lines are drawn parallel to the sides, the points of intersection of these parallels with the sides adjacent to their corresponding sides lie on a circle.*

Let A_n, A_1, A_2, A_3 be four successive vertices of the inscribed cyclic n-gon, and let M_1 on A_nA_1, M_2 on A_2A_3, M_3 and M_4 on A_1A_2 be such that KM_1 and KM_2 are parallel to A_1A_2, KM_3 to A_2A_3, and KM_4 to A_nA_1.

Since the line A_2K is a symmedian of triangle $A_1A_2A_3$, the line M_2M_3 will be antiparallel to the side A_1A_3, and consequently will be perpendicular to the line OA_2 as well as to the line LP, L and P being the midpoints of the segments OK and KA_2. But since P is also midpoint of the segment M_2M_3, we will have $LM_2 = LM_3$. For the same reason we will have $LM_1 = LM_4$. But the trapezoid $M_1M_2M_3M_4$ is inscriptible in a circle, since

$$\measuredangle M_2M_3M_4 = \measuredangle A_1A_3A_2 = \measuredangle A_1A_nA_2 = \measuredangle M_1M_4M_3.$$

The center of this circle is the point L.

If we consider the vertices A_1, A_2, A_3, A_4, the corresponding trapezoid will also be inscriptible in a circle which will have the points M_2, M_4 and the center L in common with the circumcircle of the preceding trapezoid. Continuing in this manner around the n-gon, the theorem is established.

7. Some formulas. Let θ be the angle $A_1A_2A_3$, $x_n, x_1, x_2, \cdots, x_{n-1}$ the distances of point K from the side $A_nA_1, A_1A_2, A_2A_3, \cdots, A_{n-1}A_n$, and $a_n, a_1, a_2, \cdots, a_{n-1}$ the lengths of these sides. As above we have

$$x_1/a_1 = x_2/a_2 = \cdots = \tfrac{1}{2} \tan \omega = 2S/\sum a_i^2.$$

On the other hand we have

$$x_1 = (KM_3) \sin \theta = \tfrac{1}{2}a_1 \tan \omega, \qquad x_2 = (KM_2) \sin \theta = \tfrac{1}{2}a_2 \tan \omega,$$

and

$$\begin{aligned}(M_2M_3)^2 &= 4(PM_2)^2 = (KM_3)^2 + (KM_2)^2 - 2(KM_3)(KM_2) \cos \theta \\ &= \tfrac{1}{4}(a_1^2 + a_2^2 - 2a_1a_2 \cos \theta) \tan^2 \omega \csc^2 \theta \\ &= \tfrac{1}{4}(A_2A_3)^2 \tan^2 \omega \csc^2 \theta = R^2 \tan^2 \omega.\end{aligned}$$

Consequently

$$\rho^2 = \tfrac{1}{4}(R^2 + R^2 \tan^2 \omega) = \tfrac{1}{4}R^2 \sec^2 \omega,$$

or

$$\rho = \tfrac{1}{2}R \sec \omega.$$

The length s of the antiparallel (M_2M_3) is

$$s = 4SR/\sum a_i^2 = R \tan \omega.$$

This shows that all these antiparallels are equal.

For the distance $(OK) = d$ we have

$$\begin{aligned}d^2 &= 4(LK)^2 = 4(LM_2)^2 + 4(KM_2)^2 - 4(KM_2)(M_1M_2) \\ &= R^2 \sec^2 \omega + a_2^2 \tan^2 \omega \csc^2 \theta - 2a_2 \tan \omega \csc \theta(x_n \csc \phi + x_2 \csc \theta),\end{aligned}$$

where $\phi = \measuredangle KM_1A_1$. But

$$2x_2 = a_2 \tan \omega = 2R \sin \theta_2 \tan \omega, \qquad 2x_n = a_n \tan \omega = 2R \sin \theta_n \tan \omega,$$

where $\theta_2 = \measuredangle A_2A_1A_3$ and $\theta_n = \measuredangle A_1A_2A_n$. Therefore

$$\begin{aligned}d^2 &= R^2 \sec^2 \omega - 4R^2 \tan^2 \omega \sin \theta_2 \sin \theta_n \csc \theta \csc \phi \\ &= R^2 \sec^2 \omega - 4R^2 \tan^2 \omega \sin \theta_2 \sin \theta_n \csc (\theta_1 + \theta_2) \csc (\theta_1 + \theta_n),\end{aligned}$$

where $\theta_1 = \measuredangle A_1A_3A_2 = \measuredangle A_1A_nA_2$.

Let $1/\lambda$ represent the anharmonic ratio of the four successive vertices A_n, A_1, A_2, A_3. Then

$$\begin{aligned}(A_nA_3A_1A_2) &= 1 - \lambda = \sin \tfrac{1}{2}(A_nA_1) \sin \tfrac{1}{2}(A_2A_3) \csc \tfrac{1}{2}(A_1A_3) \csc \tfrac{1}{2}(A_nA_2) \\ &= \sin \theta_2 \sin \theta_n \csc (\theta_1 + \theta_2) \csc (\theta_1 + \theta_n),\end{aligned}$$

and we have

$$d^2 = R^2 \sec^2 \omega - 4R^2(1 - \lambda) \tan^2 \omega = R^2 - (3 - 4\lambda)s^2.$$

Finally, we have

$$(O\Omega_1) = (O\Omega_2) = sd(s^2 + R^2)^{-1/2}$$

and

$$(K\Omega_1) = (K\Omega_2) = Rd(s^2 + R^2)^{-1/2}.$$

In the case where $n=3$, that is, when the polygon is a triangle, the above formulas become

$$\rho^2 = \tfrac{1}{4}R^2 \sec^2 \omega = R^2(a^2b^2 + b^2c^2 + c^2a^2)/(a^2 + b^2 + c^2),$$
$$s = abc/(a^2 + b^2 + c^2), \qquad d^2 = R^2 - 3s^2.$$

These are the formulas which were given by Lemoine in 1873†.

COROLLARY. *The first Lemoine circle is the same for all the cyclic n-gons of the same kind inscribed in the same circle.*

8. Tucker circles. The first Lemoine circle of a cyclic n-gon may be considered as a particular case of the circles about to be mentioned and which may, from the analogous circles in the geometry of the triangle, be called Tucker circles.

THEOREM 4. *If two inscriptible cyclic n-gons of the same kind are directly homothetic, having the Lemoine point K for center of homothety, then each side of one cuts the sides of the other which are adjacent to the corresponding parallel side, in points on a circle.*

The proof of this theorem is analogous to that of Theorem 3.

THEOREM 5. (Second Lemoine circle) *If, through the Lemoine point K of an inscriptible cyclic n-gon $A_1A_2A_3 \cdots A_n$, we draw antiparallels to the diagonals A_1A_3, A_2A_4, A_3A_5, $\cdots$, the intersections of each of them with the sides adjacent to the corresponding diagonals lie on a common circle.*

This follows because each antiparallel has for length

$$2s = 2R \tan \omega = \text{constant},$$

and has K as midpoint.

COROLLARY. *The second Lemoine circle is the same for all the cyclic n-gons of the same kind inscribed in the same circle.*

9. Generalization of the preceding theorems. We conclude by generalizing some of the above theorems.

THEOREM 6. *The center and axis of collineation of any projectivity π on a circle are the Lemoine point and line of the figure formed by the chords which join, in pairs, the corresponding points of π.*

Let the two point sets $(A_1, A_2, A_3, A_4, \cdots)$ and $(A_1', A_2', A_3', A_4', \cdots)$

† Lemoine, Nouvelles Annales de Mathématiques, 1873, p. 364, and Association française pour l'avancement des sciences, 1873, p. 91, VI.

be projective on a circle, and let K and u be their center and axis of collineation.

Consider any two chords A_1A_1' and A_2A_2'. The lines A_1A_2' and $A_1'A_2$ intersect in a point M on the axis u. If A_{12} is the point of intersection of the chords A_1A_1' and A_2A_2', then for all points on the line MA_{12} the ratio of the distances to the two considered chords is equal to the ratio of the lengths of these two chords.

The polar of M with respect to the circle passes through the center of collineation K and through the point A_{12}. It is, moreover, the fourth harmonic of the two chords and the line MA_{12}. Consequently A_1A_2, KA_{12}, $A_2'A_1'$ are concurrent in a point N, and the ratio of the distances of any point on NA_{12}, and therefore of the point K, from the two chords, is equal to the ratio of the lengths of these same chords. Obviously one arrives at the same conclusion for all pairs of chords.

We see, then, that the point K is the Lemoine point of the figure formed by the chords A_1A_1', A_2A_2', $\cdots$. Similarly, the line u is the Lemoine line of the same figure.

COROLLARY. *The figure formed by the chords mentioned above possesses Brocard points, a Brocard circle, and first and second Lemoine circles.*

The Brocard points are the foci of the conic C which envelops the chords in question.

The projections of the Brocard points on the chords lie on the principal circle of the conic C.

As for the radii of the Lemoine circles, the distances of the Lemoine and Brocard points from the center of the circumcircle, and the value of tan ω, we have the same corresponding formulas.

Definitions

1. For a 6-gon with successive vertices A_1, A_2, A_3, A_4, A_5, A_6, we have a 6-gon of the first kind (the figure $A_1A_2A_3A_4A_5A_6$), a 6-gon of the second kind (the figure formed by the two triangles $A_1A_3A_5$, $A_2A_4A_6$), and a 6-gon of the third kind (the figure formed by the three double lines A_1A_4, A_2A_5, A_3A_6).

2. If, in the plane of a polygon, there exists a point K such that the distances of the point to the sides of the polygon are proportional to the lengths of these sides, this point is called the *Lemoine point* of the polygon. If, moreover, the polygon is inscribed in a circle, the polar of K with respect to the circle is called the *Lemoine line* of the polygon.

3. If, in the plane of a polygon, there exist two points Ω_1 and Ω_2 such that

$$\begin{aligned} &\sphericalangle\, \Omega_1A_1A_2 = \sphericalangle\, \Omega_1A_2A_3 = \cdots = \sphericalangle\, \Omega_1A_nA_1 \\ &= \sphericalangle\, \Omega_2A_2A_1 = \sphericalangle\, \Omega_2A_3A_2 = \cdots = \sphericalangle\, \Omega_2A_1A_n = \omega, \end{aligned}$$

these points are called *Brocard points* of the polygon.

4. A circle passing through two successive vertices of a polygon and tangent at one of these points to either the preceding or the following side of the polygon is called an *adjoint circle* of the polygon.

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THE SIEVE OF ERATOSTHENES AND THE MÖBIUS STRIP

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The Möbius strip is formed from a long rectangle of paper by rotating one end of the strip 180° relative to the other end and joining the two ends together. The resulting strip is a one-sided surface. It may likewise be shown that the strip will be one-sided for all rotations which represent odd multiples of 180°, but will be two-sided for all rotations which represent even multiples of 180°.

The ideal plane strip, prior to deformation into the Möbius figure, may be considered as a prism whose base has only two sides. From the practical standpoint, however, the strip has a finite thickness. This consideration forms the basis for a similar study of prisms of 3, 4, 5, $\cdots$, N sides when deformed in a manner analogous to that of the Möbius strip.

Suppose, for example, that the two bases of a regular triangular prism are joined after one base has been rotated about the long axis of the prism by 120° with respect to the other base. All three sides of the prism are now continuous. The same will be true for a similar rotation of 240°. However, for rotations of 0° and 360°, the figure will present three sides. These are the only possibilities, since they will be repeated upon further twisting, depending upon whether the total rotation of one base relative to the other is $360\ N°$, $360\ N°+120°$, or $360\ N°+240°$, where N is an integer.

In the case of the square prism, there may be 1, 2, or 4 "sides" depending upon the extent of the twist prior to the joining of the ends. If one base is rotated relative to the other base by 0° or by any multiple of 360°, four sides will of course be presented. If, however, the rotation is represented by $90°+360\ N°$ or $270°+360\ N°$, there will be only one side. For all rotations represented by $180°+360\ N°$ two sides will result. These possibilities will be repeated indefinitely with continued twisting of the prism prior to the joining of the ends.

By a generalization of this process, the possibilities may be derived for any prism of number of sides S when one base is rotated by multiples of $360°/S$, relative to the other base. The following array gives the number of surfaces obtained for prisms up to ten sides, when twisted by $0\cdot(360°/S)$, $1\cdot(360°/S)$, $2\cdot(360°/S)$, $\cdots$, $S(360°/S)$. The first number in each horizontal represents the number of sides of the regular prism at 0° rotation. Successive numbers from left to right represent the number of surfaces obtained for the first, second, etc. multiples of $360°/S$.

It is apparent that every horizontal repeats itself both diagonally and vertically. It is also obvious that each row of the array may be generated from purely numerical considerations by setting down, from left to right, the highest factor

common to the number of sides represented by that row and the successive integers 1, 2, 3, $\cdots$, and so on. For example, in the 6-horizontal, the numbers 1, 2, 3, 2, 1 are the highest common factors respectively of 6 and 1, 6 and 2, 6 and 3, 6 and 4, and 6 and 5.

2 1 2
3 1 1 3
4 1 2 1 4
5 1 1 1 1 5
6 1 2 3 2 1 6
7 1 1 1 1 1 1 7
8 1 2 1 4 1 2 1 8
9 1 1 3 1 1 3 1 1 9
10 1 2 1 2 5 2 1 2 1 10

From the above considerations it is to be noted that the only integer occurring in a row representing a prime number, other than itself, is 1. The requisite condition for the occurrence of a prime horizontal, therefore, is the alignment of 1's in all prime diagonals. This eliminates every second number after 2, every third number after 3, every fifth number after 5, and so on, thereby duplicating the Sieve of Eratosthenes.

A NON-SINGULAR POLYHEDRAL MÖBIUS BAND WHOSE BOUNDARY IS A TRIANGLE

BRYANT TUCKERMAN, Cornell University

A Möbius band is a non-orientable surface which can be defined abstractly by proper identification of the opposite ends of a rectangle, with a sense opposite to that which would make the rectangle into a cylinder. Its boundary is a single simple closed curve.

A physical model of a Möbius band is customarily made by pasting together the ends of a long narrow strip of paper after twisting one end through 180° with respect to the other. The boundary of the usual model is an unknotted closed space curve. Another model of the Möbius band in three-dimensional space is the "cross-cap," of which the boundary may be a circle. However, the cross-cap has a line of self-intersections, that is, it is singular.

It does not seem well-known that there are non-singular models (that is, homeomorphs) of the Möbius band in three-dimensional Euclidean space, such that the boundary is a plane curve, for example a circle. However, such models are discussed and illustrated in *Anschauliche Geometrie*, by D. Hilbert and S. Cohn-Vossen (Berlin, Springer, 1932; New York, Dover, 1944), pp. 278–279, figs. 307–308. (The cross-cap is shown in fig. 310, and the usual model of the Möbius band is on p. 269.) Those models have curved surfaces.

The purpose of this note is to present a convenient *polyhedral*, non-singular model of a Möbius band, in three-dimensional Euclidean space, whose boundary lies in a plane and is in fact a triangle. Its seven vertices are the six vertices and the centroid of a regular octohedron. Its nine faces are six of the equilateral triangular faces of the octohedron, and one large and two small isosceles right triangular faces each containing the centroid of the octohedron. The boundary is a triangle consisting of two edges and a diagonal of the octohedron. (The triangle could be deformed to a circle by an isotopic, but naturally not polyhedral, deformation of the space.) By bisection of the large right triangle, the model can be made into a simplicial complex having ten 2-cells, seventeen 1-cells, and seven 0-cells.

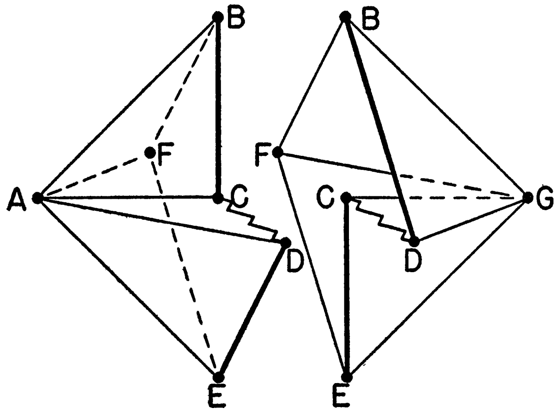

FIG. 1

Figure 1 shows the model after being cut into two pieces which were then pulled apart. Along the back the cut is along two edges of the octohedron; in the front the cut (shown jagged) is through the large right triangle. This decomposition makes it evident that the model represents a Möbius band. The triangular boundary is indicated by heavy lines. It will be noted that the Möbius band meets the plane of its boundary both inside and outside the boundary.

This model is related to the "heptahedral" model of the projective plane (ibid., pp. 266 ff.). Both are subsets of the figure whose faces are the faces of a regular octohedron together with three square faces whose vertices are vertices of the octohedron. But this model of the Möbius band is not a subset of the heptahedron, nor is it singular as is the heptahedron.

The model is readily constructed from equilateral and isosceles right triangular faces joined as in Figure 2. The jagged line corresponds to the jagged cut in Figure 1, and can be omitted. The heavy lines will be the boundary of the model. In each triangular face, the + or − sign at an edge represents the direction of the bend at that edge in the finished model, as viewed from the side of the face which is on top in Figure 2. To construct the model, bend along the interior

edges, downward (say) where marked +, upward where marked −, and juxtapose and seal together the two edges of each of the four pairs marked AC, GC, BF, EF.

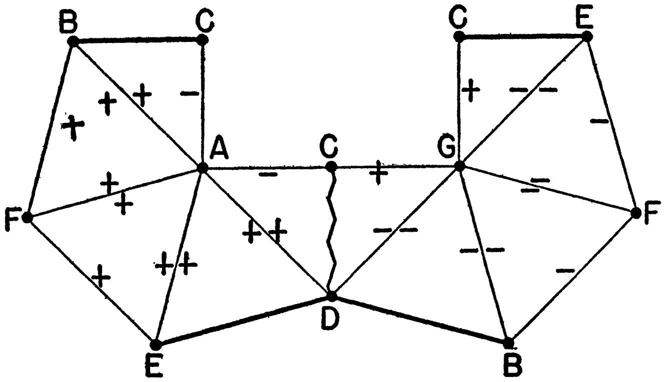

FIG. 2

Two modifications of this model, having fewer faces, may be mentioned. If, in Figure 1, the vertices B and E are displaced toward the rear, then the vertex F and the four triangular faces which meet at F may be replaced by two triangular faces ABE and GBE. There results a model having seven triangular faces. On the other hand, if the vertex C is displaced toward the front, then the vertex F and the four triangular faces which meet at F may be replaced by a square face $ABGE$, while the triangular face AGD becomes a reëntrant quadrilateral face $ACGD$. There results a model having six faces, namely four triangular faces, a square face, and a reëntrant quadrilateral face. In both of these modifications, the boundary of the Möbius band is a reëntrant plane quadrilateral.

It is a pleasure to acknowledge my indebtedness to Professor H. Hopf for interesting conversations with him while he was in this country a year ago, since the idea for this note arose in one of these discussions.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

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ONE-SIDED MAXIMA AND MINIMA

J. D. MANCILL, University of Alabama

1. Introduction. Repeated inquiries from students and colleagues concerning the subject of this discussion, doubtless due to the incomplete treatment of it in the literature, have prompted this attempt to give a complete analysis of the

problem of one-sided (unilateral) extremes for functions of a single variable. Almost without exception, the text books on the calculus restrict their discussion of maxima and minima of functions to the interior points of the range of definition of the function. The author knows of no text which treats fully sufficient conditions for an end-point of the range to be a maximum or a minimum. When the end-point extremes are mentioned they are frequently referred to as absolute extremes, since the usual definition of relative extreme excludes the end-points. But this is undesirable since the end-point extreme may not be absolute. It is better to state the definition of relative extreme so as to include the end-points, and develop necessary conditions and sufficient conditions that an end-point be a relative maximum or minimum. Then the procedure, even in elementary calculus, would be to test the end-points along with the zeros of the first derivative for relative extremes, and then determine the absolute extremes of the function from all the extremes on the closed range.

In a recent note Oakley* notes that the end-point extremes may be transformed into interior point extremes by a suitable transformation on the independent variable. Although this explains why a point may be an end-point maximum in one formulation of a problem and an interior maximum in another, such a transformation is not a practical way of handling end-point maxima. For first, it renders even the simple problems very complicated, especially when it comes to checking the sufficient conditions for a maximum or a minimum. Second, it is entirely unnecessary. This is due to the fact that an end-point is in general an extreme and one only needs to determine whether it is a relative maximum or a relative minimum. Some examples will illustrate these remarks.

2. **Functions of one variable.** We shall consider a single-valued function

$$y = f(x), \qquad a \leqq x \leqq b,$$

defined on the closed interval (a, b). The following definitions will be needed:

DEFINITION 1. $f(x)$ belongs to the class C of continuous functions at $x=x_1$, $a<x_1<b$ if and only if $\lim_{x\to x_1} f(x)=f(x_1)$.

DEFINITION 2. $f(x)$ belongs to the class of functions C at the end-point a if and only if $\lim_{x\to a^+} f(x)=f(a)$ and at the end-point b if and only if $\lim_{x\to b^-} f(x)=f(b)$.

DEFINITION 3. $f(x)$ belongs to the class of functions C^n if and only if $f^{(n)}(x)$ belongs to C.

DEFINITION 4. $f'_+(x_1)=\lim_{h\to 0^+}[f(x_1+h)-f(x_1)]/h$, and similarly for $f'_-(x_1)$.

DEFINITION 5. $f(x)$ belongs to the class of functions D' on (a, b) if and only if $f(x)$ belongs to C on (a, b) and $f'(x)$ belongs to C on (a, b) except at a finite number of points x_1 at which the derivatives $f'(x_1+)$ and $f'(x_1-)$, on the right and left of x_1 respectively, exist, where $f'(x_1+)=\lim_{x\to x_1^+} f'(x)$ and similarly for $f'(x_1-)$.

DEFINITION 6. $f(x)$ has an *absolute maximum* at a point x_1 of the interval (a, b) if and only if $f(x_1)\geq f(x)$ for all values of x in (a, b). A function $f(x)$ has a

* End-point maxima and minima, this MONTHLY, vol. 54 (1947), pp. 407–409.

relative maximum at a point x_1 of (a, b) if and only if $f(x_1) \geqq f(x)$ for all values of x on (a, b) in a certain neighborhood of x_1.

Obvious modifications of the inequalities are necessary for the definition of absolute and relative minima. A well-known property of continuous functions states that a function $f(x)$ which is continuous on the closed interval (a, b) takes its absolute maximum and minimum values at least once in that interval. Therefore, one must consider the end-points of the closed interval in determining the absolute extremes of the function on that interval.

The reader should distinguish between $f'_+(x_1)$ and $f'(x_1+)$. This is easily done by considering the function $x^2 \sin (1/x)$. In this case $f'_+(0)=0$ but $f'(0+)$ does not exist.

The geometric meaning of Definition 5 is that the curve defined by the function $y=f(x)$ on the range (a, b) has a finite number of corners, or in other words, the curve is composed of a finite number of sub-arcs each of class C'.

THEOREM 1. *If a continuous function $f(x)$ assumes a relative maximum value at $x=x_1$ of the interval (a, b); and if $f'_-(x_1)$ exists, then $f'_-(x_1) \geqq 0$; and if $f'_+(x_1)$ exists, then $f'_+(x_1) \leqq 0$. The inequalities are reversed if x_1 is a minimum.*

This theorem follows immediately from Definition 3 and the inequalities

$$[f(x_1 + h) - f(x_1)]/h \geqq 0 \qquad \text{for } h < 0,$$
$$[f(x_1 + h) - f(x_1)]/h \leqq 0 \qquad \text{for } h > 0,$$

for the case when x_1 is a maximum.

COROLLARY (FERMAT'S THEOREM). *If a function $f(x)$ assumes a relative maximum or a minimum at $x=x_1$, $a<x_1<b$ at which $f(x)$ is differentiable, then $f'(x_1)=0$.*

The corollary follows from the fact that $f'(x_1)$ exists if and only if $f'_-(x_1)$ and $f'_+(x_1)$ exist and are equal. The end-points are excluded from the corollary since the left-hand derivative at $x=a$ and the right-hand derivative at $x=b$ are not considered in the theorem.

THEOREM 2. *A continuous function $f(x)$ assumes a relative maximum at $x=x_1$; x_1 interior to (a, b) if $f'_-(x_1)>0$ and $f'_+(x_1)<0$: at $x=a$ if $f'_+(a)<0$; and at $x=b$ if $f'_-(b)>0$. Similar conditions with inequalities reversed are sufficient for a minimum.*

These conclusions follow immediately from Definition 4. The function $f(x) =1-|x|$, $-1 \leqq x \leqq 1$ affords a simple illustration of these results. In this example $f'_-(0)=1$ and $f'_+(0)=-1$. Therefore, $f(x)$ has a relative maximum at $x=0$ which is also the absolute maximum on that range.

THEOREM 3. *If $f(x)$ is of class C^n within (a, b), $f^{(n)}(a+)$ exists, and*

$$f^{(k)}(a) = 0, \qquad k = 1, \cdots, n-1,$$
$$f^{(n)}(a+) < 0,$$

then $f(x)$ has a relative maximum at $x=a$. Similar conditions with the inequalities

reversed are sufficient for a minimum.

From Taylor's formula and the hypotheses of the Theorem, we have

$$f(x) - f(a) = (x - a)^n f^{(n)}(t)/n!, \qquad a < t < x.$$

The sign of the right member of this equation depends only upon the sign of $f^{(n)}(t)$, since $x > a$. Also, t approaches a with x and therefore, if x is sufficiently near a the sign of the right member is negative.

As an example consider the function $f(x) = -x^{3/2}$, $0 \leqq x \leqq 1$. It is easily seen that $f'(0) = 0$ and $f''(0+) = -\infty$. Therefore $f(x)$ has a relative maximum at $x = 0$ which is also the absolute maximum on that range.

THEOREM 4. *If $f(x)$ is of class C^n within (a, b), $f^{(n)}(b-)$ exists and*

$$f^{(k)}(b) = 0, \qquad k = 1, \cdots, n-1,$$
$$f^{(n)}(b-) < 0, \qquad n \text{ even},$$
$$f^{(n)}(b-) > 0, \qquad n \text{ odd},$$

then $f(x)$ has a relative maximum at $x = b$. Similar conditions with the inequalities reversed are sufficient for a minimum.

It follows from the hypotheses of the theorem and from Taylor's formula that

$$f(x) - f(b) = (x - b)^n f^{(n)}(t)/n!, \qquad x < t < b.$$

In this case the sign of the right member of this equation depends upon the sign of $f^{(n)}(t)$ and whether n is even or odd since $x < b$. Therefore, the conclusions of the theorem follow as in the proof of Theorem 3.

As an example consider the function $f(x) = -(-x)^{5/2}$, $-1 \leqq x \leqq 0$. It is easily seen that $f'(0) = f''(0) = 0$, and $f^{(3)}(0-) = +\infty$. Therefore, $f(x)$ has a relative maximum at $x = 0$ which is also the absolute maximum on that range.

We may say that $f(x)$ on (a, b) has a *left-hand maximum* at $x = x_1$ if $f(x_1) \geqq f(x)$ for all values of x in a certain right hand neighborhood of x_1, for example in Theorem 3. Likewise, we may say $f(x)$ has a *right-hand maximum* at $x = x_1$ if $f(x_1) \geqq f(x)$ for all values of x in a certain left-hand neighborhood of x_1, for example in Theorem 4. Obviously then, if *x_1 is interior to the interval (a, b) and $f(x)$ has a left-hand and a right-hand maximum at x_1, at which $f(x)$ is continuous, then $f(x)$ has a maximum at $x = x_1$.* For example, the function

$$f(x) = -(-x)^{5/2}, \qquad -1 \leqq x \leqq 0,$$
$$= -x^{3/2}, \qquad 0 \leqq x \leqq 1,$$

has a maximum at $x = 0$. The following well-known theorem follows easily:

THEOREM 5. *If $f(x)$ is of class C^{2n} on (a, b), x_1 is interior to (a, b), and*

$$f^{(k)}(x_1) = 0, \qquad k = 1, \cdots, 2n-1,$$
$$f^{(2n)}(x_1) < 0,$$

then $f(x)$ has a maximum at $x=x_1$. Similar conditions with inequalities reversed are sufficient for a minimum.

It may also be remarked here that if a continuous function $f(x)$ has a left-hand maximum and a right-hand minimum (or vice-versa) at x_1 interior to (a, b), then $f(x)$ has an inflection at x_1. If $f'_-(x_1) \neq f'_+(x_1)$, then $f(x)$ has an abrupt inflection at x_1. For example the function

$$\begin{aligned} f(x) &= -(-x)^{5/2}, && -1 \leqq x \leqq 0, \\ &= x^{3/2}, && 0 \leqq x \leqq 1, \end{aligned}$$

has an inflection at $x=0$.

3. **An application.** As a physical application, let us consider one of the problems treated by Oakley in his note already referred to:

A man is in a boat at P one mile from the nearest point A on shore. He wishes to go to B which is farther down the shore M miles from A. If he can row r miles an hour and walk w miles an hour, toward what point C should he row in order to reach B in least time?

Assume the shore line to be straight with the point B to the right of A and let $x=AC$. Then the time $t(x)$ that it takes to go from P to B is

$$t(x) = (1 + x^2)^{1/2}/r + (M - x)/w, \qquad 0 \leqq x \leqq M,$$

whence

$$t'(x) = x/r(1 + x^2)^{1/2} - 1/w,$$

and $t'(0)<0$. Therefore, $t(x)$ has a relative maximum at $x=0$ for all positive values of r, w, and M, and consequently the man would not row towards A. If $r \geqq w$, then $t'(M)<0$ and $t(x)$ has a relative minimum at $x=M$. This is the absolute minimum since in this case $t'(x) \neq 0$ on $0 \leqq x \leqq M$. Therefore, the man should row directly to B. If $r<w$, then $t'(w)=0$ has the unique solution

$$x_C = r/(w^2 - r^2)^{1/2}.$$

It is easily shown that in this case $t(x_C)<t(M)$ if $0<x_C<M$, and consequently the man should row to x_C.

There is still one other case which was not considered by Oakley, that is when $x_C=M$, or when $r=wM/(1+M^2)^{1/2}<w$. In this case $t'(M)=0$ and $t''(M)>0$. It follows from Theorem 4 that $t(x)$ has a relative minimum at $x=M$ and this is the absolute minimum since $t'(x)$ vanishes only at x_C. Therefore, the man should row directly to B in this case.

It is the author's firm conviction that if maxima and minima problems, even in elementary calculus, are analyzed as exemplified in these pages, the student will meet no difficult or astonishing "answers." This is due in part to the fact that the student will not be led to the erroneous conclusion that maxima and minima occur only at the vanishing of the first derivative.

FERMAT'S PRINCIPLE AND CERTAIN MINIMUM PROBLEMS

A. V. Baez, Stanford University

The object of this note is to show the relation between Fermat's Principle and the end-point minimum problems discussed by Oakley.*

It is known that of all possible paths connecting two points A and B in space a light ray leaving A chooses the path which enables it to reach B in the least time. This is Fermat's Principle from which the laws of reflection and refraction follow. They can be used to solve certain minimum problems without differentiation. Before discussing Oakley's problems consider a simpler one.

A farmer has to walk from his house to a straight river, fill his pail and carry it to his barn in the least time. At what point should he fill his pail? The solution by geometrical optics is obvious. His path to and from the river must make equal angles with the normal to the river.

Suppose he always walks to the river at a speed v_1 and away from it at a speed v_2. Where then should he dip his pail? Snell's Law immediately suggests that his path of approach to the river should make an angle θ_1 and that away from the river an angle θ_2 with the normal so that

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}.$$

The problem which follows was discussed by Oakley, and his notation is used, together with the above equation which expresses Snell's Law.

Problem. *A man is in a boat at P one mile from the nearest point A on shore. He wishes to go to B which is farther down the shore M miles from A. If he can row r miles an hour and walk w miles an hour, toward what point C should he row in order to reach B in least time?*

Let $AC=x$. Snell's Law indicates that

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{r}{w}.$$

Use $\theta_2=90°$ and geometry to obtain

$$\frac{x}{\sqrt{1+x^2}} = \frac{r}{w}.$$

Solving for x we have

$$x = \frac{r}{\sqrt{w^2 - r^2}}$$

which Oakley obtains by calculus. The optical analysis makes clear why the result is independent of M(θ_1 is the "critical" angle). Optically it is also apparent that if $r<w$ and $r \geqq M\sqrt{w^2-r^2}$ or if $r \geqq w$ (giving $\sin\theta_1 \geq 1$) the man should go straight to B.

* End-Point Maxima and Minima, this Monthly, vol. 54 (1947) pp. 407–409.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 816. *Proposed by P. L. Chessin, New York, N. Y.*

A announces a two digit number from 01 to 99. B reverses the digits of this number and adds to it the sum of its digits and then announces *his* result. A continues in the same pattern. All numbers are reduced modulo 100, so that only two digit numbers are announced. What choices has A for the initial number in order to insure that B will at some time announce 00?

E 817. *Proposed by E. V. Hofler, Colgate University*

If the graph for a polynomial of the fourth degree has two real points of inflection, then the secant through these two points and the curve will bound three distinct areas. Show that two of these areas are equal and the largest area is equal to the sum of the other two.

E 818. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find four different fractions, each of the form $m/(m+1)$, such that their sum is an integer. (For example: $1/2+2/3+6/7+41/42=3=1/2+2/3+8/9+17/18$.)

E 819. *Proposed by H. F. Sandham, Trinity College, Ireland*

If
$$S_n = 1/1 + 1/2 + \cdots + 1/n,$$
prove that
$$\gamma < S_p + S_q - S_{pq} \leqq 1,$$
where γ is Euler's constant.

E 820. *Proposed by Kaidy Tan, Chip-Bee Institute, Amoy, Fukien, China*

If ABC is an equilateral triangle, and P is any point on the circumference of the inscribed circle, prove synthetically that $(PA)^2+(PB)^2+(PC)^2$ is constant.

SOLUTIONS

Eadem Numero Mutata Resurgo

E 784 [1947, 412]. *Proposed by R. E. Gaines, University of Richmond*

Show that the locus of the intersection of two successive perpendicular

tangents to a logarithmic spiral is another logarithmic spiral.

Solution by Ernest Trost, Zürich, Switzerland. We shall establish a more general result. Let $\alpha \leqq 90°$ be the constant angle between a tangent t to the logarithmic spiral and the radius $r = A^\theta$ drawn through the point of contact, $P(r, \theta)$. If $\beta \leqq 90°$ be the *constant* angle between t and another tangent t_1 at $P_1(r_1, \theta_1)$, then $\theta_1 - \theta = \beta$, and the angles β, $180° - \alpha$, $180° - \beta$, α of the quadrangle Q with sides r, t, t_1, r_1 are constant. Since $r_1 = rA^\beta$, Q is unaltered in form as P moves on the spiral L. Thus the locus of each point of Q is a spiral congruent with L after a suitable turning about the pole.

Also solved, by analytical methods, by Ragnar Dybvik, R. T. Hood, Roscoe Woods, and the proposer. The above solution appears in a footnote of a paper by Trost, *Zur Theories der isoptischen Kurven*, to appear in *Nieuwe Archief voor Wiskunde.*

Equilateral Triangle About a Regular Polygon

E786 [1947, 471]. *Proposed by Michael Goldberg, Washington, D.C.*

Suppose that an equilateral triangle is circumscribed about a regular n-gon, where $n = 3k \pm 1$, so that one side of the n-gon lies on one of the sides of the triangle. Show that the angle subtended by this side of the n-gon at the opposite vertex of the triangle is $2\pi/3n$.

Solution by the Proposer. Let D be a vertex of the given n-gon touched by a side of the circumscribing equilateral triangle. Let P be the foot of the perpendicular from the center O of the n-gon upon the base AB, where A is chosen on the same or opposite side of PO according as $n = 3k - 1$ or $3k + 1$. Let the angle AOP be 3θ. Locate C on PO, and the point E, so that $EO = EC = OA$ and the angles EOC and ECO are equal to θ. Then the points C, E, A are collinear. Join C to D. Then

$$\begin{aligned}\text{angle } DOE &= \text{angle } DOC \pm \theta \\ &= k\pi/n \pm \pi/3n = \pi(3k \pm 1)/3n = \pi/3,\end{aligned}$$

since $3k \pm 1 = n$. Therefore triangle DOE is equilateral, and it follows in all cases that angle DCO is $\pi/6$, and that C is the vertex of the circumscribing equilateral triangle.

Also solved by Roger Lessard.

Editorial Note. The above solution proves that if regular polygons having $3k \pm 1 (k = 1, 2, \cdots)$ sides be inscribed in the same circle $O(A)$, and all have a common vertex A, then the equilateral triangles circumscribed about these polygons and having one side coincident with a side of the polygon issuing from A, have their opposite vertices on the trisectrix with center O and double point A.

Elaboration of Heron's Formula

E 787 [1947, 471]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a triangle ABC, show that

$$s^4 + (s-a)^4 + (s-b)^4 + (s-c)^4 - a^4 - b^4 - c^4 = 12S^2,$$

where a, b, c are the sides, s the semi-perimeter, and S the area.

I. *Solution by Leo Moser, University of Manitoba.* Replacing s and S by their known values in terms of a, b, c, it becomes necessary and sufficient to show that

$$\left(\frac{a+b+c}{2}\right)^4 + \left(\frac{-a+b+c}{2}\right)^4 + \left(\frac{a-b+c}{2}\right)^4 + \left(\frac{a+b-c}{2}\right)^4 - a^4 - b^4 - c^4$$
$$= 12\left(\frac{a+b+c}{2}\right)\left(\frac{-a+b+c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{a+b-c}{2}\right).$$

If we consider the left side as a polynomial of degree four in a, and note that it vanishes for $a=b+c$, $a=b-c$, $a=-b+c$, $a=-b-c$, then we get the identity apart from the numerical factor $12/16$ on the right side. This may be determined by taking, say, $a=1$, $b=c=0$.

The result is essentially the same as that given (with proof) in art. 808, p. 534, of Todhunter's *Algebra*, 1885 edition.

II. *Solution by M. LeLeiko, Rutgers University.* In the identity

$$(p+q+r)^4 + p^4 + q^4 + r^4 = (p+q)^4 + (q+r)^4 + (r+p)^4 + 12pqr(p+q+r),$$

set $p=s-a$, $q=s-b$, $r=s-c$. Then $p+q+r=s$, $p+q=c$, $q+r=a$, $r+p=b$. With the aid of Heron's formula the result now follows.

Similar results may be obtained by making the same substitutions in the identities

$$(p+q+r)^3 + p^3 + q^3 + r^3 = (p+q)^3 + (q+r)^3 + (r+p)^3 + 6pqr$$

and

$$(p+q+r)^2 + p^2 + q^2 + r^2 = (p+q)^2 + (q+r)^2 + (r+p)^2.$$

Also solved by Daniel Block, W. E. Buker, P. L. Chessin, Monte Dernham, Ragnar Dybvik, J. C. Eaves, J. F. Heyda, Meyer Karlin, B. R. Leeds, Roger Lessard, Eleanor Ranking, O. M. Rasmussen, Kaidy Tan, C. W. Trigg, and the proposer.

Several solvers made the observation that a, b, c may be *any* numbers (not only those capable of representing the sides of a triangle) if we replace s by $(a+b+c)/2$ and S^2 by $s(s-a)(s-b)(s-c)$.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4295. *Proposed by Irving Kaplansky, University of Chicago*

Show that any group with more than two elements admits an automorphism other than the identity.

4296. *Proposed by H. S. Wall, University of Texas*

It is known that when the continued fraction

$$x = p - \cfrac{q}{p - \cfrac{q}{p - \cdots}}$$

converges, then its value is the numerically larger root of the equation $x^2-px+q=0$. On the other hand, Newton's formula for computing the roots by successive approximation is

$$x_{k+1} = x_k - \frac{x_k^2 - px_k + q}{2x_k - p} = \frac{x_k^2 - q}{2x_k - p}, \qquad k = 0, 1, 2, \cdots.$$

Show that if x_0 is an approximant of the continued fraction, then $x_1, x_2, x_3, \cdots$ are approximants of the continued fraction.

4297. *Proposed by Paul Erdös, Syracuse University*

Put $\phi_k^{(n)} = \sum r_1 r_2 \cdots r_k$, where the r's run through all integers $\leqq n$, and the r's are all different. $\phi_n^{(n)} = n!$ It is well known that $\phi_{n-1}^{(n)} \not\equiv 0 \pmod{\phi_n^{(n)}}$. Prove that if n is sufficiently large $\phi_k^{(n)} \not\equiv 0 \pmod{\phi_n^{(n)}}$ for any k. (This is not always true; e.g. $n=3$, $k=1$.)

4298. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a tetrahedron $ABCD$, the incenter of which is I, the perpendicular at A to the faces ACD, ADB, ABC respectively cut the planes ICD, IDB, IBC in A_1, A_2, A_3. (1) The perpendicular through A to the plane $A_1A_2A_3$ passes through the point of contact of the inscribed sphere with the opposite face BCD. (2) The analogous property is true for a triangle ABC.

4299. *Proposed by R. C. Lyness, Crakemarsh, Uttoxeter, Staffordshire, England*

If the difference between two consecutive cubes is a square, then it is the square of the sum of two consecutive squares; e.g., $8^3-7^3=(3^2+2^2)^2$.

SOLUTIONS

Six Spheres Associated with a Tetrahedron

3981. [1941, 69] *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let S_i, $(i=1, 2, 3, 4, 5, 6)$, be the spheres of similitude of the spheres with centers at the vertices of the tetrahedron $A_1A_2A_3A_4$ such that the square of the radius of any one of the latter is equal to one-half of the sum of the squares of the edges of the opposite face. (1) Examine the relative positions of the spheres S_i. (2) Show that these six spheres are orthogonal to the circumspheres of $A_1A_2A_3A_4$ and of its anticomplementary tetrahedron. (3) The powers of the extremities of an edge of $A_1A_2A_3A_4$ with respect to the sphere S_i whose center is on the opposite edge are independent of the length of that last edge. (4). The spheres S_i' symmetric to the spheres S_i with respect to the midpoint of the edges on which they are centered intersect in two points collinear with the circumcenter of $A_1A_2A_3A_4$.

*Solution by R. Bouvaist, Vincelles, Saône-et-Loire, France.** We shall designate the sphere (S_j) whose center ω_{ik} is on the edge A_iA_k by (ω_{ik}). Let $A_2A_3=a$, $A_1A_4=a'$, $A_1A_3=b$, $A_2A_4=b'$, $A_1A_2=c$, $A_3A_4=c'$, and let h_i be the length of the altitude drawn from A_i. If the tetrahedron $A_1A_2A_3A_4$ is taken as the reference tetrahedron, the equation in normal tetrahedral coördinates of the circumsphere (0) of $A_1A_2A_3A_4$ may be written as

$$S \equiv \sum \frac{a^2yz}{h_2h_3} = 0.$$

Let R_i denote the radius of the given sphere of center A_i.

(1) The six points ω_{ik} are coplanar. We have

$$\overrightarrow{A_1\omega_{12}} = \frac{R_1^2}{R_1^2-R_2^2}\overrightarrow{A_1A_2}, \qquad \overrightarrow{A_1\omega_{13}} = \frac{R_1^2}{R_1^2-R_3^2}\overrightarrow{A_1A_3},$$

$$\overrightarrow{A_1\omega_{14}} = \frac{R_1^2}{R_1^2-R_4^2}\overrightarrow{A_1A_4}.$$

The plane (P) determined by ω_{12}, ω_{13}, ω_{14} has as its equation

$$P \equiv \sum \frac{R_1^2x}{h_1} = 0$$

which is satisfied by the coördinates of all six points ω_{ik}. It may be noted that (P)

* Translated and corrected by W. E. Byrne, Virginia Military Institute.

$$\bar{\omega}(A_3) = -\frac{1}{R^2 - R_2^2}[b^2R_2^2 - a^2R_1^2]$$
$$= \frac{b^2(a'^2 + b^2 + c'^2) - a^2(a^2 + b'^2 + c'^2)}{a'^2 + b^2 - (a^2 + b'^2)}$$

which does not depend on c.

(4) Let (ω'_{ik}) be symmetrical to (ω_{ik}) with respect to the midpoint of the edge A_iA_k. The centers ω'_{ik} are located in a plane (P'), the reciprocal transversal plane of (P) with respect to the tetrahedron $A_1A_2A_3A_4$.‡ The equation of (P') is:

$$P' \equiv \sum \frac{x}{n_1R_1^2} = 0.$$

The sphere (ω'_{ik}) is orthogonal to (O), since the extremities of the diameter of (ω'_{ik}) on A_iA_k are harmonic conjugates with respect to A_i and A_k. All the spheres (ω'_{ik}) have two points M_1, M_2 in common, situated on the diameter of (O) perpendicular to (P'). Both M_1 and M_2 satisfy the relation

$$\overline{MA_1}\cdot R_1 = \overline{MA_2}\cdot R_2 = \overline{MA_3}\cdot R_3 = \overline{MA_4}\cdot R_4.$$

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

An Introduction to Analytical Geometry. Vol. II. By A. Robson. Cambridge, at the University Press; New York, The Macmillan Company, 1947. 8+215 pages. $2.50.

This book is the second volume of a work on Analytic Geometry. Since frequent reference is made to volume I and it is written in a terse style, it would not by itself prove suitable as a text for a course in geometry in most American schools. However, it could be a valuable reference to teachers of courses in advanced analytic, analytic projective, or synthetic projective geometry.

The contents of this book include material that is frequently covered in courses in real, complex, synthetic, analytic, metric, or projective geometry, a much wider range of topics than is dealt with in many courses in geometry. There are ten chapters, the first of which is numbered seventeen, entitled Homography, and deals with projectivities and perspectivities, including pro-

‡ Court, ibid., p. 122.

jectivities on a conic. Since theories of cross ratio, pole and polar, and parametric representation are developed in volume I, free use is made of these concepts. Chapter eighteen (the second in this volume) deals mainly with the theory of involutions from the algebraic point of view, and includes applications to conic sections. Proofs of Desargues' and Brianchon's theorems which utilize the theory of involutions are also presented. The next chapter is entitled General Geometry and contains a brief discussion of the different types of geometry and the coördinates used in each. Miscellaneous properties of conic sections are included in this chapter. Ranges on a Conic is the title of chapter twenty, which is a continuation of the discussion of projectivities on a conic begun in chapter seventeen. The twenty-first chapter deals with systems of conics, their classification and some of the properties of systems of conics. Chapter twenty-two is entitled Reciprocation and primarily deals with reciprocation with respect to a general conic (duality), reciprocation with respect to a circle being considered a special case of the general theory. The problem of the classification of curves represented by equations of the second degree is dealt with in chapter twenty-three. Chapter twenty-four is entitled Foci and Confocals, and includes a general definition of foci as well as a brief discussion of the properties of confocal conics. The last two chapters are entitled Normals and Evolutes, and Special Homogeneous Coordinates and deal with these topics, especial emphasis being placed on areal coördinates.

The virtues of this book include (1) a large number of well chosen illustrative examples, (2) a long list of exercises for the student, (3) examples showing how well known theorems can be proved by several different modes of procedure, and (4) examples illustrating how methods customarily associated with one type of geometry can be utilized to advantage in some other type of geometry. Its faults include (1) so many references to volume I that it can not be read satisfactorily by those who are not conversant with that work, (2) a style more terse than many American readers are accustomed to, and (3) (which some may consider a virtue), a vagueness in hypotheses regarding the type of geometry (metric, projective, real, complex, etc.) in which a given proof of a theorem is valid.

R. G. SANGER

NEW BOOKS RECEIVED

Les Principes Mathématiques de la Mécanique Classique. By M. Brelot. Grenoble, Arthaud, 1945. 62 pages. 120 fr.

Die Bewegungsgruppen der Kristallographie. By J. J. Burckhardt. Basel, Birkhauser, 1947. 186 pages. 29 fr.

Analytic Mechanics. By A. E. Currier. Annapolis, U. S. Naval Institute, 1948. 10+306 pages. $4.75.

Elements of Nomography. By R. D. Douglass and D. P. Adams. New York, McGraw-Hill Book Co., 1947. 9+209 pages. $3.50.

Calculus and its Applications. By R. D. Douglass and S. D. Zeldin. New York, Prentice-Hall, Inc., 1947. 8+568 pages. $5.15.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

EDITORIAL NOTE. Club reports have been received in almost overwhelming numbers and it has been impossible thus far to include all of them on these pages. If the report of your chapter has not as yet appeared, it is probably due to the fact that your Editor has used the reports in the order in which he has received them.

In order to record the activities of your chapter for the year 1947–48, the club reports should be sent in immediately. When writing the report please follow the form which is used on these pages. The material will be included for publication in the order in which it is received, provided the reports are neatly written in the form prescribed.

CLUB REPORTS, 1946–47

Pi Mu Epsilon, University of Pennsylvania

The Pennsylvania Alpha Chapter of Pi Mu Epsilon, inactive during the war, was reorganized at the close of the fall semester of 1946–47. At that time sixteen undergraduates were elected to membership. Interest is growing rapidly among mathematics and science students on campus. During the year the following talks were given by members of the faculty and students:

Algebraic geometry, by Dr. I. S. Cohen

Polyhedra and the four-color problem, by Dr. Hans Rademacher

How to draw a tree (Fibonacci series), by Mr. W. J. Turanski

Farey series and lattice points, by Mr. Morris Newman.

The officers for the coming year are: President, William J. Turanski; Secretary, Mary-Elizabeth Hamstrom; Treasurer, Anthony Penico.

Pi Mu Epsilon, Syracuse University

The Pi Mu Epsilon Chapter at Syracuse University started the year with the initiation banquet at which thirty-eight students and faculty members were initiated. The speaker for the evening was Dr. S. S. Cairns, head of the mathematics department of the University, who gave a talk on *Peculiarities of polyhedra.*

Other papers given during the year included:

Irrational numbers, by Dr. Bruns

Operational calculus, by Dr. L. L. Lindsey, retiring head of the Applied Science Mathematics Department.

The group experimented this year by giving a Pi Mu Epsilon dance on April 25, 1947 for members and their guests. The decorations were illustrated mathematical problems hung on the walls, and each member wore a name tag. At the last meeting of the year, twenty-eight students were initiated.

The following officers were elected for the year 1947–48: Director, Joyce E. Jones; Vice-Director, Richard Stitt; Secretary, Elizabeth Gillespie; Treasurer, Donald Schick.

Officers for the group for the year 1946–47 were: Director, June Kath; Vice-Director, Sara D. Gray; Secretary, Joyce E. Jones; Treasurer, Jane Berkman.

Mathematics Club, Syracuse University

Papers presented by the members of the faculty were:

Quality control, by Dr. Hicks

Theory of complex numbers, by Dr. Bruns

Unsolved problems, by Dr. Rosenbloom

Four-color problem, by Dr. A. Gelbart

The mathematics of Mickey Mouse, by Dr. S. S. Cairns.

The annual Christmas party in the first term, and a hot-dog roast in the second comprised the outstanding social events of the year. At another meeting movies of various mathematical operations were shown.

Officers who served during the year are: President, Lorraine Zerveck; Vice-President, Sara Gray; Secretary, Teresa Hastings; Treasurer, Milton Fisher; Publicity Head, Elizabeth Gillespie.

Officers elected for the year 1947–48 are: President, Teresa Hastings; Vice-President, Elizabeth Gillespie; Secretary, Wallace Roher; Treasurer, John Dolphin; Publicity and Social Chairmen, Annalyse Haas and David Ashkar.

Mathematics Club, Oshkosh State Teachers College

The *Mathematics Club* of the Oshkosh State Teachers College resumed monthly meetings this year after a period of suspended activities. Members of the club presented the following papers:

Applications of mathematics to meteorology, by Robert Wonders

Use of Mathematics in the construction of a radio, by Alan Brunka

History of the normal probability curve, by Elsie Gandt

History of logarithms and the slide rule, by Clarence Bittner

Early history of mathematics, by Roy Matzdorf

The life and work of Einstein, by Kermet Jones.

The club made a study of motion pictures dealing with mathematics and the following films were shown:

Frequency curves

Gallup poll

Geometry brought to life

Locus

Rectilinear coördinates

Slide Rule

Origin of mathematics

Einstein's theory of relativity

One meeting was devoted to ma
tivities closed with a picnic.

Officers for 1946–47 were: President, James W. Hartford; Vice-president, Harvey Stangel; Secretary, Louise M. Richter; Treasurer, Donald W. Derber; Faculty Adviser, Professor May M. Beenken.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

REPORT OF CONFERENCE OF THE AMERICAN COUNCIL ON EDUCATION

The annual Conference of Constituent Member Representatives of the American Council on Education was held in Washington, D. C., January 23–24, 1948. At this conference Professor A. A. Bennett of Brown University represented the Mathematical Association of America. A summary of Professor Bennett's report of the conference follows.

The first session was devoted to discussing a report on a questionnaire on Universal Military Training, issued by the American Council on Education and dealing with the recommendations of the President's Advisory Commission on Universal Training. The following essentials for national security were recommended: 1) a strong, healthy, educated population, 2) need for scientific research and development, 3) a coördinated intelligence service, 4) a regular Army, Navy, Air Force, and Marine Corps of high mobility and striking power, 5) industrial mobilization and stock piling. Less than a quarter of nearly every group polled were in favor of the proposed plan for universal military training.

Other sessions were devoted to the discussion of the following topics: 1) the extension of the social security act to apply to non-taxed educational organizations and systems; 2) discussion of "The Report of the President's Commission on Higher Education." This "Report" was urgently recommended for detailed study by all the constituent organizations.

SUMMER COURSES

The following institutions announce advanced courses in mathematics for the summer of 1948:

Catholic University of America. June 28 to August 6: Professor Finan, theory of numbers; Dr. McBrien, theory of equations; Professor Ramler, synthetic projective geometry; Professor Rice, advanced calculus, calculus of observations; Mr. Slud, partial differential equations, calculus of variations.

Columbia University. July 6 to August 13: Professor Kasner, survey of modern mathematics, differential geometry; Professor Koopman, partial differential equations of mathematical physics; Dr. Levi, introduction to algebra; Professor Lorch, introduction to mathematical logic; Professor Murray, theory of functions of a complex variable; Professor Ritt, differential equations; Professor Wolfowitz, probability, statistical inference.

Iowa State College. June 14 to July 21: theory of equations; elementary differential equations I; analytical mechanics I; non-euclidean geometry; history of mathematics; advanced calculus I; vector analysis; functions of a real variable; advanced probability. July 21 to August 27: differential equations II;

analytical mechanics II; advanced calculus II; Laplace transform and operational mathematics, potential theory.

Northwestern University. The following advanced courses are offered during the six-week session, June 25 to August 7, and during the nine-week session, June 25 to August 28: fundamental concepts of analysis; determinants and matrices; non-euclidean geometry; theory of statistics; econometrics; introduction to the theory of numbers; algebra for teachers; the history and teaching of mathematics; differential equations of mathematical physics; introduction to modern algebra; vector analysis; independent study; seminar; thesis.

Ohio State University. June 21 to September 3: Professor Alden, theory of equations; Professor Bamforth, projective geometry, theory of matrices; Professor Mickle, differential geometry, continuous groups; Professors Miller and Alden, advanced calculus; Professors Sealander and Alden, differential equations; Professor Sealander, vector analysis.

Stanford University. June 17 to August 28: Professor Polya, elementary mathematics from higher point of view, selected topics from the theory of functions of a complex variable; Professor Schoenberg, advanced calculus, partial differential equations of physics and engineering; Professor Spencer, advanced calculus, elementary tensor calculus; Staff, seminar, advanced reading and research.

State University of Iowa. June 8 to August 4: Professor Chittenden, differential equations; Professor Conkwright, elementary group theory; Professor Cosby, Jr., calculus of variations; Professor Craig, matrices and determinants, sequential analysis; Professor Knowler, mathematics of business and industry; Professor Price, teaching of mathematics; Professor Woods, pure geometry; Professor Wylie, astronomy, reading in astronomy; Staff, research.

Teachers College, Columbia University. July 6 to August 13; Professor Clark, teaching arithmetic in the elementary school; Professor Fehr, professionalized subject matter in advanced secondary school mathematics, teaching algebra in secondary schools; Dr. Lazar, history of mathematics, logic for teachers of mathematics; Mr. Mirick, elementary mechanics, teaching geometry in secondary schools; Professor Reeve, teaching and supervision of mathematics-junior high school, teaching and supervision of mathematics-senior high school. In addition, on consecutive Thursdays beginning on July 8, there will be five special lectures and discussions pertaining to the reorganization and teaching of mathematics in the post-war world.

University of Buffalo. June 14 to September 4: Professor Pound, analytic mechanics, functions of a complex variable. July 6 to August 14: Professor Gehman, functions of a real variable; Professor Schneckenburger, non-euclidean geometry.

University of Chicago. June 29 to September 4: Mr. Crabtree, integration in abstract spaces; Professor Kaplansky, introduction to modern higher algebra; Professor MacLane, introduction to algebraic topology; Professor Santaló, integral geometry; Professor Schilling, class-field theory; Professor Segal,

mathematical foundations of quantum theory; Professor Stone, theory of functions of a complex variable; Professor Weil, introduction to algebraic geometry, seminar on current literature.

University of Illinois. Professor Bourgin, algebra; Professor Brahana, theory of groups; Professor Chanler, geometry; Professor Day, analysis in function spaces, analysis; Miss Hildebrandt, the teaching of general mathematics in high school; Professor Mendel, introduction to higher geometry; Professor Munroe, introduction to higher analysis; Professor Schubert, introduction to higher algebra; Professor Vaughan, fundamental concepts.

University of Michigan. June 21 to August 13: The following advanced courses will be offered in addition to the standard courses in differential equations, theory of equations, advanced calculus, mechanics and statistics: Professor Bartels, Fourier series and vector analysis; Professor Brauer, higher algebra and seminar in algebra; Professor Copeland, probability; Professor Craig, analytic theory of frequency functions; Professor Dwyer, analysis of variance and computational methods; Professor Fischer, intermediate mathematics of life insurance; Professor Hay, advanced mechanics; Professor Hildebrandt, functions of a complex variable and orthogonal functions; Dr. Jones, teaching of algebra, history of geometry and teaching of elementary collegiate mathematics; Professor Kaplan, elementary function theory with applications; Dr. Lockhart, synthetic projective geometry; Professor Myers, differential geometry; Professor Rainville, intermediate differential equations and number theory; Professor Thrall, theory of vector spaces; Professor Wilder, foundations of mathematics and topology of polyhedra and the n-sphere.

University of Minnesota. June 15 to July 23: Professor Carlson, advanced analytic geometry, solid analytic geometry; Professor Hatfield, mathematical recreations, mathematics of small vibrations; Professor Loud, differential equations, numerical methods in computation. July 26 to August 28: Professor Bearman, vector analysis, the calculus of finite differences; Professor Olmstead, intermediate calculus.

University of North Carolina. June 10 to July 20: Professor Brauer, functions of a complex variable; Professor Cameron, introduction to modern algebra; Professor Garner, history of mathematics; Professor Henderson, elements of non-euclidean geometry; Professor Hill, elementary mathematical statistics; Professor Hoyle, differential equations, and advanced calculus I; Professor Winsor, college geometry. From July 21 to August 28: Professor Browne, algebraic invariants; Professor Hobbs, theory of equations; Professor Lasley, differential geometry; Professor Linker, differential equations (continued); Professor Mackie, advanced calculus I (continued).

University of Pennsylvania. Professor Beal, modern analytic geometry; Professor Caris, theory of numbers; Professor Cohen, differential equations, theory of functions of a complex variable.

University of Virginia. June 28 to August 21: Mr. Floyd, differential equations and applied mathematics; Professor Hedlund, advanced analysis; Profes-

sor Whyburn, analytic topology.

University of Wisconsin. June 25 to August 20: Professor Arnold, mathematics of elementary statistics; Professor Bruck, college geometry, determinants and matrices; Professor Langer, Lie theory of differential equations; Professor MacDuffee, survey of the foundations of arithmetic, tensor analysis; Mrs. Sokolnikoff and Professor Specht, higher mathematics for engineers; Staff, mathematical applications.

University of Wyoming. June 14 to July 16: Professor Barr, methods in secondary mathematics; Professor Bristow, advanced calculus; Professor Schwid, ordinary differential equations; Professor Smith, vector analysis. July 19 to August 20: Professor Neubauer, history of mathematics; Professor Varineau, partial differential equations, curve fittings.

REPRINTS AVAILABLE

Reprints of the following articles may be secured by writing to the office of the Secretary-Treasurer, University of Buffalo, Buffalo 14, New York. They will be distributed free to members and subscribers as long as the supply lasts.

"A List of Mathematical Books for Schools and Colleges" (1917);

"A Suggested List of Mathematical Books for Junior College Libraries" (1925);

"Benjamin Peirce, 1809–1880." By R. C. Archibald (1925);

"Report of the Committee on Assigned Collateral Reading in Mathematics" (1928);

"Report of the Committee on College Entrance Requirements in Geometry" (1931);

"Collegiate Mathematics Needed in the Social Sciences" (1932);

"Readings in the Literature on Teaching with Special References to Mathematics" (1935);

"Advanced Preparatory Mathematics in England, France and Italy" by W. D. Cairns (1935);

"The Ph.D. Degree and Mathematical Research," by R. G. D. Richardson (1936);

"Herbert Ellsworth Slaught," by W. D. Cairns and G. A. Bliss (1938);

"It Can't Happen Here," a mathematical musical farce, by A. Marie Whelan (1938);

"Report of the Committee on Tests" (1940).

PERSONAL ITEMS

Professor R. W. Brink represented the Association at the dedication of a new Science Hall on March 8–9, 1948 at the College of St. Thomas, St. Paul, Minnesota.

Professor P. A. Caris was appointed a delegate of the Association to the meeting of the American Academy of Political and Social Science at Philadelphia on April 2–3, 1948.

Professor J. W. Cell represented the Association at the inauguration of

John D. Messick as president of the East Carolina Teachers College, Greenville, North Carolina on March 6, 1948.

Professor B. P. Reinsch was a representative of the Association at the inauguration of J. Hillis Miller as president of the University of Florida, Gainesville, Florida on March 5, 1948.

Dr. John H. Curtiss, chief of the National Applied Mathematics Laboratories of the National Bureau of Standards, was elected a fellow of the American Statistical Association at the Association's annual meeting, December 28–31, 1947.

Dr. Mina Rees and Dr. Warren Weaver were included on the list of American citizens receiving British awards "in recognition of valuable services rendered to the Allied war effort in various fields of scientific research and development." The list was announced by Lord Inverchapel, the British Ambassador, on February 11, 1948. Both Dr. Rees and Dr. Weaver were awarded the King's Medal for Service in the Cause of Freedom.

Stanford University announces the following: Professor Harold Davenport from University College, London is at Stanford University for the year 1947–48 as Visiting Professor; Professor Ainsley H. Diamond of Oklahoma Agricultural and Mechanical College is teaching part-time at Stanford University during the spring quarter, 1947–48; Professor Rhoda Manning is on leave from Oregon State College and is teaching part-time during the year 1947–48 at Stanford University; Professor M. Schiffer of Harvard University is giving a series of lectures from February 1 to June 15, 1948. Other lectures have been given recently at Stanford University by Professor Harold Cramer of the University of Stockholm, Sweden; Professor M. Fréchet, Institut Henri Poincaré, Paris; Professor Marcel Riesz of the University of Lund, Sweden.

Professor Richard Brauer of the University of Toronto has been appointed to a professorship at the University of Michigan.

Dr. R. S. Burington is Chief Mathematician and Director of the Evaluation and Analysis Group, Bureau of Ordnance, Navy Department, Washington, D. C.

Associate Professor T. F. Cope of Queens College has been promoted to a professorship.

Professor H. A. Davis of West Virginia University has been appointed Head of the Department of Mathematics.

Dr. Henri A. Jordan has been appointed to an associate professorship at Colby College.

Associate Professor H. E. Jordan was reported erroneously to have left the University of Kansas.

Professor G. E. Raynor of Lehigh University will become Head of the Department of Mathematics on July 1, 1948.

Professor J. B. Reynolds, head of the Department of Mathematics of Lehigh University, will retire on June 30, 1948.

Mr. Emil D. Schell has accepted an appointment as chief of the Mathe-

matics and Electronic Computor Branch, Office of the Comptroller, United States Air Forces.

Assistant Professor L. W. Swanson of Coe College has been promoted to an associate professorship.

Professor John L. Synge has been appointed to a Senior Professorship in the School of Theoretical Physics at the Dublin Institute for Advanced Studies. He is resigning from his present position as Head of the Department of Mathematics at Carnegie Institute of Technology as of the end of the academic year, and will take up his duties in Dublin on September 1.

Assistant Professor Leo Zippin of Queens College has been promoted to an associate professorship.

The University of Michigan announces the appointment of S. W. Hahn as instructor.

Mr. A. L. McCarty, formerly of City College of San Francisco, died on June 20, 1947. Mr. McCarty, a charter member of the Association, was one of the organizers of the Northern California section.

Professor Emeritus A. B. Turner of the College of the City of New York died on February 5, 1948 at the age of seventy-five years. He was a charter member of the Association.

Professor G. E. Wahlin of the University of Missouri died on February 11, 1948. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

OCTOBER MEETING OF THE INDIANA SECTION

The fall joint meeting of the Indiana Section of the Mathematical Association of America and the Mathematics Section of the Indiana Academy of Science was held at Ball State Teachers College, Muncie, Indiana on October 17, 1947.

Sixty-four persons attended the meeting, including the following twenty-one members of the Association: W. L. Ayres, Juna L. Beal, G. E. Carscallen, K. W. Crain, Olive M. Draper, P. D. Edwards, E. L. Godfrey, G. H. Graves, H. F. S. Jonah, M. W. Keller, F. C. Leone, P. M. Pepper, J. C. Polley, D. H. Porter, C. K. Robbins, A. E. Ross, L. S. Shively, W. O. Shriner, Florence A. Wirsching.

Announcement of the 1948 national meeting of the National Council of Teachers of Mathematics to be held in Indianapolis on April 2–3, 1948, was made by M. H. Ahrendt, of Anderson College.

No business meeting was held. The time and place of the spring meeting for 1948 was to be arranged by the officers of the Section.

The following papers were presented.

1. *Pythagorean triangles*, by John Funderburg and G. A. Jackson, introduced by the Secretary.

In the euclidean plane, an assemblage of points representing Pythagorean triangles is shown to consist of a parabolic lattice set and sets that are multiples of this set. A subset contains all primitive triangles. It is demonstrated that this assemblage of points can be located on systems of hyperbolas, circles, and straight lines. The approximate values of certain constants (including roots of prime numbers, transcendental e and π) are shown to be obtained by the use of convergent or oscillatory series, of which the members are represented by points included in this assemblage of points.

2. *Some special sums of cotangents*, by H. F. S. Jonah, Purdue University.

The speaker illustrated a method for summing certain finite sums of cotangents. The original sums arose in a research project in electrical engineering.

3. *Mathematics teacher training in relation to the proper teaching of undergraduate algebra*, by A. E. Ross, University of Notre Dame.

The speaker discussed a fundamental approach to the teaching of algebra in college, as well as the problems of training teachers to carry out such a program of instruction successfully.

4. *A study of factors related to engineering mathematics at Purdue University*, by Paul Irick, Purdue University, introduced by the Secretary.

The speaker showed the relation between grades in mathematics and various factors such as position of student in graduating class, average grade in high school mathematics, number of high school units in mathematics, and grades on different tests given during orientation period. The study followed the students through the first two years of college mathematics.

5. *A mathematical theory of religion*, by G. H. Graves, Purdue University.

Due to the studies of Whitehead, Russell, Keyser, and others, it is now generally recognized that mathematics has no particular subject matter but is concerned with constructing logical systems on postulates suggested by any field of interest. Religion is a promising field in this connection, for in religion, we constantly observe conclusions and decisions, and hence conduct and character, resulting from postulates held as convictions by an individual or by a society. Just as geometry, for instance, has gained greatly in clearness and in range by a study of its foundations and the recognition of incompatible systems which are nevertheless consistent individually, so it may be expected that different religions can gain in clearness and in tolerance by studying their fundamental postulates with a view to eliminating contradictions and non-essentials, and to tracing the connections of the characteristics of individuals or societies with their fundamental postulates.

6. *A reduced set of postulates for hyperbolic geometry*, by Rev. H. F. DeBaggis, C.S.C., University of Notre Dame, introduced by the Secretary.

The speaker presented a minimal set of postulates for hyperbolic geometry. Independence examples were given for all of the postulates.

7. *The postulates of a tri-operational algebra*, by Rev. F. L. Brown, C.S.C. University of Notre Dame, introduced by the Secretary.

In this paper the speaker presented a set of postulates for a tri-operational algebra, showing the independence of each of the postulates. He presented the minimal set satisfying these postulates and derived a few elementary consequences. (These consequences are among those published by the author in the *Reports of a Mathematical Colloquium*, Issue 5–6, Issue 7, Notre Dame, Indiana.)

8. *Functional representation of partially ordered additive groups*, by Ky Fan, University of Notre Dame, introduced by the Secretary.

For any compact Hausdorff space Ω, the totality $C(\Omega)$ of all real continuous functions defined on Ω may be considered as a partially ordered additive group (p.o.a.g.). Any subgroup G of $C(\Omega)$ which contains all constant functions is obviously a p.o.a.g. with the following three properties: (1) G contains a sub-group R which is group-order-isomorphic to the totally ordered additive group of all real numbers; (2) The sub-group R contains an element e such that for any element f of G, the relation $ne \geqq f$ holds for some natural number n; (3) If for some pair of elements f, g of G, $nf+g \geqq 0$ holds for all natural numbers n, then $f \geqq 0$. Conversely, for any abstract p.o.a.g. G with properties (1), (2), (3), there exists a compact Hausdorff space Ω such that G is group-order-isomorphic to a sub-group G' of the p.o.a.g. $C(\Omega)$ formed by all real continuous functions on Ω, where G' contains all constant functions.

9. *Geometric illustrations of abstract complexes*, by Charles Brumfiel, Ball State Teachers College, introduced by the Secretary.

The speaker gave two and three dimensional examples of abstract topological complexes. The incidence matrices of an n-complex completely determine its topology. Methods were explained for calculating topological invariants, Betti numbers, and torsion coefficients, by means of the incidence matrices.

P. M. PEPPER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirtieth Summer Meeting, Madison, Wisconsin, September 6–7, 1948.

Thirty-second Annual Meeting, Columbus, Ohio, December 31, 1948.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Illinois Institute of Technology, Chicago, May 14–15, 1948
INDIANA
IOWA
KANSAS
KENTUCKY, Berea, May, 1948
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, San Francisco, January 29, 1949
OHIO
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA, Philadelphia, November 27, 1948
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
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CONTENTS

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PRACTICAL COMPUTATIONAL METHODS IN THE SOLUTION OF EQUATIONS*

RUFUS OLDENBURGER, De Paul University

1. Introduction. The basically simple techniques of this paper greatly shorten the labor in computing solutions of ordinary linear differential equations with constant coefficients and algebraic equations arising in applied problems. The reader is especially referred to Section 3 on complex roots.

We shall be concerned with some practical methods of approximating the roots of the equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0 \tag{1}$$

in which the coefficients $a_1, a_2, \cdots, a_n$ are real. This equation is associated with the linear differential equation

$$\frac{d^ny}{dt^n} + a_1\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_ny = 0. \tag{2}$$

We often wish to solve (2) with the initial conditions

$$t = 0,\ y = A_1,\ \frac{dy}{dt} = A_2, \cdots,\ \frac{d^{n-1}y}{dt^{n-1}} = A_n \tag{3}$$

in which $A_1, A_2, \cdots, A_n$ are real.

From probability considerations we can say that if (2) is a differential equation which represents a physical system it is very unusual for two roots of the corresponding equation (1) to be equal or almost equal. It is therefore assumed in this paper that the roots $\alpha_1, \alpha_2, \cdots, \alpha_n$ of (1) are distinct. Then there are numbers $B_1, B_2, \cdots, B_n$ such that the solution of(2) and (3) is

$$y = B_1e^{\alpha_1t} + B_2e^{\alpha_2t} + \cdots + B_ne^{\alpha_nt}. \tag{4}$$

If i is the imaginary unit, there are real numbers u, v such that $\alpha_1 = u + iv$. Then the term $B_1e^{\alpha_1t}$ in (4) is $B_1e^{ut}(\cos vt + i \sin vt)$. If $B_1 \neq 0$, and u is negative, the term B_1e^{ut} approaches zero as t increases beyond all bounds; otherwise B_1e^{ut} does not approach zero. For this reason, if t is time, (2) is said to be the differential equation of a stable physical system if and only if the real parts of the roots of (1) are all negative. In the design of any piece of equipment the question of stability is an important one.

If $n \geqq 3$ the equation (1) should be solved by trial in preference to any standard method with prescribed steps. Some information about the nature of the roots should be obtained before carrying out the actual solution. For example, Descartes rule of signs may be used. If one has a problem in transient phenomena the Routh† stability criterion may be employed. If $n \leqq 5$ in (1), then

* Presented to the Illinois Section of the Association, May 24, 1947.

† H. S. Wall, Polynomials whose zeros have negative real parts, this MONTHLY, vol. 52, No. 6, 1945, pp. 308–332.

synthetic division to find real roots may be used at once.

We shall employ the term "large" in this paper. This is a purely relative term of comparison between the given quantity said to be large and other quantities with which it is being compared. In some problems the given quantity is large if it is at least 10 times each of the other quantities involved; sometimes the factor is at least 100. The concept of "large" is used to throw away terms, and what terms may be thrown away depends on the errors one is allowed to make in the problem. When we say that a real root is *numerically large* we mean that this root is numerically so large compared to the remaining roots we can simplify the algebraic expressions of this paper by keeping only the terms of highest power in the numerically large root. To simplify algebraic expressions we shall also refer to complex roots with numerically large real or imaginary parts. Here we compare the numerically large part with the other part of the root, the absolute values of the remaining roots, and the quantities $A_1, \cdots, A_n$.

If there are numerically large real roots of (1), they should be found by the methods of Section 2. By the process of Section 3 the complex roots should be located.

In the case of a stable transient system the roots with numerically large real or imaginary parts may be discarded.† G. A. Philbrick‡ has developed an analyzer of physical systems based on the idea that the transient performance is determined by only a few of the roots of the corresponding equation (1), so that physical systems can ordinarily be described by differential equations of low order.

2. Numerically large real roots. In the solution of (1) it is advisable first to find approximately the numerically large real roots, and to use the reduced equation after each such root is found. Probability considerations show that an advantage in removing a numerically large real root is that the reduced equation often has relatively small coefficients compared to the coefficients in (1).

We shall explain later how to test the closeness of an approximation to a numerically large real root of (1). We note here that if (1) has a numerically large real root, the equation

$$x - a_1 = 0 \tag{5}$$

might give a good approximation to α_1. It is the author's experience that this approximation generally is not good.

It will now be proved, as is well known, that a numerically large real root α_1 of (1) approximately satisfies the equation

$$x^2 + a_1 x + a_2 = 0. \tag{6}$$

† Discarding roots with numerically large real or imaginary parts has been successful in the author's experience except in the case of gas turbines where numerically large real roots of an equation (1) correspond to the temperature in the combustion chamber which can vary rapidly, and the essentially smaller roots correspond to the speed of the wheel which changes relatively slowly.

‡ Presented to the SAE National Aeronautics Meeting, Oct. 2–4, 1947, Los Angeles, California, in a paper entitled, Electronic analog studies for turbo-prop control systems.

Let σ_1 be the sum of the remaining roots $\alpha_2, \alpha_3, \cdots, \alpha_n$ of (1). It follows that $a_1 = -(\alpha_1+\sigma_1)$. Let σ_2 be the sum of the products of $\alpha_2, \alpha_3, \cdots, \alpha_n$ in pairs. Then if $|\alpha_1|$ is large enough relative to $|\alpha_2|, \cdots, |\alpha_n|$ the quantity σ_2 is small compared to $\alpha_1\sigma_1$. Therefore a_2 is approximately $\alpha_1\sigma_1$. Then (6) is approximately $x^2-(\alpha_1+\sigma_1)x+\alpha_1\sigma_1=0$. Obviously α_1 satisfies (6) approximately.

There are equations such that (6) does not give a good approximation, but

$$x^3 + a_1x^2 + a_2x + a_3 = 0 \tag{7}$$

determines α_1 to sufficient accuracy. If (5), (6), or (7) do not give α_1 to sufficient accuracy we may form similar equations using more coefficients of (1). In the author's experience this has not been necessary.

The author has not found the usual synthetic division process satisfactory for finding numerically large real roots. As an example consider the synthetic division

$$\begin{array}{rrrr|l} 1 & 121 & 2030 & 1010 & \underline{-101} \\ & -101 & -2020 & -1010 & \\ \hline 1 & 20 & 10 & 0 & \end{array}$$

which shows that -101 is a root of a certain cubic. A computer would ordinarily not guess the exact value -101. Suppose that in attempting to find the root -101 he tries -90. Ordinary synthetic division follows.

$$\begin{array}{rrrr|l} 1 & 121 & 2030 & 1010 & \underline{-90} \\ & -90 & -2790 & 68,400 & \\ \hline 1 & 31 & -760 & 69,410 & \end{array}$$

Since the remainder 69,410 is large compared to the entry 1010, he would be inclined to think that he is very far from the actual root, when he is really quite near.

The synthetic division used above is left to right synthetic division. Suppose now that the computer tries right to left synthetic division with the value -90. Keeping one decimal place he obtains the following.

$$\begin{array}{rrrr|l} 1 & 121 & 2030 & 1010 & \underline{-90} \\ & & -2018.8 & & \\ \hline & 22.4 & 11.2 & & \end{array}$$

The remainder 11.2 is obtained by dividing -1010 by -90, the entry -2018.8 is 11.2 minus 2030, and the first remainder 22.4 is the result of dividing -2018.8 by -90. Since $90+22.4$ is near 121 the computer knows that -90 is near a root.

It is the author's experience that if one makes a guess $\alpha_1+\epsilon$ to a numerically large real root α_1 of (1), even a small error ϵ will yield a large remainder when left to right synthetic division is used, and it is necessary for the computer to work with large entries compared with the remainder and the entries obtained

in right to left synthetic division. It is a simple matter to justify these conclusions on purely mathematical considerations. For the sake of brevity these will be omitted.

Probability considerations show that there is a good chance that equation (1) will have a real root which dominates the others in absolute value.*

The preceding discussion will now be illustrated with the equation

$$x^6 + 228x^5 + 6{,}000x^4 + 37{,}000x^3 + 210{,}000x^2 + 320{,}000x + 274{,}000 = 0 \tag{11}$$

which arose in the study of a new diesel electric locomotive. The actual process used in the industrial solution will be given. Since the coefficient of x^5 is large compared to the leading coefficient it appears that there is probably a real root with large numerical value. Solution of (6) yields the approximation $\alpha_1 = -198$. In the following synthetic division of the left member of (11) by -198 we start at the right and round off numbers to convenient values.

1	228	6,000	37,000	210,000	320,000	273,000	$\lfloor -198$
		−5,800	−36,000	−208,000	−319,000	−273,000	
	29	182	1,050	1,610	1,380	0	

The fifth remainder 1,380 is obtained by dividing 274,000 by 198, the fourth by dividing 319,000 by 198, and so on. The first remainder is approximately 29 instead of the required value, $a_1 + (-198)$, that is, 30. We therefore try -199 as shown.

1	228	6,000	37,000	210,000	320,000	273,000	$\lfloor -199$
		−5,800	−36,000	−208,000	−319,000	−273,000	
	29	181	1,040	1,600	1,370	0	

The first remainder is 29; also here $a_1 + (-199)$ is 29. Therefore -199 is the more satisfactory approximation to the large root of (11) under consideration.

We now proceed to the reduced equation

$$x^5 + 29x^4 + 181x^3 + 1{,}040x^2 + 1{,}600x + 1{,}370 = 0 \tag{12}$$

for which the root -23 is readily found by the method just described.

3. Complex roots. There will now be described a modified synthetic division process which ordinarily leads rather rapidly† to complex roots of (1), in contrast to standard techniques in common use. Conjugate complex roots of (1) determine a quadratic factor of the given polynomial on the left of (1). There-

* Several thousand equations of type (1) arising in practical problems were solved. These represented rather well all of the major fields of engineering. In almost every case there was a real root numerically large compared to the other roots, and it was readily found by synthetic division performed from right to left.

† Sixth and eighth degree equations which would have taken a day by Graeffe's method have always been solved by the author in less than 30 minutes for two or three significant digits.

fore division of this polynomial by a general quadratic function, x^2+ex+f, will be explained. The forms of the remainder in this division will be established by induction.

Let r_i, s_i denote the pair of remainders after the ith division. Therefore, clearly,

$$\begin{aligned} r_1 &= a_1 - e, & s_1 &= a_2 - f, \\ r_2 &= e^2 - f - a_1 e + a_2, & s_2 &= ef - a_1 f + a_3, \\ r_3 &= -e^3 + 2ef + a_1 e^2 & s_3 &= -e^2 f + f^2 + a_1 ef \\ &\quad - a_2 e - a_1 f + a_3, & &\quad - a_2 f + a_4. \end{aligned}$$

These results suggest that the remainders after the ith division have the form

$$\begin{aligned} r_i &= (-1)^i[e^i - (i-1)e^{i-2}f + \cdots], \\ s_i &= (-1)^i[e^{i-1}f - (i-2)e^{i-3}f^2 + \cdots], \end{aligned} \tag{13}$$

in which all terms in the a's and terms of lower degree in e than those exhibited have been omitted. The next row in the division will be

$$r_i, \qquad er_i, \qquad fr_i.$$

Therefore

$$r_{i+1} = s_i - er_i, \qquad s_{i+1} = a_{i+2} - fr_i.$$

Hence

$$\begin{aligned} r_{i+1} &= (-1)^{i+1}[e^{i+1} - ie^{i-1}f + \cdots], \\ s_{i+1} &= (-1)^{i+1}[e^i f - (i-1)e^{i-2}f^2 + \cdots], \end{aligned}$$

in which all terms in the a's and terms of lower degree than those shown have been dropped. Since the formulas (13) have been verified when $i=2$ (as well as 3), it follows by induction that they are valid if $i=2, \cdots, n-1$.

It will now be assumed that $f\neq 0$. It follows from (13) that, as polynomials in e, one of r_{n-1}, s_{n-1} is of odd degree in e. Now a real polynomial equation of odd degree has a real root. Therefore, if $f\neq 0$, there is an e such that one of the final remainders r_{n-1}, s_{n-1} vanishes. Such a root of $r_{n-1}=0$ or $s_{n-1}=0$ can be found by studying the sign of the polynomial. There may be such a root when $f=0$.

It will now be assumed that there are numbers e, f such that $s_{n-1}(e, f)=0$. In the zw-plane

$$s_{n-1}(z, w) = 0 \tag{14}$$

is a curve, and (e, f) is a point on this curve. If there is a point (e, f) such that (e, f) is on (14) and also on

$$r_{n-1}(z, w) = 0, \tag{15}$$

then an approximation (e_1, f_1) to (e, f) may be found by trial.

Thus if for a given value of e, one of the remainders $r_{n-1}(e, f)$, $s_{n-1}(e, f)$ is of odd degree in f, we can find a point (e, f) on one of the curves (14), or (15) by trial of values of f.

Unless the problem is a particularly difficult one, it is possible without studying the signs of the polynomials $r_{n-1}(e, f)$, $s_{n-1}(e, f)$ to find rather quickly a point (e_1, f_1) near a point of intersection (e, f) of the curves (14) and (15).

It should be noted that either or both of the curves $r_{n-1}(z, w)=0$ or $s_{n-1}(z, w)=0$ may have more than one branch. For example if $n=4$, then $r_3(z, w)=0$ is a cubic curve with $z=a_1/2$ as asymptote. All branches of (14) and (15) should be tested to find a point (e_1, f_1) which approximately satisfies (14) and (15).

We return to the locomotive example of Section 2. After the factors corresponding to the roots -199 and -23 of the locomotive equation (11) have been removed, the reduced equation

$$x^4 + 6x^3 + 42x^2 + 67x + 60 = 0 \tag{16}$$

is obtained. Equation (16) will be solved by the methods described above. Examination of (16) with trial divisors shows that synthetic division by a real number yields a positive remainder so that there are no real roots of (16). The Routh test shows that the real parts of the roots are negative so that the numerical value of such a real part cannot exceed 6 (*i.e.* minus the sum of the roots). As a first guess we try $(e_1, f_1)=(2, 0)$. The table for division is

$$\begin{array}{r|rrrrr}
 & & 1 & 4 & 34 & \\ \hline
2,0 & 1 & 6 & 42 & 67 & 60 \\
 & & 2 & 0 & & \\ \cline{3-4}
 & & 4 & 42 & & \\
 & & & 8 & 0 & \\ \cline{4-5}
 & & & 34 & 67 & \\
 & & & & 68 & 0 \\ \cline{5-6}
 & & & & -1 & 60
\end{array}$$

The quotient is $x^2+4x+34$, and the remainder $-x+60$.

If $e_1=1.9$ instead of 2, the remainder $2x+60(r_3=2,\ s_3=60)$ is obtained. Since $f=0$ in each case, and since $r_3(2, 0)<0$, $r_3(1.9, 0)>0$, there is a point $(e_1, 0)$ on the curve (15) for e_1 between 1.9 and 2.

We shall now show how a pair of values (e_1, f_1) is chosen so that (e_1, f_1) is approximately on both curves (14) and (15). If $(e_1, f_1)=(2, 1)$ the table at the top of the following page is obtained. Thus s_3 has decreased from 60 to 27. Next from $(e_1, f_1)=(2, 2)$ the remainder $-5x-4$ is obtained. Thus $r_3(2, 2)=-5$, and $s_3(2, 2)=-4$. Since these numbers are both small it is true that x^2+2x+2 is an approximation to a quadratic factor of the left side of (16). Since the remainders r_3 and s_3 are both negative a new approximation to (e_1, f_1) would be $(1.8, 1.8)$. The remainder is now $0.8x+1.4$, and the quotient is $x^2+4.2x+32.6$.

Further trial with neighboring values shows that for two-digit accuracy the expression $x^2+1.8x+1.8$ may be used as an approximation to a factor of the left side of (16).

$$
\begin{array}{r|rrrrr}
 & & 1 & 4 & 33 & \\
\hline
2,1 & 1 & 6 & 42 & 67 & 60 \\
\cline{1-1}
 & & 2 & 1 & & \\
\cline{2-3}
 & & 4 & 41 & & \\
 & & & 8 & 4 & \\
\cline{3-4}
 & & & 33 & 63 & \\
 & & & & 66 & 33 \\
\cline{4-5}
 & & & & -3 & 27
\end{array}
$$

Approximate roots of the locomotive equation (11) are therefore

$$-199, \qquad -23, \qquad -2.1 \pm 5.3i, \qquad -0.9 \pm i. \tag{17}$$

4. Simplification of solutions for transients. The approximate solution of the differential equation (2) corresponding to the algebraic equation (11) is of the form

$$\begin{aligned} y = {} & B_1e^{-199t} + B_2e^{-23t} + e^{-2.1t}(C_3 \cos 5.3t + C_4 \sin 5.3t) \\ & + e^{-0.9t}(C_5 \cos t + C_6 \sin t). \end{aligned} \tag{18}$$

We shall show that if t represents time (as it does in this example) we are justified in discarding the first two terms on the right of (18).

We consider a solution (4) of (2) and (3) in which $n \geqq 2$, α_1 is real and the solution is stable (the real parts of $\alpha_1, \cdots, \alpha_n$ are negative). The B's satisfy the equations

$$\sum_{i=1}^{n} B_i = A_1, \quad \sum_{i=1}^{n} \alpha_i B_i = A_2, \cdots, \quad \sum_{i=1}^{n} \alpha_i^{n-1} B_i = A_n. \tag{19}$$

If Δ is defined as the determinant of the coefficients of the B's in (19), then $\Delta \neq 0$ because Δ is a product of the differences $\alpha_i - \alpha_j$ for $i \neq j$.* We note that Δ is the determinant in the denominators of the solution of (19) for $B_1, \cdots, B_n$. The determinant Δ is of degree $n-1$ in α_1, whereas α_1 does not occur in the determinant which is the numerator of B_1. This is not generally true for the numerators of $B_2, \cdots, B_n$. It follows that if α_1 is sufficiently large numerically (compared to $A_1, \cdots, A_n$ as well as the other roots), then the coefficient B_1 in (4) is generally small in comparison with the other B's. Hence as t increases the term $B_1e^{\alpha_1 t}$ decreases to zero rapidly compared to the other terms, and may be dropped from the solution. We remark that for transients we are concerned with what happens for increasing t.

* L. Weisner, Introduction to the Theory of Equations. Macmillan, 1938. pp. 55, 56.

If the roots α_1 and α_2 are conjugate complex roots of (1), the notation $u+vi$ will be used for α_1. The numerators of B_1 and B_2 corresponding to these roots are of lower degree in v than the determinant Δ. If v is large compared to $|u|$, the absolute values of the remaining roots, and $A_1, \cdots, A_n$, the quantities B_1 and B_2 are in general relatively small compared to the remaining B's. Therefore if $n \geqq 3$, and v is large enough in the above sense, and further $|u|$ is not small numerically compared to the real parts of the other roots of (1), the first two terms of (4) can be discarded. The same is true if $|u|$ is large compared to v, the absolute values of the remaining roots, and $A_1, \cdots, A_n$.

TWO TEASERS FOR YOUR FRIENDS

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You may test the native mathematical ability and the mathematical training of your friends with the following variations of a very old problem. Our first problem has a simple but significant catch to it which a bright ten-year old may discover at once, but which most of your non-mathematical colleagues will miss; the second has a catch for the professional mathematician. Try them on your friends. We present the problems as they might arise at a dinner table.

PROBLEM 1. I am sitting at a table four feet wide. I place my knife at one edge of the table, then half way across the table, then half way from there to the other edge, then half way from there toward that edge, and so on indefinitely. How long a time is it before the knife is at that edge? We understand that we are talking about a mathematical knife having no width, that the measurements indicated at the various steps are mathematically exact, and that the problem is strictly mathematical.

Of course most of your friends will tell you that the knife never reaches that far edge. But your bright ten-year-old may say that there was nothing about time in the data and hence he can reach no conclusion about how long a time it would take. Your colleagues may be taken aback by this remark.

PROBLEM 2. Add to the data of Problem 1 the statement that it took one second for the knife to be moved in the first step, a half a second for the second step, a fourth of a second for the third step, and so on, with the general rule that it took $1/2^{n-1}$ seconds for the nth step, where n is a positive integer. Now what is the answer to the question of Problem 1?

This goes to the heart of a very old problem. Most of your friends, after a little thought, will say that the proper answer is *two seconds.* What do you say? My answer is given on page 377.

A MACHINE FOR PLAYING THE GAME NIM

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1. Introduction. The game Nim is played as follows. Starting with an arbitrary number of objects arranged in separate piles, two persons play alternately, removing one or more objects from any one pile in a given turn. The person taking the last object wins, for the case which we have designated as the *normal case*, but loses in the *reversed case*. An interesting feature of the game is that there is a complete mathematical theory for it, by which a player can infallibly win if given a choice of first or second move. This theory is of such a nature that the computations required can be carried out by simple electrical circuits. It is the purpose of the present article to describe one method of doing so. To prevent the problem from becoming too trivial, we shall add the restriction that no relays or other devices are allowed; that is, the part of the circuit concerned with computation can contain ordinary switches only. In the early stages of the design this was a somewhat annoying limitation; but with the design completed, one finds, after all, that introduction of relays will not lead to any obvious simplification.

Nim was originally played with three piles. In this form it is said to have had its origin in the Orient, perhaps in China, and to be of great antiquity. In 1906 the theory for an arbitrary number of piles was published [1]. This theory, which also takes account of the reversed case, is summarized in Section 2 of the present article. In [2] is described a more general game, in which one can remove objects from k piles in each turn, rather than from only one pile. The theory is given for the normal case. In 1940, E. U. Condon* obtained a patent on a machine for playing the normal case of the game forming the subject of this article [3]. The circuit appears to be quite different, however, from that considered here, and makes extensive use of relays. A model of this machine for four piles with a maximum of seven objects in each was actually built as an exhibit for the New York World's Fair. Since that machine contained over a ton of equipment, while the present one weighs only about five pounds, it is felt that this article will be of interest.

In 1941, a machine was designed by the author for playing the normal and reversed cases with $k=1$. Relays were used, but otherwise the machine was equivalent in principle to that of the present article. In 1941–42 the author extended the theory to include the reversed case for arbitrary k, and proved that the conditions previously employed, if slightly modified, are both necessary and sufficient for correct moves. These elementary results were preliminary to the design of a machine for playing the normal and reversed cases with both k and n arbitrary. The design of such a machine was completed in 1941–42 [4]. It may be noted that the two cases, $k=1$ and $k=k$, differ only trivially as far as the theory is concerned; but, by this slight increase in generality, the problem of mechanization is greatly complicated. Here we assume $k=1$, with no restriction on n, the number of piles.

* Now head of the National Bureau of Standards.

2. Mathematical theory. Although the theory of play is well known, it must be reviewed here, since it forms the basis for the proposed design. First, one must count the objects in each pile and write these numbers in the binary system. For example, if there are four piles containing 7, 12, 10 and 4 objects, respectively, one would write down the set of numbers

$$\begin{array}{r} 111 \\ 1100 \\ 1010 \\ 100. \end{array} \tag{1}$$

Next, add corresponding digits, starting with those at the right. The sum of the unit digits is 1 in the present case; the sum of the two's digits is 2, the sum of the four's digits is 3, and the sum of the extreme left-hand digits is 2. If these sums of corresponding digits are all even, then the position is said to be a safe position, but if one or more of these sums is odd, the position is not safe. The position shown, for example, is not safe.

Starting with a position which is not safe, one can always leave a safe position by playing on one pile only. This operation, which is of importance in the sequel, is carried out as follows. First, one must pick the leftmost column of all those columns which gives an odd sum. In our example, the column in question would be the second column from the left, that is, the column for which the sum is 3. Since the sum of digits in this column is odd, at least one of the digits in it must be a 1. Pick any pile which gives a 1 in this position; for example, the first, second, or fourth piles would be permissible in our special case. Change the 1 in the *odd farthest left* (as we shall call the column in question) to a 0, and adjust the remaining digits in such a way that the other sums are all even. The new binary number thus obtained represents the number remaining in that pile after the completion of the move. The number of objects is actually decreased as required. Also the resulting position is safe.

We have shown that a position which is not safe can always be turned into a safe position by legitimate operations. On the other hand, the reader will easily verify that a person confronted with a safe position can never leave a safe position by moving, as required, in one pile only. Given the proper initial conditions, it follows that a player can always leave his opponent confronted with a safe position, and that he himself will never be confronted with such a position. Since the number of objects continually decreases, eventually the position must be reached for which all piles are empty. This position is safe, however, and hence it will be the opponent's turn to play when it arises.

The theory as described above is for the case in which the object of the game is to make the last move. Because the theory is slightly simpler, this case has been termed the *normal case*. One may also play a game, which we have designated as the *reversed case*, in which one's purpose is to force the opponent to take the last object. Only a trivial modification of the theory is required for this case, namely, to leave an odd instead of an even number when the game has so

far progressed that just one pile has more than one object. An opportunity for this modification will always arise, since the position in question must occur eventually, and is not safe. This reversed case, which leads to a trivial modification of the theory, can be dealt with by an equally trivial modification of the circuit; and hence the machine is designed to play that form of the game as well as the other.

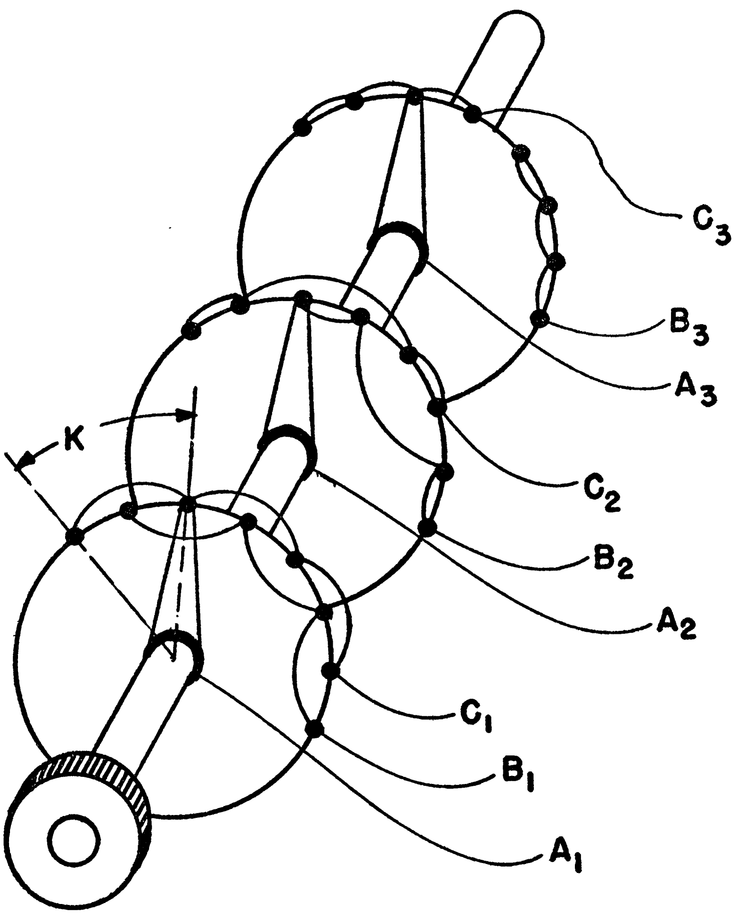

FIG. 1a

3. Conversion of numbers to the binary system. Our first task is to convert the number of objects in each pile to the binary system. Let us suppose that these numbers are introduced by changing the contact position on a rotary switch of conventional design, and that the switch is supplied with as many layers of contacts ("pies") as there are digits in the binary number required for representing the maximum number to be put in that pile. The connections for pies representing the units', two's, and four's digits are then as shown in Figure

1a, and the general arrangement for the jth pie, which gives the jth digit, is shown in Figure 1b. A rigorous proof that the circuit has the properties described can be given.

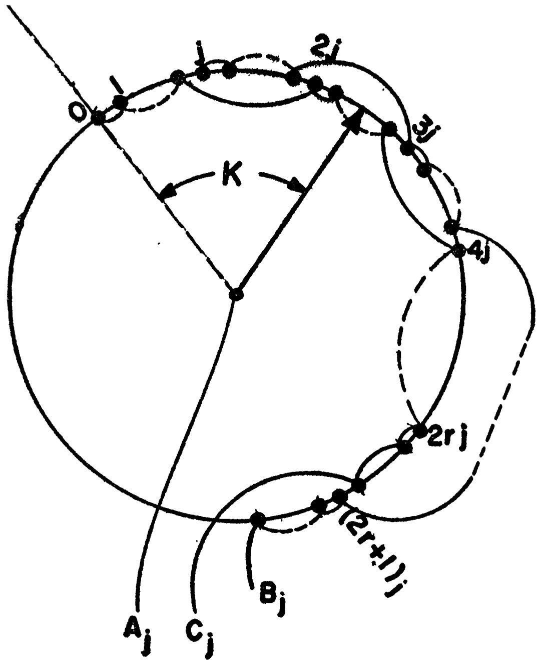

FIG. 1b

4. Even and odd. According to the theory, the next step is to find the sum of corresponding digits. Such a problem is quite difficult to solve by any simple arrangement of switches, and we inquire, therefore, whether finding the sum is really necessary. In the course of the calculation the only use made of the sum is to determine whether it is even or odd, and hence the problem may be revised as follows: Given a series of n switches (one for each pile) to determine whether an even number or an odd number of the switches are closed. Stated in this way, the problem can be solved by a simple circuit.

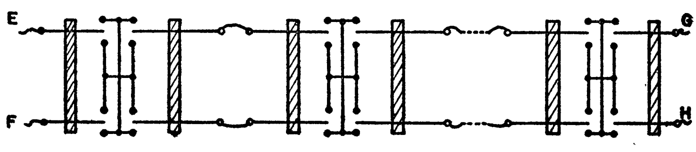

FIG. 2

Let us notice that the evenness or oddness is reversed when any one switch of the series reverses its position. Hence the original problem, to determine whether a sum is even or odd, now becomes the simpler problem of arranging switches to

reverse connections in this way. A solution is given in Figure 2. The switches are double-pole, double-throw, one switch being required for the jth digit in each pile. It is easy to verify that changing the position of any one of these switches has the effect of interchanging the terminal wires G, H; in other words if G was

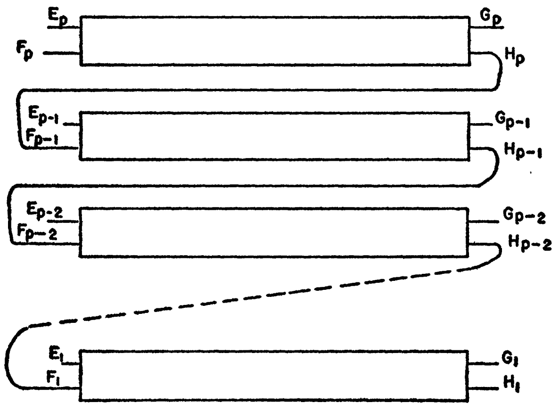

FIG. 3

initially connected to E, and H to F, then changing the position of a switch will leave the circuit with G connected to F, and H to E. Also, if all switches are in the up position, then E, G and F, H will be connected. It follows that E and G will be connected, and F and H will be connected, whenever an even number of switches are in the down position; but the connections will be interchanged whenever an odd number of switches are in the down position. Hence the problem is solved.

5. Odd farthest left. By the foregoing circuits we can determine whether the sums of corresponding digits are even or odd. We must next pick out the odd sum that is farthest left. Such a selection may be made by combining the separate circuits as indicated in Figure 3. From the discussion of Figure 2, we know that the wires H_j in Figure 3 will be neutral or connected to F_j, according as the sum of the jth digits is even or odd, and the opposite is true of the wires G_j. Hence, in the circuit as a whole, a given wire will be connected to F_p if and only if j is the largest subscript for which the sum is odd. For the wires F_j will be connected to F_p if and only if all the sums for greater j are even; and G_j will be connected to F_j if and only if the sum for that j is itself odd.

We remark in passing that the wires E_j for $j>1$ are not really used, and E_1 itself is needed only for the reversed case. Hence the first double-pole switch shown in Figure 2 may generally be replaced by a single-pole switch. This simplification, which is made in Figure 4, is not made in Figure 2 because the circuit (for $j=1$) will be required later in the complete form.

6. Completion of machine for normal case. Having selected the proper column in the general array corresponding to (1) we are required next to select the

piles for which the binary numbers have a 1 in this column; in other words, we must select the proper rows. To this end let us add an extra set of switches which will be on if the jth digit is 1, off otherwise. That such switches can be constructed follows from Section 3. If these switches are connected in series with the previous arrangement for finding the odd farthest left, and the whole is connected in series with an indicating lamp, then the lamp will light if and only if there is a 1 in the odd farthest left. The complete circuit is shown in Figure 4.

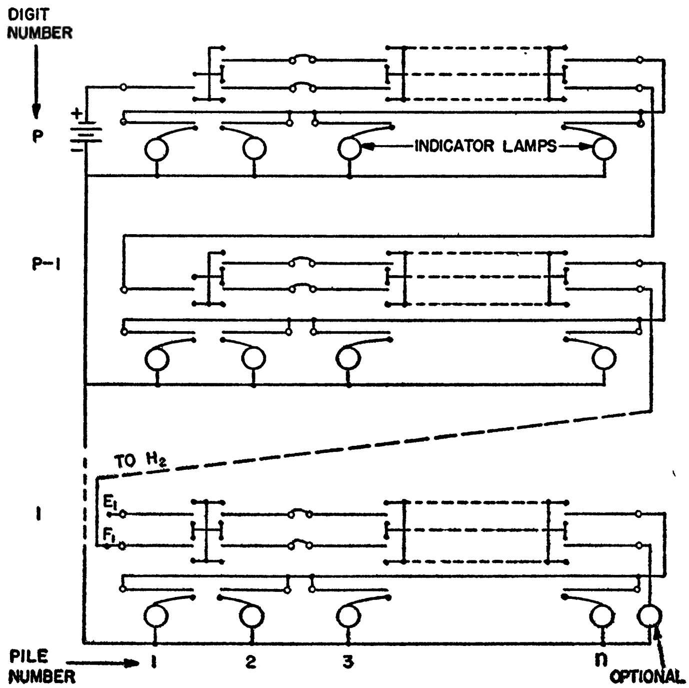

FIG. 4

For a given pile there will be several lamps, one for each digit in the binary representation, and these lamps cannot be replaced by a single one without upsetting the operation of the circuit. However, they may be arranged in a compact group, so that they function like a single lamp as far as indicating purposes are concerned. Introduction of relays at this stage would eliminate the need for using separate lamps in this way, but it is not evident that any advantage would be gained. The light labeled "optional" in Figure 4 goes on whenever it is the opponent's move. Although not really necessary in the present context, the possibility of such a connection would be useful if mechanization of the machine, for automatic operation, were undertaken.

An experimental machine has been designed for a maximum of seven objects in each of four piles. Hence there are four switches, and each switch has eight positions. Since any number not exceeding 7 can be written in the binary system with three digits, there are three indicator lights for each pile. These are

grouped behind a translucent screen covering a large round hole above each switch. Neon bulbs are used partly because of their low current requirement and partly because they will not light if two are in series. The input is 110 volts, a.c.

To use the machine, one specifies the initial position by adjusting the switches. For example, if the four piles are to have 2, 3, 5, and 7 objects, respectively, then the switches would be set to 2, 3, 5 and 7. If the position is safe no lights will light, and it is the opponent's move. If, however, the position is not safe, which is true of our example, then the indicator lights will be on for those piles, and only those piles, in which a correct move can be made. In the special case cited, the lights over the first, second, and fourth switches would be on. The operator of the machine selects any one of these permissible piles and turns its switch to the left (that is, reduces the number of objects) until the light for that pile is off. It follows from the theory that all the other lights will be off, and thus the move is completed. The opponent makes his move by adjusting whichever switch he pleases, whereupon the operator of the machine repeats the procedure just described. This process is continued until the game is over.

It should be observed that the move is completely specified by the machine; the only contribution made by the operator is to select the pile he prefers, when a correct move can be made in more than one pile, and to make the adjustments indicated. No calculation on his part is required at any time. The operation of selecting the alternative, when several possibilities are presented, could be eliminated by a circuit similar to that of Figure 3; that is, one could so arrange matters that, for example, only the leftmost light of the series would go on. The present arrangement is believed preferable because it allows the machine to indicate all possible correct moves.

7. Concluding remarks. To make the machine play the reversed case, we arrange a circuit which moves the wire labeled "to H_2" in Figure 4 from F_1 to E_1, when it first happens that no pile has more than one object. In the experimental machine a switch is provided for distinguishing the normal and reversed cases.

It is believed that the number of switches used is the minimum for a design of this general type; the present machine has four switches with about 10 pies each and eight contacts in each pie. For the general case the number of pies is approximately proportional to $n \log m$ and the number of contacts on a given pie is $m+1$, if m is the maximum number of objects in a given pile.

References

1. Bouton, C. L., Nim, a Game with a Complete Mathematical Theory, Ann. Math. Sec. Ser. v. 3, No. 1, Oct. 1901.

2. Uspensky and Heaslet, Elementary Number Theory, McGraw-Hill, 1939.

3. Condon, E. U., Edgewood, Tawney, G. L., Derr, W. A., Wilkinsburg, Westinghouse Electric and Manufacturing Co., Pittsburgh, Pa., A Machine to Play the Game of Nim, U. S. Patent no. 2,215,544, Sept. 24, 1940.

4. Redheffer, R. M., A Machine for Playing Generalized Nim, B.S. Thesis in Mathematics, 1943, Mass. Inst. Tech.

FURTHER REMARKS ON THE APPROXIMATION OF NUMBERS AS SUMS OF RECIPROCALS

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1. Bibliographic notes. After publication of the author's recent paper on this subject,† several persons called his attention to the fact that some of his results were known previously. Beginning with their leads, a thorough search of the literature pertaining to R- and $\bar{R}$-expansions gave rise to the following bibliography which is believed to be exhaustive:

(a) O. Perron. *Irrationalzahlen.* Walter de Gruyter, Berlin; 2nd ed. 1939, 1st ed. 1921; pp. 116–122 for both editions. Discusses "Sylvester series" (in the 1921 edition they are called "Engel series of the second kind") which are essentially the same as R-expansions. But a "Sylvester series," in addition to including the R-expansions, also includes the cases where in $\sum 1/N_i$, beyond a certain value $i=i_0$, one always has $N_{i+1}=N_i^2-N_i+1$. Thus a "Sylvester series" yields an infinite series alternative to the terminating R-expansion for a rational number, where the last term of that R-expansion is $1/(N_{i_0}-1)$. Theorem 49 on p. 119 is essentially another statement of the author's Theorems I and II on p. 136.†

(b) J. J. Sylvester. On a Point in the Theory of Vulgar Fractions, *Amer. Jour. of Math.*, III, 1880, pp. 332–335; also, Postscript to Note on a Point in Vulgar Fractions, *ibid.*, pp. 388–389. Discusses "Sylvester series" (see (a)) or "sorites" (Sylvester). On p. 333 are statements (but without proofs) which are essentially the author's Theorems I and II.†

(c) G. Stratemeyer. Stammbruchentwickelungen für die Quadratwurzel aus einer rationalen Zahl, *Mathematische Zeitschrift*, XXXI, 1930, pp. 767–768. Expresses a quadratic irrationality by means of an R-expansion.

(d) G. Stratemeyer. Entwicklung positiver Zahlen nach Stammbrüchen, (Dissertation) *Mitteilungen des Mathematischen Seminars der Universität Giessen*, XX Heft, 1931, pp. 3–10, 25. Studies series of the form $\gamma=a+\sum_{\nu=1}^{\infty}\epsilon_\nu/q_\nu$, where the ϵ_ν's are $+1$ or -1. However this is not the $\bar{R}$-expansion, the discovery of which Stratemeyer seems to have missed; at no step is his sequence dictated by the idea of coming closest to the true value; in fact, those q_ν's satisfy $q_{\nu+1} \geqq q_\nu(q_\nu \pm 1)$, similar to the R-expansion. Stratemeyer obtains results essentially equivalent to the author's Theorems I and II, but in slightly more general form, involving the ϵ_ν's. There is included the derivation of his R-expansion for quadratic irrationalities, given in (c). Stratemeyer also gives the first five terms of the R-expansion for the decimal part of π.

(e) W. Sierpiński. Sur un algorithme pour développer les nombres réels en séries rapidement convergentes, *Bulletin International de l'Académie des Sciences*

* National Applied Mathematics Laboratories, National Bureau of Standards.

† The Approximation of Numbers as Sums of Reciprocals, this MONTHLY, March, 1947, pp. 135–142. The reader's attention is called to an obvious error at the bottom of p. 135, where $1/(a_i^2+a_i)$ should read $1/(a_i^2-a_i)$, and to an error on p. 139, 3rd line, where $1/(a_i^2-a_i)$ should read $1/(a_i^2+a_i)$. Also, note that the present article employs the notation $1/N_i$ in place of $1/a_i$ which was used previously for the ith partial fraction.

de Cracovie, Classe des Sciences Mathématiques et Naturelles. Série A, Sciences Mathématiques, 1911, pp. 113–117. Studies the third type of expansion into reciprocals mentioned on pp. 138, bottom, and 139, top, of the author's paper† (which was not labeled) and proves the same theorem concerning the irrationality of an expansion having an infinite number of terms. But Sierpiński states the closeness of the ith approximation correctly as $1/a_i(a_i+1)$ whereas the author† gave it incorrectly as $1/a_i(a_i-1)$. In actual magnitude, that is still about the same as the closeness of approximation of an R-expansion, since as long as i does not denote the last term, $a_i+1=a_i'$ where after $i-1$ terms, $1/a_i$ would be the continuation as a Sierpiński expansion and $1/a_i'$ the continuation as an R-expansion. (Of course, the preceding statement is not to be taken to mean that $a_i+1=a_i'$ for every a_i and a_i' in the R- and Sierpiński expansions respectively).

(f) F. Engel. *Verhandlungen der* 52. *Versammlung deutscher Philologen und Schulmänner*, Marburg, 1913. On pp. 190–191 is an abstract of his lecture on expansions into reciprocals. There is an error on p. 190 in the statement of the criterion of rationality, which should read $q_{n+v+1}=q_{n+v}^2-q_{n+v}+1$, (the $+1$ was omitted).

(g) J. H. Lambert. *Beyträge zum Gebrauche der Mathematik und deren Anwendung*, Zweyter Theil, Berlin, 1770. Chapter III, Verwandlung der Brüche. On pp. 99–104 is a definition, with illustrations, of the $\bar{R}$-expansion, with a statement of the rapidity of convergence as "double the number of places" at each step; but no proofs of anything are given. The first three terms in the $\bar{R}$-expansion of the decimal part of π are given correctly, but the fourth is given incorrectly as $-\dfrac{1}{44338\ 40233\ 62059}$ instead of $+\dfrac{1}{15164\ 89608\ 87729}$.

(h) O. Spiess. Über eine Klasse unendlicher Reihen, *Archiv der Mathematik und Physik*, Ser. 3, XII, 1907, pp. 124–134. Deals with R-expansions; contains a proof of Theorem II; develops R-expansions for certain quadratic irrationalities (see (c) and (d)).

(i) E. Cahen. Note sur un développement des quantités numériques, qui présente quelque analogie avec celui en fractions continues, *Nouvelles Annales de Mathématiques*, Ser. 3, X, 1891, pp. 508–514. Treats R- and Sierpiński expansions (see (e)), the latter twenty years before Sierpiński. Hence if a single name should be given to the hitherto-unnamed third type of expansion, it is "Cahen-expansion." Proves finiteness of expansions for rational numbers, gives inequalities for remainders, and states a theorem concerning uniqueness similar to Theorem I.

2. Comparison with simple continued fractions. It is of interest to compare the number-theoretic aspects of R-expansions with that of the Euclidean algorithm which arises from the development of the fraction a/b into a simple

† *loc. cit.*

continued fraction. Thus the Euclidean algorithm arising from

$$\text{(A)} \qquad \frac{a}{b} = \cfrac{1}{M_1 + \cfrac{1}{M_2 + \cdots + \cfrac{1}{M_m}}}$$

can be expressed as

$$\text{(A}'\text{)} \qquad \begin{aligned} b &= aM_1 + r_1 \\ a &= r_1 M_2 + r_2 \\ r_1 &= r_2 M_3 + r_3 \\ &\cdots\cdots\cdots\cdots \\ r_{m-3} &= r_{m-2} M_{m-1} + r_{m-1} \\ r_{m-2} &= r_{m-1} M_m. \end{aligned}$$

The r_i's are a decreasing set of positive integers and r_{m-1} is the well known greatest common divisor (g.c.d.).

Now consider the R-expansion

$$\text{(B)} \qquad \begin{aligned} &\frac{a}{b} = \frac{1}{N_1} + \frac{1}{N_2} + \cdots + \frac{1}{N_n} \\ &\frac{a}{b} - \frac{1}{N_1} = \frac{aN_1 - b}{bN_1} = \frac{r_1}{bN_1}, \qquad \text{where } r_1 < a. \end{aligned}$$

Repetition of this argument on r_1/bN_1 in place of a/b leads to

$$\begin{aligned} \frac{r_1}{bN_1} &= \frac{1}{N_2} + \frac{r_2}{bN_1N_2}, & r_2 &< r_1, \\ \frac{r_2}{bN_1N_2} &= \frac{1}{N_3} + \frac{r_3}{bN_1N_2N_3}, & r_3 &< r_2, \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

until there is reached

$$\begin{aligned} \frac{r_{n-2}}{bN_1N_2 \cdots N_{n-2}} &= \frac{1}{N_{n-1}} + \frac{r_{n-1}}{bN_1N_2 \cdots N_{n-1}}, & r_{n-1} &< r_{n-2}, \\ \frac{r_{n-1}}{bN_1 \cdots N_{n-1}} &= \frac{1}{N_n} + 0. \end{aligned}$$

The above system can be expressed as

$$
(\mathrm{B}') \qquad \begin{aligned}
aN_1 &= b + r_1 \\
r_1N_2 &= bN_1 + r_2 \\
&\cdots\cdots\cdots\cdots \\
r_{n-2}N_{n-1} &= bN_1N_2 \cdots N_{n-2} + r_{n-1} \\
r_{n-1}N_n &= bN_1N_2 \cdots N_{n-1}.
\end{aligned}
$$

Inspection of the foregoing system reveals that any r_i is a multiple of the g.c.d. of a and b, and this is noteworthy for r_{n-1}. Hence $r_{n-1}=1$ is sufficient but not necessary for a and b to be relatively prime. When $r_{n-1}=1$, the last denominator is $bN_1N_2 \cdots N_{n-1}$, and conversely. Thus in the R-expansion of a/b, $N_n=bN_1N_2 \cdots N_{n-1}$ is a sufficient but not necessary condition for a and b to be relatively prime.

A practical result of r_{n-1} being a multiple of the g.c.d., since in both the above and the Euclidean algorithm $r_i<r_{i-1}$, is that as a rule, for a/b fewer partial fractions are required in an R-expansion than in the simple continued fraction. For example,

$$
\frac{17}{90}=\frac{1}{6}+\frac{1}{45}, \quad \text{but} \quad \frac{17}{90}=\cfrac{1}{5+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{2}}}}.
$$

Of course, there could be exceptions where the r_i's in the Euclidean algorithm decrease so rapidly as to reach in fewer steps a lower last value (which is the g.c.d.) than r_{n-1} in the R-expansion.

3. Closeness of approximation of R-expansions.

LEMMA 1. *For $i>1$, $N_i>N_1 \cdots N_{i-1}$, where $1/N_i$ is the ith term in the R-expansion.*

In (B'), since $b>a$,

$$
N_i = \frac{a+I}{a-I(i)} N_1 \cdots N_{i-1} + \theta(i), \tag{1}
$$

where $I>0$, $I(i)>0$, $0 \leqq \theta(i) < 1$. The right side of (1) is greater than

$$
\left(\frac{a}{a-1}\right) N_1 \cdots N_{i-1} + \theta(i) > N_1 \cdots N_{i-1}.
$$

(The converse is obviously not true, namely in $\sum 1/N_i$, if $N_i>N_1 \cdots N_{i-1}$, that does not imply that $\sum 1/N_i$ is an R-expansion. For example, consider $1/4+1/5+1/24$.) From Lemma 1 there readily follows the result stated by the

author without proof in the earlier article, namely.

THEOREM IV. *If p/q is an approximation to x obtained by the R-expansion, then the remainder $x-p/q$ is less than $1/q$.*

Proof. Let p/q arise from the first n terms of the R-expansion, where

$$x = \frac{1}{N_1} + \frac{1}{N_2} + \cdots + \frac{1}{N_n} + \text{remainder}. \tag{2}$$

The remainder is less than $1/(N_n^2 - N_n)$. (The argument is strengthened if p/q is reduced to lower terms since $1/q' > 1/q$ when $q' < q$; so consider the unreduced case $q = N_n N_{n-1} \cdots N_1$.) For $n=1$, the theorem is obvious. For $n>1$, Lemma 1 leads to

$$N_n - 1 \geqq N_{n-1} \cdots N_1, \tag{3}$$

from which

$$q = N_n N_{n-1} \cdots N_1 \leqq N_n(N_n - 1) = N_n^2 - N_n. \tag{4}$$

Hence

$$\frac{1}{q} \geqq \frac{1}{N_n - N_n} > \text{remainder}. \tag{5}$$

That $x - p/q < 1/q$ is a best possible result is obvious from

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \cdots. \tag{C}$$

For suppose that there were some $\theta < 1$ such that $x - p/q$ were always less than θ/q. Let $L(m)$ be a number arbitrarily close to 1, obtained by taking enough terms, that is, m in number, on the right side of (C), and then stopping, so that there are a definite number of terms in some R-expansion. (Note that (C) taken as an infinite series is not an R-expansion, even though it is a "Sylvester series"). Then for $n=2$, $L(m) - 5/6$ would have to be less than $\theta/6$ for a fixed θ less than 1, which is impossible.

4. Closeness of approximation of $\overline{\text{R}}$-expansions.

THEOREM V. *If p/q is an approximation to x obtained by the $\overline{R}$-expansion, then*

$$\left| x - \frac{p}{q} \right| < \frac{1}{2q}.$$

except when $x = 3/4$, when the $\leqq$ relation may hold.

It is instructive, at the price of redundancy, to prove two versions of this theorem, first a weaker version establishing the $\leqq 1/2q$ relation for all approximations p/q, and subsequently the stronger version yielding the sharper $<1/2q$ relation in all but one case.

LEMMA 2 (for weak version). *For $i>1$, $N_i>N_1 \cdots N_{i-1}$, where $1/N_i$ is the absolute value of the ith term in the $\overline{R}$-expansion.*

If the $\overline{R}$-expansion begins with 1, this first term may be deleted without affecting the following argument, because of the obvious facts that (a) if $1-1/N_1 \pm \cdots$ is an $\overline{R}$-expansion of x, then $1/N_1 \mp \cdots$ is an $\overline{R}$-expansion of $1-x$, and (b) if the original $\overline{R}$-expansion begins with 1, proving $N_i>N_1 \cdots N_{i-1}$ for the deleted expression proves it for the original expression by shift of index so that $N_1=1$ and the new N_i becomes the original N_{i+1}. Thus one need consider an $\overline{R}$-expansion beginning with $\frac{1}{2}$ or less. In the $\overline{R}$-expansion, let all terms $\pm 1/N_i$ be considered with only $+$ signs. From $1/N_1 \leqq 1/2$, and from $N_{i+1} \geqq 2(N_i^2-N_i)$, each $N_{i+1}=N_i^2-N_i+\epsilon_i$, $\epsilon_i>1$. Then, by the uniqueness theorem for R-expansions, the $1/N_i$'s are the partial fractions belonging to some R-expansion and must satisfy $N_i>N_1 \cdots N_{i-1}$, by Lemma 1.

Now consider the first n terms of the $\overline{R}$-expansion. For $n=1$,

$$\left| x - \frac{p}{q} \right| \leqq \frac{1}{2q} \cdot$$

For if the $\overline{R}$-expansion commences with 1, the remainder $<1/2$ in absolute value; if it begins with $1/2$ followed by $\mp 1/4$, the next term must be $\pm$, so that

$$\left| x - \frac{1}{2} \right| < \frac{1}{4},$$

whereas if there is no third term, x must be $1/2+1/4$, or

$$\left| x - \frac{1}{2} \right| \leqq \frac{1}{4};$$

if it begins with $1/3$ (or $1/I$, $I>3$), the next term $\leqq 1/12$ (or $\leqq 1/2(I^2-I)$), so that

$$\left| x - \frac{1}{3} \right| < \frac{1}{6}\left(\text{or } < \frac{1}{2I}\right).$$

For $n>1$, by Lemma 2 and equations (3), (4) and (5) of Section 3, together with the inequality for the remainder after n terms in an $\overline{R}$-expansion, namely,

$$\left| x - \frac{p}{q} \right| \leqq \frac{1}{2(N_n^2 - N_n)}, \tag{6}$$

there follows

$$\left| x - \frac{p}{q} \right| \leqq \frac{1}{2q}, \tag{7}$$

which is the weakened version of Theorem 5.

The proof of Theorem V hinges upon the following inductive lemma.

LEMMA 3. *In an $\overline{R}$-expansion, whenever $N_i > N_1 \cdots N_{i-1}+1$, $i>1$, then also $N_{i+1} > N_1 \cdots N_i+1$.*

Proof.

$$\begin{aligned} N_{i+1} \geqq 2N_i^2 - 2N_i &= N_i(2N_i - 2) > N_i\{2(N_1 \cdots N_{i-1})\} \\ &= N_1 \cdots N_i + N_1 \cdots N_i > N_1 \cdots N_i + 1. \end{aligned}$$

Now consider any $\overline{R}$-expansion. (a) If it begins with 1, the next term cannot be $-1/3$, so it is either $-1/4$ or $-1/(4+I)$. Since $4>1+1$, Lemma 3 applies to each succeeding approximation, and hence (3), (4), (5) and (7) hold with the $<$ sign in place of $\leqq$. (b) If it begins with 1/2, the next term cannot be $\pm 1/3$ but may be $+1/4$. (If that is all, one has the exceptional case of 3/4 where $\leqq$ holds, although when 3/4 is written as the alternative $\overline{R}$-expansion $1-1/4$, there is no exception). If 1/4 is followed by $- \cdots$ (cannot be $+ \cdots$) then $<$ applies at 1/2. At $1/2+1/4$, the absolute value of the next term $\leqq 1/24$, so $<$ holds. Beyond the term 1/4, since $4>2+1$, Lemma 3 is employed, and again (3), (4), (5) and (7) hold with $<$ signs. If it begins with 1/2 followed by $\pm 1/(4+I)$, the same reasoning (Lemma 3, then (3), (4) and so on) again results in $<$. (c) If it begins with 1/3 or $1/(3+I)$, the same line of reasoning holds; *e.g.* after 1/3 there must follow $\pm 1/12$ or $\pm 1/(12+I)$, $|\text{error}| < 1/11 < 1/6$, then for $n>1$, Lemma 3 followed by (3), (4), and so on, to obtain $<1/2q$ in (7). Thus Theorem V is established.

That Theorem V contains the best possible inequality to hold for all $\overline{R}$-expansions, is apparent in considering the particular $\overline{R}$-expansion $1/2-1/4+1/I$. If instead of $<1/2q$, the closeness were expressed by $<1/(2+\delta)q$, for fixed δ, one would be led to a contradiction, since the I could be chosen sufficiently large so that the error after one term, where $q=2$, would be arbitrarily close to 1/4.

It is of interest to compare the closeness of approximation of a fraction p/q obtained by either the R- or $\overline{R}$-expansion with a decimal approximation p'/q', where q' is a power of 10. On the basis of Theorems IV and V, there is a superficial resemblance between R-expansion approximations and "chopped" decimal approximations, as well as between $\overline{R}$-expansion approximations and "rounded" decimal approximations. But there is a fundamental difference between the use of successive decimal approximations to a given number, and the employment of either the R- or $\overline{R}$-expansion, which is more favorable to the latter. For after $\pm 1/N_n$, in the next partial fraction $\pm 1/N_{n+1}$, the N_{n+1} may be very much larger than the minimum value that could arise in an R- or $\overline{R}$-expansion, say about k times that minimum. In such cases, by obvious modifications of equations (3), (4), (5) and (7), based upon the strengthening of Lemma 3 with a factor of k on the righthand side of the inequality, it follows that not only the closeness of the nth but the closeness of all ensuing approximations p/q will be about $1/kq$ or $1/2kq$ for the R- or $\overline{R}$-expansion respectively.

NOTE ON A PAPER BY L. S. JOHNSTON

IVAN NIVEN, University of Oregon

In a paper entitled *Denumerability of the Rational Number System* in the February, 1948, issue of this MONTHLY, L. S. Johnston has exhibited a one-to-one correspondence between the positive integers and the rational numbers, with a specific construction provided. The purpose of this note is to give another constructive correspondence.

For any positive integer n, let r be the number of zeros occurring in the representation of n in the binary scale of notation. These r zeros divide the representation into $r+1$ blocks of 1's, some blocks being perhaps empty: for $j=2, 3, \cdots, r$ let a_j denote the number of 1's between the $(j-1)$th and the jth zeros, counted from the right; define a_1 and a_{r+1} as the number of consecutive 1's on the right and left ends of the representation. E.g., if n is 110010 in the binary scale, then $r=3$ and $(a_1, a_2, a_3, a_4)=(0, 1, 0, 2)$. If $n=1$, then $r=0$, and $a_1=1$. Note that n determines a unique set of values $a_1, a_2, \cdots, a_{r+1}$, with positive a_{r+1}, and conversely.

Now define $b_j=-a_j/2$ for even a_j, and $b_j=(a_j+1)/2$ otherwise: this is a one-to-one correspondence between the non-negative integers a_j and all integers b_j. Finally, if $p_1, p_2, p_3, \cdots$ are the consecutive primes 2, 3, 5, $\cdots$, we define

$$f(n) = \prod_{j=1}^{r+1} p_j^{b_j}. \tag{1}$$

Thus to any positive integer n corresponds a unique positive rational number $f(n)$. Conversely, any positive rational number f, except 1, can be written uniquely in the form (1) with $b_{r+1}\neq 0$. The exponents b_j determine a unique set a_j, with $a_{r+1}\neq 0$, and these give a unique positive integer n. E.g., if $f=33/49$, then $(b_1, b_2, b_3, b_4, b_5)=(0, 1, 0, -2, 1)$ and $(a_1, a_2, a_3, a_4, a_5)=(0, 1, 0, 4, 1)$, so that $n=1011110010$ in the binary scale.

We now obtain a one-to-one correspondence between the positive integers and all positive rational numbers, by letting $n+1$ correspond to $f(n)$, written $n+1\leftrightarrow f(n)$, for all positive integers n, and in addition we let $1\leftrightarrow 1$. On the other hand, if we want a one-to-one correspondence between the positive integers and all rational numbers, we can use the correspondences $1\leftrightarrow 0$, $2\leftrightarrow 1$, $3\leftrightarrow -1$, $2n+2\leftrightarrow f(n)$, $2n+3\leftrightarrow -f(n)$, where n ranges over all positive integers.

A GENERALIZATION OF EULER'S ϕ-FUNCTION

V. L. KLEE, JR., University of Virginia

The purpose of this note is to point out an interesting generalization of Euler's ϕ-function, and to show that to a large extent its properties parallel those of the ϕ-function. Proofs are omitted, but may be supplied readily by the reader. It is suggested that many of the theorems stated below would be suitable as exercises for students in a Number Theory class.

We call an integer kth-*power-free* if it is not divisible by the kth power of any integer >1. If k and n are positive integers we denote by $\Phi_k(n)$ the number of integers h in the set $1, \cdots, n$, for which the greatest common divisor (h, n) is kth-power-free. Φ_1 is the ϕ-function.

In establishing for Φ_k properties analogous to those of ϕ one can follow the order and method of proof of Landau [Elementare Zahlentheorie] to obtain the following results:

$$\sum_{d|n} \Phi_k(d^k) = n^k;$$

$$\Phi_k(n^k) = n^k \sum_{d|n} \mu(d)/d^k;$$

$$\Phi_k(n) = n \prod_{p \text{ prime and } p^k|n} (1 - p^{-k});$$

$$\Phi_k(\prod p_i^{a_i}) = \prod_{a_i<k} p_i^{a_i} \cdot \prod_{a_i \geqq k} p_i^{a_i-k}(p_i^k - 1),$$

for distinct primes p_i; Φ_k is multiplicative.

From these results one obtains the following:

$\Phi_k(mn)/[\Phi_k(m)\Phi_k(n)] = Q\Phi_k(P)/P\Phi_k(Q)$, *where* P *is the product of the kth powers of all primes* p *for which* $p^k \nmid m$ *and* $p^k \nmid n$ *but* $p^k | mn$, *and* Q *is the product of the kth powers of all primes* q *for which* $q^k | m$ *and* $q^k | n$.

$\Phi_k(n)$ *is odd if and only if* $n = 2^e m$, *where* m *is odd and kth-power-free and either* $e=0$ *or* $e=k$.

The analogue of the Fermat theorem which holds for Φ_k *is that if* a *and* n *are coprime, then* $n/(a^{\Phi_k(n)}-1, n)$ *is kth-power-free.*

The reader will be able to supply other results of similar nature.

As a generalization, different from that above, of the relation $\phi(n) = n\sum_{d|n}\mu(d)/d$, we have $\Phi_k(n) = n\sum_{d|n}\mu^k(d)/d$, where $\mu^k(n)$ is defined as follows: $\mu^k(1)=1$; if $n = p_1^{a_1}p_2^{a_2} \cdots p_t^{a_t}$ is the canonical factorization of n, then $\mu^k(n) = (-1)^t$ if $a_1 = a_2 = \cdots = a_t = k$, and $\mu^k(n) = 0$ if for some i, $a_i \neq k$.

It may be of interest to note the following properties of the functions μ^k: μ^k *is multiplicative;* $\mu^1(n^1) = \mu^2(n^2) = \cdots$; $\sum_{d|n^k}\mu^k(d)/d^j = \sum_{d|n^j}\mu^j(d)/d^k$; $\sum_{d|n}\mu^k(d)$ *is* 1 *or* 0 *according as* n *is or is not kth-power-free.*

We note in conclusion a further possible generalization. If A is an arbitrary set of integers and k and n are positive integers, denote by $\Phi_{(A,k)}(n)$ the number of residue classes R modulo n for which $r \in R$ and $a \in A$ implies that $(r+a, n)$ is kth-power-free. $\Phi_{(A,1)}$ is Lucas's generalization [*Theorie des Nombres*, p. 402]. By a simple adaptation of the argument of Dickson [*Introduction to the Theory of Numbers*, p. 7] the following result can be established.

$\Phi_{(A,k)}$ *is multiplicative. If* $x = \prod p_i^{a_i}$ *is the canonical factorization of* x, *then* $\Phi_{(A,k)}(x) = \prod(p_i^{a_i} - r_i p_i^{a_i-k})$, *where* $r_i = 0$ *for* $a_i < k$, *and for* $a_i \geqq k$, r_i *is the number of distinct residues modulo* p_i^k *of the integers in* A.

CORRECTIONS

V. L. Klee, *Some remarks on Euler's totient*, this MONTHLY, vol. 54, 1947, p. 332. The last line should be replaced by:

$n+2$ are prime, n is twice a Mersenne prime, or n has the form $4p$, where p and $2p+1$ are prime.

R. W. Hamming, *Subseries of monotone divergent series*, this MONTHLY, vol. 54, 1947, pp. 462–463. The theorem which was proved in the note should be stated correctly as follows:

THEOREM. *In order that an increasing sequence $I(n)$ of positive integers may be such that $\sum d_{I(n)}$ is divergent whenever $\sum d_n$ is a divergent series of positive terms for which $d_1 \geqq d_2 \geqq \cdots$, it is necessary and sufficient that $I(n)/n$ be a bounded sequence.*

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

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EQUATIONS AND LOCI IN POLAR COÖRDINATES

R. W. WAGNER, Oberlin College

The purpose of this note is to further publicize and to urge a wider application of an idea discussed by C. Fox* to clarify the difficulties which are encountered in seeking the intersections of loci defined by equations in polar coördinates.

Everyone who has worked with polar coördinates is aware of the fact that a point has an infinite number of coördinates. Thus, the point with coördinates (r, θ) has the other coördinates $([-1]^n r, \theta+n\pi)$, $n=0, \pm 1, \pm 2, \cdots$. Writing the coördinates in this way implies that the common conventions that r may be any real number and that θ is measured in radians are being used.

The locus or graph of an equation is usually taken to mean the set of all points for which some pair of coördinates satisfy the equation.

Fox's idea is this: *The lack of uniqueness in the coördinates of a point leads, in general, to a lack of uniqueness in the equation of a locus.* This principle is overlooked or ignored in the majority of text books which the author has examined. A few books† mention it and use it to find the points of intersection of polar loci.

* C. Fox, The Polar Equations of a Curve, Math. Gazette, vol. 15 (1931), pp. 486–7.

† They are the analytic geometry texts by Curtiss and Moulton (1930), by Middlemiss (1947), by Nathan and Helmer (1947), by Randolph and Kac (1946), by Sisam (1926), by Nowlan (1932), and by Mason and Hazzard (1935).

However, none of these books use the principle to its full extent, for example, to clarify the tests for symmetry or to expedite the plotting of curves.

To investigate the validity of this principle, consider the locus defined by the equation

$$F(r, \theta) = 0. \tag{1}$$

A point will be on this locus if, and only if, for some integer n ($n=0, \pm 1, \pm 2, \cdots$)

$$F([-1]^n r, \theta + n\pi) = 0. \tag{2}$$

The locus defined by assigning any permissible value to n in equation (2) will be the same as the locus defined by equation (1). For, any point which has a pair of coördinates which satisfy (1) will have a pair which satisfy (2), and conversely.

The equations obtained from (2) by assigning various values to n may be algebraically equivalent. This cannot occur unless $F(r, \theta)$ is a periodic function of θ with a period which is a rational multiple of π. If this period be $p\pi/q$, each equation obtained from (2) is algebraically equivalent to one of at most $2p$ equations which are algebraically non-equivalent. For example, all alternative forms for the equation

$$r - a \cos \theta - b \sin \theta = 0$$

are algebraically equivalent because

$$(-1)^n r - a \cos (\theta + n\pi) - b \sin (\theta + n\pi) = \begin{cases} r - a \cos \theta - b \sin \theta & (n \text{ even}) \\ -r + a \cos \theta + b \sin \theta & (n \text{ odd}). \end{cases}$$

For another example, there are four algebraically non-equivalent forms for the equation of the locus defined by

$$r - \cos (\theta/2) = 0. \tag{3}$$

The other forms are

$$-r + \sin (\theta/2) = 0, \qquad r + \cos (\theta/2) = 0, \qquad \text{and} \qquad -r - \sin (\theta/2) = 0.$$

The student should be made aware of the lack of uniqueness of the polar equation of a locus from the start. The first step in the discussion of the locus of an equation should be to find the alternate forms of the equation. The students are delighted to learn that they can graph several equations at once!

Plotting. By using the alternate equations one can frequently shorten the table values or the work required to prepare it by listing a value of r for each of the alternate equations and restricting θ to the interval $0 \leqq \theta \leqq \pi$. For example, to plot the graph of equation (3), the table of values would have values of θ in one column and values of r in four parallel columns. Each of these latter columns would show the values of r obtained from one of the alternate equations.

Symmetry. The usual tests for symmetry of a polar locus are based on condi-

tions which are sufficient to show symmetry. The failure of the test to show symmetry does not show the lack of symmetry. By utilizing the alternate equations of the locus one can state a necessary and sufficient test for symmetry which requires a single substitution for each type of symmetry.

The locus of $F(r, \theta)$ *is symmetrical with respect to* (i) *the pole,* (ii) *the polar axis or* (iii) *the perpendicular to polar axis at the pole if, and only if,* (i) $F(-r, \theta)=0$, (ii) $F(r, -\theta)=0$, *or* (iii) $F(-r, -\theta)=0$ *is algebraically equivalent to* $F(r, \theta)=0$ *or one of its alternate forms.*

After a type of symmetry has been found, some of the r-columns of the table of values may be omitted and the symmetry used to complete the curve.

Intersections. The points of intersection are found by solving simultaneously all the pairs of equations made up of an equation for the first locus and an equation for the second locus. This can be systematically accomplished by first introducing an integer valued parameter into each equation, by then restricting the values of the parameters to sets of values which yield algebraically non-equivalent equations, and finally solving the two equations simultaneously in terms of the two parameters. If there are but few alternate forms for each equation, it may be simpler to solve the various pairs of equations than to solve the equations involving the parameters.

This procedure does not include the pole among the points of intersection unless the two curves have a common tangent at the pole. In general, an intersection at the pole is most readily found by noting that both curves pass through the pole.

Example. Find the intersections of the cardioid and the limaçon whose equations are

$$r = 2 + 2\cos\theta, \qquad r = 10\cos\theta - 2.$$

Each of these equations has an alternate form, namely,

$$r = 2\cos\theta - 2, \qquad r = 10\cos\theta + 2.$$

If one solves the two given equations simultaneously, he finds the points on the inner loop of the limaçon $(3, \pm\pi/3)$. The simultaneous solutions of the two alternate equations are the same points but the coördinates appear in the form $(-3, \pm 2\pi/3)$. The simultaneous solutions of a pair of equations made up of one given equation and the alternate of the other equation are the points on the outer loop of the limaçon $(2, \pm\pi/2)$, which may appear in the form $(-2, \pm\pi/2)$. The pole is also an intersection.

Conics. If the equation of a conic

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 \tag{4}$$

is changed to polar coördinates, the polar coördinate equation is factorable. If the conic passes through the pole, one factor leads to the equation $r=0$ and the other factor leads to the unique equation of the conic. If the conic does not pass through the pole, it has two equations and these equations are obtained by

equating the factors to zero individually.

The result of substituting $x=r\cos\theta$ and $y=r\sin\theta$ into equation (4) is

$$(5)\quad (A\cos^2\theta+2B\cos\theta\sin\theta+C\sin^2\theta)r^2+2(D\cos\theta+E\sin\theta)r+F=0,$$

which can be abbreviated to

$$(6)\qquad Hr^2+2Gr+F=0.$$

When θ is increased by π, H is not changed and the sign of G is changed. If the conic passes through the pole, $F=0$ and equation (6) may be factored into

$$r=0 \quad\text{and}\quad Hr+2G=0.$$

This second equation has no alternate forms because the change which is made to find an alternate equation merely changes the sign of both terms. If the conic does not pass through the pole, $F\neq 0$ and, after multiplication by H, (6) may be factored into

$$Hr+G+\sqrt{G^2-HF}=0 \quad\text{and}\quad Hr+G-\sqrt{G^2-HF}=0.$$

Each of these equations is an alternate form of the other because the substitution which is used to find alternate forms changes each one into the other. Also they are the only algebraically non-equivalent forms of the equation.

This illustrates a situation which should be true in general. If the rectangular equation of a locus be changed to polar coördinates, the polar equation is factorable and the factors are a complete set of alternate equations for the locus.

MAXIMUM UNCERTAINTY AS A SIMPLE EXAMPLE OF A NON-DISTRIBUTIVE ALGEBRA

L. C. GREEN, Haverford College

Practical examples of non-distributive algebras simple enough for classroom use are difficult to find. It is the purpose of this note to point out that the computation of the maximum uncertainty in the elementary theory of errors provides such an example. By the elementary theory is meant that part of the theory of errors in which the *absolute* uncertainty of the sum or difference of two quantities is taken as the sum of their *absolute* uncertainties and the *percentage* uncertainty in the product or quotient of two quantities is taken as the sum of their *percentage* uncertainties. So long as the same measure of uncertainty is used throughout a discussion, it makes no difference in the application of these rules which one of the various measures is used, that is, average deviation, standard deviation and so forth.

Consider the illustration of determining the maximum uncertainty in the expansion of a rod of soda glass $100.00\pm.01$ cm. in length if the temperature increases from $20.0\pm.1$°C to $22.5\pm.1$°C and if the coefficient of linear expansion for soda glass is $(1.2\pm.1)\cdot 10^{-5}$ per degree C. from the formula

$$\Delta L=Lk(T_2-T_1).$$

If we evaluate the parenthesis first we obtain the result $(3.0\pm.5)\cdot 10^{-3}$ cm. but if we make use of the distributive law we find $(3.0\pm 4.5)\cdot 10^{-3}$ cm. By the latter method the uncertainty in the expansion would appear to be 50% greater than the expansion itself. Clearly the first result is the one of interest.

In order to make the situation explicit let us define a number pair (x, u) where x represents some measured quantity and u is its absolute uncertainty. The fundamental operations then take the form

$$(x_1, u_1) \pm (x_2, u_2) = (x_1 \pm x_2, u_1 + u_2)$$
$$(x_1, u_1)\cdot(x_2, u_2) = (x_1x_2, u_1x_2 + u_2x_1).$$

It is evident that addition and multiplication are both associative and commutative. On the other hand we have

$$(x_1, u_1)[(x_2, u_2) \pm (x_3, u_3)] = (x_1[x_2 \pm x_3], u_1[x_2 \pm x_3] + [u_2 + u_3]x_1).$$

In contrast the usual distributive law would yield

$$(x_1, u_1)[(x_2, u_2) \pm (x_3, u_3)] = (x_1[x_2 \pm x_3], u_1[x_2 + x_3] + [u_2 + u_3]x_1).$$

This non-distributive character has its origin in the fact that with the content assigned above to the symbols, $-(x, u) = (-x, u)$. If we add the rule of reckoning that $-(x, u)$ is to be replaced by $(-x, u)$ whenever it appears, the algebra becomes distributive.

Finally for their curiosity, one may write the relations:

$$\frac{(x_1, u_1)}{(x_2, u_2)} = \left(\frac{x_1}{x_2}, \frac{u_1x_2 + u_2x_1}{x^2}\right),$$
$$(x, u)^n = (x^n, nux^{n-1}),$$
$$(x, u)^{1/n} = \left(x^{1/n}, \frac{1}{n}ux^{(1/n)-1}\right).$$

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 821. *Proposed by B. H. Brown, Dartmouth College*

A sports promoter hired nine not too scrupulous athletes, and formed three

cross-country teams A, B, and C, of three men each, which he took on tour for a series of dual and triple meets. If a team could win, it always would; but a losing team could be bribed to run more slowly. Except for this failing, the athletes always ran as automata, and always finished in the same order, with no ties. (In a cross-country meet the winner receives 1 point, the next man 2 points, *etc.* The team score is the sum of the points received by its members, and low score wins.)

In dual meets, team A beat team B, B beat C, and C beat A, much to the annoyance of the promoter who unjustly accused the men of dishonesty.

In an honest triple meet, team A won, whereupon team C went to team B with the following dishonest but logical offer: "No conceivable dishonesty on the part of our C team can enable your B team to beat or even tie A; but if you will slow down, our C team can beat A, and we will divide the profits with you."

Determine completely the composition of each team in terms of the relative excellence of its members.

E 822. *Proposed by W. R. Ransom, Tufts College*

Every factorial that can be expressed as the difference of two squares can be so expressed in two different ways.

E 823. *Proposed by Max LeLeiko, Rutgers University*

Find four distinct non-zero integers a, b, c, d such that $a^2+b^2+c^2+d^2$

$$= (1/5)\left[(a+b)^2+(a+c)^2+(a+d)^2+(b+c)^2+(b+d)^2+(c+d)^2\right]$$
$$= (1/7)\left[(a+b+c)^2+(a+b+d)^2+(a+c+d)^2+(b+c+d)^2\right]$$
$$= (1/3)(a+b+c+d)^2.$$

E 824. *Proposed by E. P. Starke, Rutgers University*

We modify the harmonic series by taking the first term positive, the next two negative, the next three positive, *etc.* Show that this modified series is convergent.

E 825. *Proposed by W. E. Patten, Spartanburg, S. C.*

If p points and no more are collinear, we say they form a (collinear) set of *multiplicity* p.

Suppose we are given the rectangular lattice of points which are the corners of the squares of an h by v checkerboard, where $h \leqq v$. Determine the number of collinear sets, for each possible multiplicity $\geqq 2$, occurring in the set of points.

SOLUTIONS

An Impossible Journey

E 788 [1947, 471]. *Proposed by Leo Moser, University of Manitoba*

Consider a map on a spherical surface where the countries are determined by n great circles of which no three are concurrent. Show that if n is a multiple of

four it is impossible to make a trip visiting each country once and only once, if travelling along a boundary or crossing at a boundary point of more than two countries is forbidden.

Solution by the Proposer. We first prove by induction that such a map may be colored with two colors, so that no two countries with common boundary have the same color. This is certainly true for $n=1$. Suppose it is true for $n-1$. When we add another great circle, it is only necessary to reverse all the colors to one side of this circle to obtain a coloring for n. Also, if we use this coloring, it is clear that for n even, diametrically opposite countries will have the same color. Hence, in this case, there will be an even number of countries of each color. We next show that these two even numbers cannot be equal. If they were equal, then the total number of countries, F, would be a multiple of four, whereas we shall show that $F=4k+2$.

Since each circle intersects each other circle twice, the number of vertices, V, is $n(n-1)$. Now every vertex belongs to four edges, and every edge has two vertices. Therefore the number of edges, E, is $2V$. Using Euler's theorem, $V-E+F=2$, we get $F=n(n-1)+2$, and if n is a multiple of four, F will be of the form $4k+2$.

Finally, we note that in any trip on a map colored by our method, one must change color with every crossing from country to country, and since the number of countries of one color differs from the number of countries of the other color by at least two it will be impossible to reach them all.

Note. In a way this is a generalization of E 711 [1946, 593], because the reciprocal map for $n=4$ of our problem is homeomorphic to the surface of the rhombic dodecahedron. The direct generalization of E 711, which is also true, states that there is no Hamilton path for any polyhedron all of whose faces are parallelograms, if the number of faces is $n(n-1)$, (see problem 4176 [1947, 169]), where n is a multiple of four.

Derivatives of tan x

E 789 [1947, 471]. *Proposed by Kaidy Tan, Chip-Bee Institute, Amoy, Fukien, China*

If $y=\tan x$, show that

$$(d^n y/dx^n)\cos^{n+1} x$$

$$= \begin{vmatrix} \cos x & 0 & \cdots \sin x \\ \cos(x+\pi/2) & \cos x & \cdots \sin(x+\pi/2) \\ \cos(x+2\pi/2) & 2\cos(x+\pi/2) & \cdots \sin(x+2\pi/2) \\ \cdot & \cdot & \cdot \\ \cos(x+n\pi/2) & n\cos(x+\overline{n-1}\,\pi/2) & \cdots \sin(x+n\pi/2) \end{vmatrix}.$$

Solution by N. J. Fine, University of Pennsylvania. Let $w=\sin x$, $z=\cos x$, so that $w=yz$. If we differentiate r times by the Leibnitz rule, we have

$$w^{(r)} = \sum_{k=0}^{r} \binom{r}{k} y^{(k)} z^{(r-k)} \qquad (r = 0, 1, 2, \cdots, n).$$

Regard this as a system of $n+1$ linear equations in the quantities $y^{(k)}$, $k=0, 1, 2, \cdots, n$. The determinant of the system, being triangular and having $z^{(0)}=\cos x$ down the main diagonal, has the value $\cos^{n+1}x$. Hence, solving for $y^{(n)}$, we find

$$y^{(n)} \cos^{n+1} x = \begin{vmatrix} z^{(0)} & \cdot & \cdot & \cdot & \cdots & w^{(0)} \\ z^{(1)} & z^{(0)} & \cdot & \cdot & \cdots & w^{(1)} \\ z^{(2)} & \binom{2}{1} z^{(1)} & z^{(0)} & \cdot & \cdots & w^{(2)} \\ z^{(3)} & \binom{3}{1} z^{(2)} & \binom{3}{2} z^{(1)} & z^{(0)} & \cdots & w^{(3)} \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ z^{(n)} & \binom{n}{1} z^{(n-1)} & \cdot & \cdot & \cdots & w^{(n)} \end{vmatrix}$$

We now observe that $z^{(k)}=\cos(x+k\pi/2)$, $w^{(k)}=\sin(x+k\pi/2)$, and the proof is complete.

Also solved by the proposer.

Editorial Note. In connection with this problem see *Successive derivatives of tan x*, by R. K. Moreley, National Mathematics Magazine, Vol. XIX, No. 6.

Algorithm for Approximating $a^{1/n}$

E 790 [1947, 471]. *Proposed by H. S. Wall, University of Texas*

Let

$$f(x, y) = \frac{a + xy(x^{n-2} + x^{n-3}y + x^{n-4}y^2 + \cdots + y^{n-2})}{x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1}},$$

where n is an integer greater than unity, and $a>0$. If $x_0>0$, $x_1>0$, and x_2, x_3, x_4, $\cdots$ are computed recurrently by means of the formula

$$x_{p+2} = f(x_p, x_{p+1}), \qquad p = 0, 1, 2, \cdots$$

then

$$\lim_{p=\infty} x_p = a^{1/n}.$$

I. *Solution by the Proposer.* We may write

$$(1) \qquad f(x, y) = f(y, x) = x - \frac{x^n - a}{x^{n-1}\left(1 + \dfrac{y}{x} + \dfrac{y^2}{x^2} + \cdots + \dfrac{y^{n-1}}{x^{n-1}}\right)}.$$

Therefore $f(a^{1/n}, y)=f(x, a^{1/n})=a^{1/n}$, and the statement is obvious if x_0 or x_1 equals $a^{1/n}$.

Formula (1) shows that $f(x, y)$ is an increasing function of y, for fixed x, if $x^n>a$, and a decreasing function of y, for fixed x, if $x^n<a$. Since $f(x, y)=f(y, x)$, we may interchange x and y in this statement.

If $x_0, x_1>a^{1/n}$, then $x_2=f(x_0, x_1)>a^{1/n}$. For, if we let $x_0 \to a^{1/n}$, x_2 must *decrease* to $a^{1/n}$. Moreover, $x_2<x_0, x_1$ inasmuch as

$$x_2 - y = -\frac{y^n - a}{y^{n-1}\left(1+\frac{x}{y}+\frac{x^2}{y^2}+\cdots+\frac{x^{n-1}}{y^{n-1}}\right)} < 0,$$

for $y=x_0, x_1$. It follows that $x_2>x_3>x_4>\cdots>a^{1/n}$, and therefore $\lim_{p=\infty}x_p =r\geqq a^{1/n}$. Since $r=f(r, r)=[a+(n-1)r^n]/nr^{n-1}$, then $r^n=a$, $r=a^{1/n}$.

If $x_0<a^{1/n}<x_1$, then $x_2=f(x_0, x_1)$ is a decreasing function of x_1. If we let $x_1\to a^{1/n}$, then x_2 must *increase* to $a^{1/n}$, so that $x_2<a^{1/n}$. Moreover, $x_0<x_2$. For, if $x_2<x_0$, then x_2 must increase through the value x_0 as $x_1\to a^{1/n}$, which is impossible inasmuch as $x_0<a^{1/n}$. In this case we must have $x_2<x_3<a^{1/n}$. If $x_1<a^{1/n}<x_0$, then $x_1<x_2<a^{1/n}$. Hence we have only to consider the case where $x_0<x_1<a^{1/n}$. But then, $x_2>a^{1/n}$, $x_1<x_3<a^{1/n}$, $x_3<x_4<a^{1/n}$, $x_5>a^{1/n}$, $x_4<x_6<a^{1/n}$, $x_6<x_7<a^{1/n}$, $x_8>a^{1/n}$, $\cdots$. Therefore, the sequence $x_1, x_3, x_4, x_6, x_7, x_9, x_{10}, \cdots$ converges to $a^{1/n}$, and $x_{3n+2}=f(x_{3n}, x_{3n+1})\to f(a^{1/n}, a^{1/n})=a^{1/n}$.

II. *Solution by Leo Moser, University of Manitoba.* The result is made intuitive by the following considerations:

(a) If $x_p=x_{p+1}$, then $f(x_p, x_{p+1})$ is the approximation obtained for $a^{1/n}$ by using Newton's method on the approximation x_p.

(b) If $x_p\neq x_{p+1}$, then $f(x_p, x_{p+1})$ is the x-coördinate of the point of intersection of the x-axis and the straight line joining the points on the curve $y=x^n-a$, having x-coördinates x_p and x_{p+1}.

This would indicate the possible lines of generalization of the given result. Thus we could obtain similar methods for solving a much larger class of equations than $y^n=a$ (n an integer). Also, we might work with more than two points on the curve at a time and pass special curves through these, say circles through three points.

Also solved by J. G. Herriot.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4300. *Proposed by Leo Moser, University of Toronto*

Let $a_1, a_2, \cdots, a_n$ be n, not necessarily distinct, elements of a group of order n. Show that there exist integers p and q, $1 \leqq p \leqq q \leqq n$, such that

$$\prod_{i=p}^{q} a_i = 1.$$

4301. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Consider similar triangles ABC, the sides BC, CA, AB of which pass through the fixed points A_1, B_1, C_1. (1) The locus of the circumcenters O of these triangles is a circle. (2) The circumcircles (O) are orthogonal to a fixed circle. (3) When the points A_1, B_1, C_1 are collinear, the envelope of the circles (O) is a cardioid. What is this envelope otherwise?

4302. *Proposed by Joseph Rosenbaum, the Milford School, Conn.*

Prove that if x and y have no common factor then every odd factor of

$$x^{2^n} + y^{2^n},$$

where n is a positive integer, is of the form $2^{n+1}m+1$.

4303. *Proposed by G. T. Williams, Cambridge, Mass.*

If

$$\theta_n = \int_0^1 x(x-1) \cdots (x-n+1)dx,$$

and

$$\phi_n = \int_0^1 x(x+1) \cdots (x+n-1)dx,$$

show that $(-1)^n\theta_n = \phi_n - n\phi_{n-1}$ and find ϕ_n in terms of θ_n.

4304. *Proposed by H. D. Grossman, New York City*

From the vertices of a regular n-gon three are chosen to be the vertices of a

triangle. Prove that the number of essentially different possible scalene triangles is the integer nearest to $(n-3)^2/12$. Compare 3893 [1940, 664].

SOLUTIONS

Squares of Special Form

4237 [1947, 112]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In which systems of numeration, of base B, are there two-digit squares $m^2=ab$ and $n^2=cd$, if $d=a+1$ and $b=c+1$? Show that, if also $b+d=B+1$, the four-digit numbers $ddbb$ and $bbdd$ are the squares of the numbers mm and nn.

Solution of E. P. Starke, Rutgers University. From the first hypothesis we have

$$(1) \qquad m^2 = aB + b, \qquad n^2 = cB + d,$$

with $d=a+1$ and $b=c+1$. It is evidently necessary that b and d be quadratic residues mod B of a and c, respectively. Upon eliminating B we have $c(m^2-b)=a(n^2-d)$ or

$$(2) \qquad cm^2 - an^2 = (c-a)(c+a+1).$$

Hence, if $(c, a)=1$, any solution m, n, of (2) will satisfy (1) and determine integral values of B. The values of m and n must be sufficiently large to provide that $B>m, n$. For example, with $a=7$, $c=1$, (2) becomes $m^2-7n^2=-54$ which leads to an infinity of solutions, $n=5$, $m=11$, $B=17$; $n=7$, $m=17$, $B=41$; and so on.

Under the additional hypothesis we have at once $a+b=B$ and $m^2=aB+b$, whence

$$\begin{aligned}(mB+m)^2 &= m^2B^2+2m^2B+m^2 = aB^3+(b+2a)B^2+(2b+a)B+b \\ &= (a+1)B^3+(a+1)B^2+bB+b = dB^3+dB^2+bB+b,\end{aligned}$$

as required. Upon interchange of b and d, the corresponding equation for nn^2 results. To find numerical examples we put (1) in the form

$$(3) \qquad m^2 - 1 = d(B-1), \qquad n^2 - 1 = b(B-1),$$

$$(4) \qquad m^2 + n^2 = (b+d)(B-1) + 2 = B^2 + 1.$$

If we put $m+1=B-1$, $m-1=d$, we have

$$b = B + 1 - d = 4, \qquad n^2 = 1 + b(B-1) = 4B - 3.$$

Since n is odd we put $n^2=(2p+1)^2$, whence

$$B = p^2 + p + 1$$

and we have solutions for all $p>1$.

We may also put $x(m-1)=y(B-1)$, $y(m+1)=dx$, with any choice of x and y. In particular if $x=2$, $y=1$, we have $B=4d-3$, $b=3d-2$, $n^2=12d^2-20d+9$

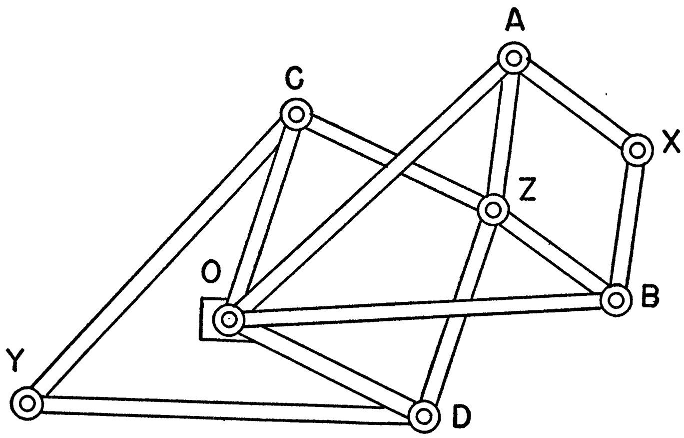
or $(6d-5)^2-3n^2=-2$ which leads to solutions such as $d=45$, $n=153$, $B=177$, $m=89$. $x=3$, $y=2$ leads to solutions like $d=140$, $n=233$, $m=209$, $B=313$. $x=4$, $y=3$ gives $d=150$, $n=175$, $m=199$, $B=265$ and other solutions.

No general formula giving all solutions appears likely. To test whether a proposed value of B is possible, we have two strong necessary conditions: (a) B^2+1 must be divisible by two or more distinct odd primes in order to be a sum of two squares in more than one way, as in (4); and (b) $B-1$ must be divisible by 4 or by two distinct odd primes, otherwise in (3) m^2, $n^2\equiv 1 \pmod{B-1}$ have only the solutions m, $n=1$, $B-2$, which will not satisfy (4).

Peaucellier Cells

4239 [1947, 168]. *Proposed by H. F. Sandham, Trinity College, Dublin, Ireland*

$AXBZ$ is a jointed rhombus connected with a fixed point O by two equal rods OA, OB. $OCZD$ is a jointed rhombus and YC, YD are equal rods. (Two Peaucellier cells, as it were "cross joined.") Prove that as Y describes a circle, X describes a conic.



Solution by the Proposer. Using the well known property of Peaucellier cells*

$$OZ\cdot OX = OA^2 - ZA^2 = m^2,$$

$$YZ\cdot YO = YC^2 - ZC^2 = n^2.$$

Therefore the following relations hold:

$$OY(OY - OZ) = OY^2 - OY\cdot OZ = n^2,$$

$$OX = \frac{OY\cdot m^2}{OY^2 - n^2}\cdot$$

* See J. D. C. De Roos, Linkages: The Different Forms and Uses of Articulated Links, Van Nostrand, 1879, pp., 87ff. See also R. C. Yates, Tools, Baton Rouge 1941, p. 184, Fig. 1; and also Yates, Curves, West Point 1946, pp. 29, 52–54.

That is, in the notation of complex variables,

$$X = \frac{Ym^2}{Y\overline{Y} - n^2},$$

or

$$Y = \theta X, \qquad \overline{Y} = \theta \overline{X}, \qquad \theta^2 X\overline{X} - n^2 = \theta m^2,$$

where

$$\theta = (Y\overline{Y} - n^2)/m^2,$$

a real quantity. Hence, if Y moves upon the circle

$$\alpha Y\overline{Y} + c\overline{Y} + \bar{c}Y + \beta = 0,$$

then X traces the conic

$$(\alpha n^2 + \beta)^2 X\overline{X} + \{n^2(c\overline{X} + \bar{c}X) - m^2\beta)\}\{c\overline{X} + \bar{c}X + m^2\alpha\} = 0.$$

Note further that if Y describes a circle through O, then the point Z describes a cissoid.

Telescopic Series

4242 [1947, 168]. *Proposed by W. O. Pennell, Exeter, New Hampshire*

Determine the sums of the following infinite series:

$$(1) \quad \frac{n+2}{n+1} - \frac{2n+2}{(n+1)(2n+1)} + \frac{3n+2}{(n+1)(2n+1)(3n+1)} - \cdots,$$

$$(2) \quad \frac{3n+7}{n+1} - \frac{8n+9}{(n+1)(2n+1)} + \frac{15n+11}{(n+1)(2n+1)(3n+1)} - \frac{24n+13}{(n+1)(2n+1)(3n+1)(4n+1)} - \cdots,$$

where n is any real number except $0, -1, -1/2, -1/3$, and so on.

I. *Solution by P. A. Clement, University of California at Los Angeles.* Consider a series of the type

$$(3) \quad S = \sum_{k=1}^{\infty} (-1)^{k+1}\left[\frac{a_k}{(n+1)(2n+1)\cdots(\overline{k-1}\cdot n+1)} + \frac{b_k}{(n+1)(2n+1)\cdots(kn+1)}\right]$$

satisfying

$$(a) \qquad a_{k+1} = b_k,$$

(b) $$\lim_{k\to\infty} \frac{b_k}{\prod_{s=1}^{k}(sn+1)} = 0,$$

(c) $n \neq -1, -1/2, -1/3, \cdots$; and $n \neq 0$ when $\lim b_k \neq 0$.

It is easy to see that the series telescopes so that the sum of the first k terms is given by

$$S_k = a_1 + (-1)^{k+1} \frac{b_k}{\prod_{s=1}^{k}(sn+1)},$$

so that, by (b) and (c), $S=a_1$.

The general term of (3) by addition becomes

$$(-1)^{k+1} \frac{a_k kn + a_k + b_k}{\prod_{s=1}^{k}(sn+1)}.$$

This is easily verified to be the general term of (1) if $a_k=b_k=1$, and of (2) if $a_k=k+2$ and $b_k=k+3$, and these specifications obviously satisfy (b) above. Hence, for the series (1), $S=a_1=1$, and for (2), $S=a_1=3$.

II. *Solution by C. F. Pinzka, Student, Rutgers University.* Let

$$S_1 = 1 - \frac{1}{n+1} + \frac{1}{(n+1)(2n+1)} - \cdots.$$

By the Cauchy ratio test, S_1 is absolutely convergent for $n \neq 0, -1, -1/2, -1/3 \cdots$. Hence S_1 and

$$1 - S_1 = \frac{1}{n+1} - \frac{1}{(n+1)(2n+1)} + \cdots$$

may be added term by term. This gives the series (1) on one side and $S_1+(1-S_1)=1$ on the other.

For (2) we take the series

$$S_2 = 3 - \frac{4}{n+1} + \frac{5}{(n+1)(2n+1)} - \cdots$$

and form (2) by adding S_2 to $(3-S_2)$ term by term. Since both S_2 and $(3-S_2)$ are seen to be absolutely convergent, the sum of (2) is $S_2+(3-S_2)=3$.

Also solved by C. B. Barker, Joshua Barlaz, Daniel Block, R. A. Bradley, Paul Brock, D. H. Browne, G. Y. Cherlin, A. B. Farnell, H. E. Fettis, H. Kaufman, Karl Itkin, J. F. Locke, Lowell Schoenfeld, F. C. Smith, F. Underwood, and the Proposer.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Probit Analysis: A Statistical Treatment of the Sigmoid Response Curve. By D. J. Finney. Cambridge, at the University Press; New York, The Macmillan Company, 1947. 13+256 pages. $3.75.

Biological assay in general is the measurement of the potency of some stimulus by the reaction it produces in living material. This kind of measurement has been used for many years in assessing the potency of vitamins, hormones, toxicants and drugs of all kinds. One type of assay which has been widely used, especially in toxicological studies, is that depending on quantal, or all-or-none response. The most obvious example of this kind of response is death. Mr. Finney has written a book about the statistical treatment of quantal assay data by means of probit analysis.

The book is arranged in ten chapters and two appendices. In the first few chapters the technique of probit analysis is introduced in its simplest form. The following chapters discuss more complicated analyses and introduce the recent developments of the technique, many of which have been due to Mr. Finney. The first appendix gives a detailed computational scheme for probit analysis and the second a brief outline of the mathematical theory of the probit method. Seven tables which will lessen the computing time for a probit analysis complete the book.

The viewpoint of the individual, untrained in statistics, who has to analyze quantal assay data is maintained throughout. Although this is the viewpoint that is emphasized, enough of the theory is given at regular intervals to enable the trained statistician to follow the theoretical basis for the methods. Examples with detailed explanations and discussion are given to illustrate each new method or variation of a method. The examples are well chosen, largely from insecticide test data.

In places the explanations of the computations seem unnecessarily detailed. This, however, is probably not a serious objection, at least from the viewpoint of workers in the applied field. One finishes reading the book with the feeling that it is a worthy companion to books by other members of the Rothamsted group of statistical workers.

S. LEE CRUMP

College Algebra. By F. S. Nowlan. New York, McGraw-Hill Book Co., Inc., 1947. 14+371 pages. $3.00.

The preface of this book states, "The main purpose in preparing the present text has been the production of an algebra text mathematically sound and at the

same time easily understood. These objectives are not incompatible and the author believes that for an average student the most difficult treatment is one that is deficient in logic and incomplete or inexact in its statement of principles." The author has been quite successful in producing a text which is mathematically sound. However, he is subject to such lapses as speaking of the addition and multiplication of equations, and a statement on page 41 implies that subtraction is an associative operation. The reviewer has some doubts as to the ease with which an average student will understand the text. These doubts arise from the large number of precisely stated definitions, the multitude of theorems and corollaries with detailed proofs, and the wide variety of methods for solving some types of problems. This feature of the book, which makes it dull reading from a student's point of view, will make it extremely valuable as a reference book for a person who is new at teaching college algebra.

The subject matter is developed on the assumption that the student already understands the elementary properties of (real) positive numbers and zero and their laws of combination. This is presumably the justification for the omission of a number of topics, such as the decimal and other scales of notation, the irrationality of such numbers as $\sqrt{2}$, approximate numbers and their use in computation. The reviewer's experience leads him to believe that most students of college algebra do not have this postulated understanding.

The multitude of topics usually included in a college algebra text to provide a wide choice of subject matter appears in this book. The one notable exception is the solution of inequalities.

There are some minor points which deserve mention. Equations and their solutions are presented in a praiseworthy manner. The comparison method for solving two simultaneous linear equations seems superfluous. Graphical methods and much of the terminology of analytic geometry are used to solve equations and to clarify the explanation of the process. The examples which illustrate the various methods for solving equations regularly include a check by substitution and set a good example for the student. Although the function concept appears early, functional notation is not used until the chapter on theory of equations. A proportion is defined in terms of a proportionality constant and not as an equality of ratios. This makes possible the inclusion of zero among the terms of a proportion. The discussion of logarithms is marred by the use of a bar to indicate a negative characteristic. (A misprint on page 165 results in the equation $\log 0.256 = 1.4082$). An earlier appearance of a principle for addition would improve the discussion of permutations, combinations, and probability.

The most novel and outstanding feature of this book is the introduction of complex numbers. The symbol $[a, b]$, with a and b real numbers, is called a complex. The complexes $[a, b]$ and $[c, d]$ are equal if, and only if, $a = c$ and $b = d$. The rational operations on complexes are defined as

$$[a, b] + [c, d] = [a + c, b + d],$$
$$[a, b] - [c, d] = [a - c, b - d],$$

$$[a, b] \cdot [c, d] = [ac - bd, ad + bc],$$

$$[a, b] \div [c, d] = \left[\frac{ac + bd}{c^2 + d^2}, \frac{bc - ad}{c^2 + d^2}\right], \qquad [c, d] \neq [0, 0].$$

It is then shown that the special complexes $[x, 0]$ behave just like real numbers and these entities are identified. It is then shown that the complex $[0, 1]$ is a square root of $[-1, 0]$, so that the system of complexes provides square roots for negative numbers. Complexes are renamed complex numbers and are endowed with a rectangular form $x+iy$ and a polar form $\rho(\cos\theta+i\sin\theta)$. Although it is surprising that this procedure has not already appeared in a book of this type, it is gratifying to find it so well presented here. A more conventional approach to complex numbers is also provided.

The book contains a four place table of logarithms, interest and annuity tables, a mortality table, and a too brief table of powers and roots of integers. The problems are carefully graduated and are such as to provide a challenge for most students. There is a lack of simple exercises for rapid drill which are useful in many connections to give a student confidence.

All in all, it is a book with a broad field of usefulness and with enough novel and unusual features to be rewarding to anyone who looks into it carefully.

R. W. Wagner

Intermediate Algebra. By J. R. Britton and L. C. Snively. New York, Rinehart and Company, 1947. 9+337 pages. \$2.00.

Entitled "Intermediate," this text leads the student from the fundamental ideas of arithmetic (in a fifteen-page chapter), through fractions, exponents, linear and quadratic equations, common logarithms, as far as progressions and the binomial theorem. Tables of powers (exponents -1, 2, 1/2, 3, 1/3) and of five-place common logarithms with proportional parts constitute the appendix, which is followed by the answers to the odd-numbered problems.

The engineering viewpoint predominates. Exercises are numerous, including occasional review lists. The word problems expose an unusually wide variety of applications and constitute one of the outstanding features of the text. Graphing and the function concept receive considerable attention, more than might reasonably be expected in an intermediate text. Another pleasing characteristic is the insistence on an adequate check of the solution of a problem.

In presenting explanations and in solving illustrative examples, the authors have earnestly striven to write clearly and amply, avoiding the pitfall of excessive brevity so often a complaint of students. Only seldom do the authors relax the usual carefulness and allow a looseness of phraseology or an inconsistency of statements. The publishers have coöperated in offering a legible, attractive format and in nearly freeing the text from typographical error.

R. A. Good

NEW BOOKS RECEIVED

Table des Solutions de la Congruence $x^4+1\equiv 0$ (*mod* p) *pour* $500\ 000<p<6\ 000\ 000$. By A. Gloden. Luxembourg, 1947. 12 pages.

Liste des Formes Linéaires des Nombres dont le Carré se Termine dans le Système Décimal par une Tranche donné de 4 *Chiffres*. By A. Gloden. Luxembourg, 1947. 15 pages.

Table des Bicarrés N^4 *pour* $3001\leqq N\leqq 5000$. By A. Gloden. Luxembourg, 1947. 17 pages.

Tables of the Bessel Functions of the First Kind of Orders Thirteen, Fourteen, and Fifteen. Prepared by the Staff of the Computation Laboratory of Harvard University. Cambridge, Harvard University Press, 1947. $10.00.

Tables of the Bessel Functions of the First Kind of Orders Ten, Eleven, and Twelve. Prepared by the Staff of the Computation Laboratory of Harvard University. Cambridge, Harvard University Press, 1947. $10.00.

Nomography. By A. S. Levens. New York, John Wiley and Sons, Inc., 1948. 8+176 pages. $3.00.

Problème Général de la Stabilité du Mouvement. By A. Liapounoff. Translated from the Russian by E. Davaux. (Annals of Mathematics Studies, No. 17). Princeton University Press, 1947. 447 pages. $3.50.

Toward General Education. By E. J. McGrath and Others. New York, The Macmillan Company, 1948. 7+224 pages. $3.00.

Analytic Geometry. by D. S. Nathan and Olaf Helmer. New York, Prentice-Hall, Inc., 1947. 10+402 pages. $3.50.

College Algebra. By H. A. Simmons. New York, The Macmillan Company, 1948. 8+619 pages. $4.00.

Modern Mathematics for Everybody. By Duane Studley. Published by the Author, 1947. 84 pages.

The Atom. Third Edition. By G. Thomson. Oxford University Press, 1948. 196 pages. $2.00.

Fundamentals of Business Mathematics. By W. R. Van Voorhis and C. W. Topp. New York, Prentice-Hall, Inc., 1948. 8+454 pages $3.75.

A Handbook on Curves and their Properties. By R. C. Yates. Ann Arbor, Edwards, 1947. 10+245 pages.

ANSWER TO THE SECOND TEASER, PAGE 342

My answer is that from the data no conclusion can be reached, as to when the knife reaches the far edge of the table. Let s ft. be the distance of the knife from the edge where it started after t seconds. The value of the dependent variable t is given for a discrete set of values of the variable s. The value $s=4$ is not included in this discrete set, and hence we do not know what t is when $s=4$. If we add further hypotheses, to the effect that t is a monotonic increasing function of s on the range $0\leqq s\leqq 4$, and is continuous at $s=4$, we might conclude that $t=2$ when $s=4$. Otherwise who knows whether t makes finite jumps!

E. J. MOULTON

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

ANNOUNCEMENT OF NAVY DEPARTMENT JOINT BOARD

The Navy Department Joint Board of U. S. Civil Service Examiners has announced that it is expanding three comparatively new, permanent laboratories in California. The program of the Naval Ordnance Test Station, China Lake, California, involves research, development and test work with ordnance equipment and explosives. The Navy Electronics Laboratory, San Diego, California, is concerned with research, testing and development of electronic control devices, detection equipment, instrumentation equipment and training aids. The activities of the Naval Air Missile Test Center, Point Mugu, California pertain to flight and laboratory testing and evaluation of guided missiles and their components.

The Board wishes to recruit qualified personnel for these laboratories. Examinations are now open in the following fields: Chemist, Mathematician, Metallurgist, Meteorologist, Physicist, Statistician, Scientific Research Administrator and Scientific Staff Assistant. Salaries range from $3397 to $9975 per annum.

Further information may be obtained from the Navy Department Joint Board of U. S. Civil Service Examiners, 1030 East Green Street, Pasadena 1, California.

THE CANADIAN MATHEMATICAL CONGRESS

The Canadian Mathematical Congress, which was organized in 1945, will hold a second national congress and seminar in 1949 at the University of British Columbia.

Any full time member of the mathematical teaching staff of a Canadian university, or any person who was a member of the Congress of 1945, or of the seminar of 1947, and any other person sponsored by a member of the Congress may become a member of the Congress on application and payment of the membership fee and acceptance by the Executive Committee. The fee for membership is $2.00 per annum.

Application for membership should be sent to: Secretary, Canadian Mathematical Congress, Engineering Building, McGill University, Montreal, Canada.

CANADIAN JOURNAL OF MATHEMATICS

The Canadian Mathematical Congress has announced the publication of a new journal, *Canadian Journal of Mathematics*, beginning January, 1949.

Manuscripts for publication in the *Journal* should be sent to the Editor-in-Chief, H. S. M. Coxeter, University of Toronto. Every paper should contain an introduction summarizing the results as far as possible in such a way as to be

understood by the non-expert.

The *Journal* will be published quarterly. The price per volume of four numbers is $6.00. This is reduced to $3.00 for members of the Mathematical Association of America. Subscriptions should be sent to: Managing Editor, G. de B. Robinson, University of Toronto, Toronto, Canada.

SUMMER COURSES

The following institutions announce advanced courses in mathematics for the summer of 1948:

Syracuse University. June 21 to July 31: vector analysis, differential equations, mathematics of statistics, probability, seminar, topics in partial differential equations, functions of a complex variable, mathematical methods of fluid dynamics. August 2 to September 4: introduction to modern algebra, advanced calculus, mathematics of statistics, advanced mathematics for engineers and physicists, probability, seminar, advanced functions of a complex variable, differential geometry.

University of California. Session II. August 2 to September 11: Professor Zariski, introduction to the arithmetic theory of algebraic varieties.

West Virginia University. Second term: Professor Stewart, modern synthetic geometry; Professor Vehse, theory of determinants and analytic geometry of space, calculus of variations; Professor Peters, theory of numbers; Professor Davis, Cremona transformations.

MATHEMATICS INSTITUTE FOR TEACHERS

A Mathematics Institute for Teachers will be held at Duke University on August 9–20, 1948. The general theme of the Institute is Mathematics at Work. The program includes lectures pertaining to the applications of mathematics given by representatives of various industries. Study groups will be organized to consider topics of interest to teachers of secondary-school and college mathematics. Programs may be obtained from Professor W. W. Rankin, Duke University, Durham, North Carolina.

PERSONAL ITEMS

The officers of Section V_1, Philosophy of Mathematics, of the Tenth International Congress of Philosophy are as follows: President, L. E. J. Brouwer; Vice-President, A. Heyting. Professor M. H. Stone and Professor H. B. Curry will be speakers. The Congress will meet in Amsterdam, Holland on August 11–18, 1948.

Assistant Professor Richard Bellman of Princeton University has been appointed to an associate professorship at Stanford University, effective September, 1948.

Associate Professor W. M. Borgman, Jr. of Wayne University has been appointed Assistant Dean of Administration.

Professor T. G. Cowling of the University College of North Wales has been appointed to a professorship at the University of Leeds.

Dr. H. D. Huskey, who was engaged in research at the British National Physical Laboratories during the year 1947, is now Chief of the Machine Development Laboratory of the National Bureau of Standards.

Dr. G. R. MacLane of Harvard University has been appointed to an assistant professorship at Rice Institute.

Professor G. M. Magee of the University of Western Ontario has accepted a position as visiting professor at Wayne University for the summer session.

Dr. Rufus Oldenburger, formerly professor of mathematics at Illinois Institute of Technology, has been appointed Chairman of the Mathematics Department of De Paul University.

Dr. C. Y. Panc of the University of Marseille has been appointed lecturer at the University of Cape Town.

Professor Joseph Pierce of Atlanta University has been appointed to an associate professorship at Wayne University.

Professor Joseph Polley of Wabash College will be a visiting professor at Wayne University for the summer session.

Dr. R. D. Schafer of the Institute for Advanced Study has been appointed to an assistant professorship at the University of Pennsylvania.

Dr. H. W. E. Schwerdtfeger of the University of Adelaide has been appointed to the position of senior lecturer at the University of Melbourne.

Professor Jabir Shibli, formerly professor of mathematics at Pennsylvania State College, has been appointed to a professorship at Presbyterian College, Clinton, South Carolina.

Associate Professor Max Shiffman of New York University has been appointed to a professorship at Stanford University, effective September, 1948.

Dr. C. V. L. Smith is now Head of the Computing Machine Section of the Mathematics Branch of the Office of Naval Research.

Miss Mary V. Sunseri of Stanford University has been promoted to an acting assistant professorship, effective September, 1948.

Assistant Professor P. M. Whitman of Tufts College has accepted a position at the Applied Physics Laboratory, Johns Hopkins University, Silver Spring, Maryland.

Professor G. F. Woodson, Jr. is now Head of the Department of Mathematics of the College of Education and Industrial Arts, Wilberforce, Ohio.

Wayne University announces the appointment of Mr. Jack Patterson as instructor.

Associate Professor Emeritus James K. Whittemore of Yale University died March 22, 1948 at the age of seventy-three years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following eighty-eight persons have been elected to membership by the Board of Governors on applications duly certified:

J. J. BARRON, Ph.D. (Wisconsin) Professor, Marshall College, Huntington, W. Va.

BARBARA B. BETTS, A.M. (Radcliffe) Editor, Mathematics Department, D. C. Heath and Company, Boston, Mass.

LOLLIE BELLE BIENVENU, M.S. (Louisiana State) Assistant, Louisiana State University, Baton Rouge, La.

L. B. BOND, B.S. (University of Washington) Instructor, Seattle College, Seattle, Wash.

R. L. BROOKS, B.S. (West Virginia Wesleyan) Graduate Student, University of North Carolina, Durham, N. C.

K. E. BROWN, Ph.D. (Columbia) Head of Department, Wagner College, Staten Island, N. Y.

J. W. BUTLER, Associate Engineer, Clinton National Laboratories, Oak Ridge, Tenn.

THOMAS CHERBAS, Student, Drexel Institute Evening School, Philadelphia, Pa.

G. Y. CHERLIN, B.S. (Rutgers) Asst. Instructor, Rutgers University, New Brunswick, N. J.

B. A. CHIAPPINELLI, Student, University of California at Los Angeles, Calif.

THERESA M. CHIAVERINI, B.A. (Michigan State) Instructor, University of Detroit, Mich.

H. B. COLEMAN, M.S. (Michigan) Assistant, University of Michigan, Ann Arbor, Mich.

A. E. COOK, A.B. (Georgetown College) Asst. Professor, Georgetown College, Georgetown, Ky.

P. R. CULWELL, M.A. (Texas) Asst. Professor, Trinity University, San Antonio, Texas

W. W. DOLAN, Ph.D. (Oklahoma) Asst. Professor, University of Oklahoma, Norman, Okla.

E. B. EILERTSEN, M.A. (California) Instructor, City College of San Francisco, Calif.

F. N. FISCH, M.A. (Colorado State) Asst. Professor, Colorado State College, Greeley, Colo.

W. T. FISHBACK, A.M. (Harvard) Teaching Fellow, Harvard University, Cambridge, Mass.

T. A. GAFFNEY, M.S. (California Institute of Technology) City College of San Francisco, Calif.

E. I. Gale, A.M. (Columbia) Asst. Professor, University of New Brunswick, Fredericton, N. B.

M. J. GERDES, President, Wayne Metal Finishing Company, Brooklyn, N. Y.

B. K. GOLD, JR., M.A. (U.C.L.A.) Instructor, Los Angeles City College, Calif.

DAVID GORDON, M.A. (Columbia) Chairman of Department, Walton High School, Bronx, N. Y.

R. P. GRAHAM, Electrical Engineer, Evanston, Ill.

J. S. GRIFFIN, JR., Student, Alabama Polytechnic Institute, Auburn, Ala.

A. G. HANSEN, M.S. (Purdue) Instructor, Purdue University, West Lafayette, Ind.

C. C. C. HARDING, B.A. (Columbia) Asst. Manager, General Section, Treasurer Division, E. I. DuPont DeNemours and Co., Wilmington, Del.

M. B. HASLAM, Student, University of Buffalo, Buffalo, N. Y.

A. H. HILL, M.S. (Wisconsin) Professor, North Dakota Agricultural College, Fargo, N. D.

A. T. HIND, JR., M.A. (Emory) Instructor, Clemson College, Clemson, S. C.

J. M. HOWELL, M.A. (U.C.L.A.) Instructor, Los Angeles City College, Calif.

KARL ITKIN, B.S. (Cooper Union) Electrical Engineer, Federal Power Commission, Washington, D. C.

S. B. JACKSON, Ph.D. (Harvard) Asso. Professor, University of Maryland, College Park, Md.

VIVIAN V. JOHNSTON, B.S. (Geneva) Instructor, Tusculum College, Greeneville, Tenn.

M. S. KLAMKIN, M.S. (Brooklyn Polytechnic Institute) Teaching Fellow, Carnegie Institute of Technology, Pittsburgh, Pa.

REV. BONAVENTURE KNAEBEL, M.S. (Catholic University) Instructor, St. Meinrad's College, St. Meinrad, Ind.

J. C. KNIPP, Ph.D. (Pittsburgh) Asso. Professor, University of Pittsburgh, Pittsburgh, Pa.

F. T. KOCHER, JR., B.S. (Bloomsburg State) Instructor, Pennsylvania State College Undergraduate Center, DuBois, Pa.

C. E. LANGENHOP, M.S. (Iowa State) Instructor, Iowa State College, Ames, Iowa

J. C. LANZ, Sc.M. (Brown) Teacher, Norristown School Board, Norristown, Pa.

C. H. LINDAHL, M.S. (Colorado) Asst. Professor, Iowa State College, Ames, Iowa

MARTIN MALTENFORT, A.M. (Montclair State) Instructor, Manhattan College, New York, N. Y.

R. W. MARSH, A.B. (American University) Graduate Student, George Washington University, Washington, D. C.

C. F. MARTIN, B.S. (U. S. Naval Academy) Instructor, University of South Carolina, Columbia, S. C.

W. J. MCCALLION, M.A. (McMaster) Lecturer, McMaster University, Hamilton, Ont.

L. H. MILLER, Ph.D. (Ohio State) Asst. Professor, Ohio State University, Columbus, Ohio

W. G. MILLER, M.A. (Florida) Asst. Professor, Clemson College, Clemson, S. C.

ABIGAIL M. MOSEY, M.A. (Syracuse) Instructor, Hobart College, Geneva, N. Y.

B. H. MOUNT, JR., M.S. (Princeton) Asst. Professor, University of Pittsburgh, Pittsburgh, Pa.

REV. J. P. MURRAY, S.J., M.A. (Boston College) Instructor, Fairfield University, Fairfield, Conn.

EUGENE ODIN, M.E. (Cornell) Senior Project Engineer, Arma Corporation, Brooklyn, N. Y.

E. F. ORMSBY, M.S. (Syracuse) Instructor, Union College, Schenectady, N. Y.

REV. E. F. O'SHEA, S.J., A.B. (Woodstock College) Instructor, University of Scranton, Scranton, Pa.

T. G. OSTROM, Ph.D. (Minnesota) Asst. Professor, Montana State University, Missoula, Mont.

W. D. PEEPLES, JR., B.S. (Howard) Graduate Asst., University of Wisconsin, Madison, Wis.

LILLIAN G. PERKINS, B.S. (South Carolina) Instructor, University of South Carolina, Columbia, S. C.

O. L. PHILLIPS, M.A. (North Texas State) Instructor, Louisiana State University, Baton Rouge, La.

C. F. PINZKA, Student, Rutgers University, New Brunswick, N. J.

J. T. PITTS, Student, Furman University, Greenville, S. C.

GERTRUDE V. PRATT, A.M. (Michigan) Central Michigan College of Education, Mt. Pleasant, Mich.

J. J. QUINN, M.S. (New York) Instructor, Bayonne Junior College, Bayonne, N. J.

ALICE B. RABON, M.Ed. (South Carolina) Instructor, University of South Carolina, Columbia, S. C.

G. P. RIGSBY, Student, California Institute of Technology, Pasadena, Calif.

R. M. ROBINSON, M.S. (Drake) Instructor, Iowa State College, Ames, Iowa

LOUIS SACKS, M.S. (Carnegie) Instructor, Carnegie Institute of Technology, Pittsburgh, Pa.

F. L. SANDER, Student, Michigan State College, East Lansing, Mich.

L. E. SCHAEFER, M.A. (Michigan State) Instructor, General Motors Institute, Flint, Mich.

N. C. SCHOLOMITI, B.S. (New York) Instructor, De Paul University, Chicago, Ill.

R. C. SEBER, B.A. (Coe) Graduate Asst., State University of Iowa., Iowa City, Iowa

E. B. SHANKS, Ph.D. (Illinois) Asst. Professor, Vanderbilt University, Nashville, Tenn.

B. I. SHOESMITH, B.S. (Chicago) Instructor, Illinois Institute of Technology, Chicago, Ill.

SISTER MARIE LORETTA, O.P., M.A. (Catholic University) Instructor, Siena Heights College, Adrian, Mich.

A. H. SMITH, Ph.D. (Brown) Asso. Professor, Purdue University, West Lafayette, Ind.

R. E. SMITH, M.A. (Pittsburgh) Asst. Professor, Duquesne University, Pittsburgh, Pa.

L. A. SNIDER, Student, George Washington University, Washington, D. C.

R. L. SNIDER, B.S. (Missouri) Instructor, Kemper Military School, Boonville, Mo.

E. V. SOMERS, M.S. (Pittsburgh) Research Engineer, Westinghouse Electric Corporation, East Pittsburgh, Pa.

ISAAY STEMPNITZKY, S.M. (Massachusetts) Research Assistant, Massachusetts Institute of Technology, Cambridge, Mass.

RUTH K. SWANSON, M.S. (Oklahoma) Instructor, Friends University, Wichita, Kan.

EDWIN TABOR, A.B. (California) 2319½ Haste St., Berkeley, Calif.

KAIDY TAN, Instructor, Chip-Bee Institute, Amoy, China

FLORENCE G. TETREAULT, B.S. (Detroit) Instructor, University of Detroit, Detroit Mich.

S. L. THOMPSON, M.A. (Michigan) Asst. Professor, Alabama Polytechnic Institute, Auburn, Ala.

LONA L. TURNER, M.A. (Michigan) Student, University of Chicago, Chicago, Ill.

BERNICE L. WARR, Student, Skidmore College, Saratoga Springs, N. Y.

L. B. WILLIAMS, S.M. (Chicago) Asst. Professor, Reed College, Portland, Ore.

R. W. YOUNG (Indiana), Instructor, Lehigh University, Bethlehem, Pa.

G. C. ZADER, B.S. (Davis-Elkins) Instructor, The Citadel, Charleston, S. C.

NOVEMBER MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The twentieth meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at Carnegie Institute of Technology, Pittsburgh, Pennsylvania, on Saturday, November 22, 1947. Professor J. B. Rosenbach presided at the morning and afternoon sessions.

The attendance was one hundred sixty-two, including the following thirty-nine members of the Association: O. F. H. Bert, J. O. Blumberg, R. C. Briant, A. M. Bryson, Helen Calkins, W. E. Cleland, H. L. Dorwart, Benjamin Epstein, F. A. Foraker, Tomlinson Fort, E. T. Frankel, Beatrice L. Hagen, Helen Harmon, B. P. Hoover, H. L. Krall, Elizabeth L. Lahti, H. R. Leifer, C. G. Maple, A. W. McGaughey, E. W. Montroll, David Moskovitz, L. T. Moston, C. E. Mullan, J. H. Neelley, E. G. Olds, Morris Ostrofsky, F. W. Owens, Helen B. Owens, J. B. Rosenbach, Edward Saibel, H. C. Shaub, R. E. Smith, F. H. Steen, Helen F. Story, J. L. Synge, J. S. Taylor, Margaret O. Taylor, C. H. Vehse, E. A. Whitman.

The officers elected at the business meeting were as follows: Chairman, J. B. Rosenbach, Carnegie Institute of Technology; Secretary-Treasurer, E. W. Montroll, University of Pittsubrgh; additional members of the Executive Committee, F. H. Steen, Allegheny College, H. L. Krall, Pennsylvania State College. The next meeting will be held on May 8, 1948, at Pennsylvania State College, State College, Pennsylvania.

The program consisted of the following papers:

1. *The United States Army experiment in higher education*, by Professor J. H. Neelley, Carnegie Institute of Technology.

Professor Neelley described the operation and achievements of Biarritz American University on whose mathematics staff he served.

2. *Averages over functions of charactersitic values of linear operators*, by Pro-

fessor E. W. Montroll, University of Pittsburgh.

Crystalline solids and polyatomic molecules in the gas phase can be represented by a set of coupled harmonic oscillators. The thermodynamic properties of such substances are determined theoretically by averaging functions of the frequencies of the normal modes of vibration of the equivalent oscillators over all the normal modes. Because of the great number of degrees of freedom of a solid, the exact calculation of frequencies of normal modes is impossible. It was shown that without too much difficulty one can obtain the traces of powers of the matrices of the secular determinants. These traces in turn give the moments of the distribution function of the normal modes. From these moments one can estimate the thermodynamic averages by mechanical quadratures. The upper bound to the error resulting from the use of a specified number of moments is given by an expression due to Markoff. Analogous procedures can be used to determine averages over characteristic values of linear operators other than matrices.

3. *Boundary problems for difference equations*, by Professor Tomlinson Fort, University of Georgia.

Professor Fort spoke on boundary problems for the linear difference equation with integral independent variable. The topic was introduced and motivated by a discussion of a weighted string vibrating in a plane with small vibrations. Known results for the equation of the second order were summarized, and brief reference to the equation of the nth order was made.

4. *Open convex surfaces which are rigid*, by Professor J. J. Stoker, New York University, introduced by the Secretary.

A two-dimensional surface in three-dimensional space is said to be rigid if it possesses as infinitesimal length-preserving deformations (with continuous third derivates) only rigid body motions. A surface is not rigid in the small in general, but it has been shown by Liebmann, Blaschke, and Weyl that a closed convex surface is, in its whole extent, rigid. It was shown that certain open convex surfaces are rigid, provided only that the infinitesimal deformations considered are bounded —a condition which is automatically fulfilled in the case of closed surfaces. The open surfaces considered belong to the class of complete surfaces, that is, surfaces on which every Cauchy sequence converges (in the sense of the Riemannian metric on the surface) to a point on the surface. Complete open convex surfaces in three-dimensional space can always be represented in the form $z=z(x, y)$ in suitably chosen Cartesian coördinates. It was shown that the following two classes of surfaces of this kind are rigid: (1) those in which $z(x, y)$ is defined over the entire xy-plane; and (2) all surfaces of revolution.

5. *Some recent trends in differential geometry*, by Professor Enrico Bompiani, University of Rome, introduced by the Secretary.

The theory of point transformations between two projective planes (or spaces) has recently attracted the attention of E. Kasner and J. DeCicco. A survey was given of some results obtained in Italy during the war by Dr. Villa and the speaker concerning the approximation of those transformations by birational or Cremona transformations, or, when that is not possible, by unirational transformations. Interest in the topic was accentuated by the development of some unexpected facts.

6. *Newtonian polynomials*, by Professor H. L. Krall, Pennsylvania State College.

Starting from Vandermonde's equation $(a+b)^{(n)}=\sum_{i=0}^{n}\binom{n}{i}a^{(i)}b^{(n-i)}$ various other formulas for Newtonian or factorial polynomials were developed. Some of these were used to obtain results concerning factorial moments.

7. *A mathematical approach to certain aspects of the breakage of solids*, by Dr. Benjamin Epstein, Carnegie Institute of Technology.

A mathematical model of certain breakage processes was constructed. The following assumptions were made: (a) any breakage process is composed of discrete steps; (b) there is a function $P_n(y)$, the probability of breakage of particles of size y in the nth step of the breakage process; and (c) there is a function $F(x, y)$, the distribution function of pieces of size $x \leqq y$, arising from the breakage of a unit mass of size y.

If, in particular, $P_n(y)$ is a constant p_n, independent of y (but possibly dependent on n), and $F(x, y)$ is such that the percentage ($<Ky$, $0 \leqq K \leqq 1$) of material arising from the breakage of a unit mass of size y is independent of y, then the distribution function $F_n(x)$ after n steps in the process is the distribution function of a product of $(n+1)$ independent random variables. Under minor restrictions this means that the size distributions are asymptotically logarithmico normal. This result may explain partially why fine grinding and crushing operations are often observed to yield logarithmico-normal distribution. Generalizations of the problem under other assumptions concerning $P_n(y)$ and $F(x, y)$ were briefly considered. Most of the results are contained in a paper in the December 1947 issue of the *Journal of the Franklin Institute.*

8. *An undergraduate program in mathematics*, by Professor Morris Ostrofsky, Duquesne University.

The undergraduate program in mathematics at Duquesne University serves the dual purpose of preparing some students for graduate work in mathematics, and training others who are in the fields of natural sciences. The first two years are identical for both groups. In the first semester of the freshman year a five hour course is offered. This course is intended to prepare students for elementary calculus and is designed to teach methods rather than mere operational techniques. It begins by developing the number system from positive integers, and includes the discussion of complex numbers, logarithms, algebraic equations, plane trigonometry, and some analytical geometry. Calculus begins the second semester of the freshman year. Integral and differential calculus are studied simultaneously. More analytical geometry is studied along with the calculus. During the first semester of the second year, calculus is continued. The second semester of the sophomore year is devoted to a course in elementary differential equations. Students majoring in mathematics, in their third year, are required to take a course in advanced calculus designed to serve as a good introduction to function theory. To round out their program they can take courses in analytical mechanics, algebra, integral equations and, possibly, topology. Science majors are offered additional courses in ordinary and partial differential equations, or a course at the level of advanced calculus but with emphasis on applications rather than rigor.

H. L. DORWART, *Secretary*

FALL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The fall meeting of Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the University of Maryland, College Park, Maryland, on December 6, 1947, with Dean E. J. Finan, Chairman of the Section, presiding at the morning and afternoon sessions.

Seventy persons attended the meeting, including the following thirty-eight members of the Association: N. H. Ball, W. Bleick, H. Campaigne, Randolph Church, C. R. Clark, A. Cohen, J. A. Duerksen, P. J. Federico, Anselm Fisher, E. E. Floyd, M. K. Fort, Jr., B. C. Getshell, Michael Goldberg, R. A. Good, Thomas Greville, G. A. Hedlund, M. A. Hyman, J. E. Ikenberry, Walter Jennings, Sidney Kaplan, A. E. Landry, Carol V. McCamman, E. J. McShane,

M. E. Meade, A. K. Mitchell, W. K. Morrill, T. W. Moore, A. B. Mewborn, J. M. Popow, O. J. Ramler, R. W. Rector, J. N. Rice, A. Tierney, C. C. Torrance, Eunice D. Schell, W. R. Utz, A. L. Whiteman, G. T. Whyburn.

The following papers were presented at the morning session:

1. *Homotheticity—an effective tool in proving certain types of theorems in geometry*, by Professor O. J. Ramler, Catholic University.

After defining homothetic rectilinear figures, the speaker proved the fundamental theorem on homothetic figures. This theorem states that lines joining corresponding points of homothetic figures are concurrent. It was used to prove that the centroid, orthocenter, circumcenter, and the center of the nine-point circle of a triangle are collinear points.

2. *Rotors in polygons*, by Dr. M. Goldberg, Bureau of Ordnance, Navy Department, Washington, D. C.

A rotor of a polygon was defined to be a closed convex curve which remains tangent to all sides of the fixed polygon during a complete rotation of the curve. Constructions for rotors were given, and several working models of them were exhibited.

3. *A trigonometric identity*, by Dr. A. L. Whiteman, Navy Department, Washington, D. C.

If $p=4n+3$ is prime, it is known that the set of integers $1, 2, 3, \cdots, (p-1)/2$ contains more quadratic residues of p than non-residues. The only known derivations of this result are transcendental. The author establishes a trigonometric identity which leads to the following two equivalent theorems. If $p=4n+3$ is prime, then $\sum_{m=1}^{p-1} \cot (\pi m^2/p)>0$ and $\sum_{m=1}^{(p-1)/2} [m^2/p] >(p-1)(p-5)/24$.

4. *The sound of a projectile moving at supersonic speed*, by Professors S. B. Jackson (introduced by the Secretary), and M. H. Martin, University of Maryland.

A projectile moving at supersonic velocity v in a plane sets up a sound wave whose wave front is the envelope of a family of circles with centers at the points of the trajectory. In the case of a projectile acted on by gravity and by a retardation $f(v)$, it was shown that the wave front turns its concave side toward the trajectory so that an observer in the plane of the motion hears the sound from a uniquely determined apparent source, provided that $f(v)$ is continuous, that $f(v)/v$ increases with v, and that the limiting velocity is less than the speed of sound.

5. *The first year course in college mathematics*, by Professor E. J. McShane, of the University of Virginia.

This address was given by invitation. The speaker gave an interesting account of the course in Freshman Mathematics required of Arts and Sciences students at the University of Virginia. The classes meet five times a week for two semesters. The first semester is devoted to algebra exclusively, beginning with elementary logic, and the derivation of the ordinary laws of algebra from a set of postulates. Emphasis is placed on the understanding and appreciation of a mathematical argument rather than the cultivation of mathematical technique and familiarity with mathematical formulae. The course becomes more conventional as the semester progresses; and by the end of the semester, the usual topics in algebra have been covered. The second semester is devoted to trigonometry and analytic geometry with a minimum of time spent on solution of triangles. The reduction of the conic sections to standard forms is omitted.

M. H. MARTIN, *Secretary*

NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The annual meeting of the Philadelphia Section of the Mathematical Association of America was held at Bryn Mawr College, Bryn Mawr, Pennsylvania, on Saturday, November 29, 1947. Professor C. J. Rees, Chairman of the Section, presided at the morning and afternoon sessions.

There were sixty present, including the following forty-one members of the Association: P. T. Bateman, T. A. Botts, H. W. Brinkmann, W. B. Campbell, P. A. Caris, V. F. Cowling, J. E. Davis, F. L. Dennis, Arnold Dresden, N. J. Fine, C. D. Firestone, J. R. Kline, V. V. Latshaw, Marguerite Lehr, M. Le Leiko, F. L. Manning, Clifford Marburger, E. A. McDougle, S. S. McNeary, A. E. Meder, Jr., Martin Moliver, Lillian Moore, F. D. Murnaghan, W. R. Murray, C. A. Nelson, J. C. Oxtoby, A. E. Pitcher, A. O. Qualley, Edward Rayher, G. E. Raynor, C. J. Rees, I. J. Schoenberg, L. L. Smail, A. W. Tucker, R. M. Walter, J. B. Walton, G. C. Webber, Anna Pell Wheeler, S. S. Wilks, H. M. Zerbe.

At the business meeting the following officers were elected for the coming year: Chairman, E. P. Starke, Rutgers University; Secretary, T. A. Botts, University of Delaware. The Program Committee for the next meeting will be: W. H. Gottschalk (Chairman), University of Pennsylvania, J. C. Oxtoby, Bryn Mawr College, and L. L. Smail, Lehigh University.

The program consisted of the following papers:

1. *On Walsh functions*, by Professor N. J. Fine, University of Pennsylvania.

The Walsh functions $\{\psi_n(x)\}$ are introduced as the completion of the Rademacher system. It is shown that the $\{\psi_n(x)\}$ may be considered the full group of characters of a certain commutative, compact, totally disconnected topological group G, just as the exponentials $\{e^{2\pi nxi}\}$ are the full character group of the reals modulo 1. The image of the group operation of G in the reals, $T_a(x)\equiv x\dotplus a$, is introduced; $T_a(x)$, for fixed a, is a measure-preserving transformation of the unit interval into itself, so that we have an invariant integral $\int_0^1 f(x\dotplus a)dx=\int_0^1 f(x)dx$. After a discussion of the Lebesgue constants of the system $\{\Psi_n(x)\}$, the following uniqueness theorem is stated: If two Walsh series (not necessarily Fourier series) converge to the same function except perhaps on a countable set, then the series are identical.

2. *Convergence criteria for continued fractions*, by Professor V. F. Cowling, Lehigh University.

Recent years have seen a greatly renewed interest in the study of convergence criteria for continued fractions. The investigations of H. S. Wall, W. T. Scott and J. F. Paydon represent one approach. Another more geometric method was introduced by W. Leighton and W. J. Thron. This latter method has been considerably extended by Thron. For an account of this approach see the joint paper by Cowling, Leighton, and Thron (*Bull. Amer. Math. Soc.* vol. 50, 1944, pp. 351–357).

3. *Vector methods in the teaching of trigonometry and analytic geometry*, by Professor F. D. Murnaghan, Johns Hopkins University.

In this paper Professor Murnaghan advocates the use of vector methods in the teaching of trigonometry and analytic geometry. The idea of direction is basic in trigonometry, and the ideas of distance and direction are basic in geometry. These concepts are most conveniently specified by

the coördinates of a vector or, equivalently, by the direction numbers of a line segment. The central formula of trigonometry, namely, the formula for the cosine of the difference of two angles, follows immediately from the expression for the magnitude of the vector from one point on the unit circle to another. The idea of the positive and negative sides of a directed line or of an oriented plane is almost intuitive when the student is taught the significance of the coefficient vector of a linear equation in two or three variables. The concept of the alternating product of two plane vectors (or of three space vectors) leads naturally to the idea of the signed area of a parallelogram (or volume of a parallelepiped) and explains to the beginner the fundamental importance of determinants in analytic geometry. A treatment of analytic geometry from the point of view of vectors may be found in the speaker's book *Analytic Geometry* (Prentice-Hall, 1946).

4. *A few concepts in modern statistical inference*, by Professor S. S. Wilks, Princeton University.

The purpose here was to discuss some of the concepts of modern statistical inference without going into mathematical details. The concepts considered included random sampling, representative sampling, statistical control, acceptance sampling, confidence limits, tolerance limits, and statistical tests of significance. They were presented in conjunction with material on slides.

J. C. OXTOBY, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirtieth Summer Meeting, Madison, Wisconsin, September 6–7, 1948.

Thirty-second Annual Meeting, Columbus, Ohio, December 31, 1948.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, San Francisco, January 29, 1949
OHIO
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA, Philadelphia, November 27, 1948
ROCKY MOUNTAIN
SOUTHEASTERN, University of Alabama, University, March 18–19, 1949
SOUTHERN CALIFORNIA, John Muir Jr. College, Pasadena, March 12, 1949
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, University of Buffalo, Buffalo, May, 1949
WISCONSIN

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CIRCULAR-ARC ROTORS IN REGULAR POLYGONS

MICHAEL GOLDBERG, Bureau of Ordnance, U. S. Navy

1. Introduction. A *rotor* for a polygon is defined here as a convex closed curve which remains tangent to all the sides of the fixed polygon during a complete rotation of the curve. Rotors are a natural generalization of the curves of constant width (also called curves of constant breadth, courbes orbiformes, Gleichdicke) which can rotate between a pair of parallel fixed lines. Since a curve of constant width which is constrained by only one pair of fixed lines is free to slide along these lines, another pair of parallel fixed lines may be added as constraints. Then the curve can rotate in a rhombus which, as a special case, may be a square.

Kakeya, in an oral communication to Fujiwara, showed a method of constructing rotors of regular polygons. Fujiwara investigated rotors more thoroughly, gave a Fourier series for describing them, derived many of their properties and described several examples of them in closed forms [1]. Among these properties he showed that the sides of the polygon must be the sides of a *regular* polygon except for the quadrilateral in which case any rhombus could serve. Hayashi derived a simpler formulation for a class of rotors for all the regular polygons [2].

A circle is a trivial and obvious case of a rotor, although universal, and it will be excluded here from further consideration as a rotor. However, rotors bounded entirely by arcs of circles are of particular interest. In addition to the intrinsic interest of the geometer for these simple rotors, there is the concern of the technician who may have occasion to employ them. The circular-arc rotors are easier to depict and to manufacture. Circular-arc rotors for the square were considered by Euler in 1778 as part of his investigations of curves of constant width. These are described extensively in a paper by Schilling [4]. A circular-arc rotor for the triangle was described in 1875 by the great kinematician, F. Reuleaux, in his *Theoretische Kinematik.* Fujiwara added an infinite series of regular rotors composed of equal arcs of radius equal to the height of the triangle as well as several irregular rotors of five unequal arcs of the same radius [1 and 3]. A circular-arc rotor for the pentagon is described by Fujiwara [3, pp. 245–246]. It seems that for only the three foregoing polygons ($n=3$, 4, 5) have any circular-arc rotors been described in the previous literature.

It is the purpose of this paper to show the construction of circular-arc rotors for all the regular polygons. The construction employs only the most elementary geometry and it serves, therefore, as a simpler introduction to the subject of rotors than any of the previous ones. Other possibilities for these circular-arc rotors are discussed in the last section of the paper.

2. The envelopes of the sides of a moving polygon. Consider the envelopes of the sides of a polygon which moves so that two of the sides, AB and AC, pass through two fixed points P and Q as shown in Figure 1. Through A draw AF_2

parallel to BD and meeting in F_2 the fixed circle which is the locus of A. The angle F_2AP is supplementary to the angle ABD and, therefore, the arc F_2P is constant and F_2 is a fixed point. Also, the perpendicular F_2G_2 from F_2 on BD is equal to the altitude of the triangle ABD. Therefore, BD always touches a circle the center of which is at F_2. This demonstration of a theorem due to Bobillier is given by Besant [5].

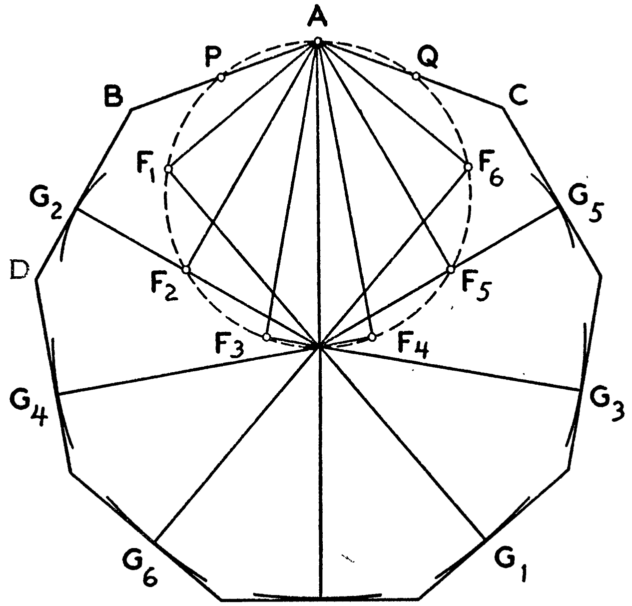

FIG. 1

Similarly, any side of the polygon taken with the sides AB and AC serves as the third side of a moving triangle. The line from A parallel to this third side cuts the circle in a point which is the center of a circular envelope of this moving third side. In general, the centers of the circular envelopes of the moving sides of the polygon are the intersections of the rays from A parallel to the sides of the polygon with the fixed circle. If the polygon is a regular polygon with an odd number of sides, the centers divide the fixed circle into the same number of equal arcs. If the polygon has an even number of sides, then the centers divide the fixed circle into half the number of equal arcs.

3. Location of the arc centers of a circular-arc rotor. Hayashi [2] exhibited a class of rotors which, for the even polygons, have an axis of symmetry, while the rotors for the odd polygons have two axes of symmetry. In the case of a circular-arc rotor, if each radius is reduced by the magnitude of the smallest radius, a rotor is obtained which has one or more sharp corners. When n is even, the rotor is of constant width. If the sharp corner is on the axis of symmetry, then this corner must be the center of the opposite arc which crosses the axis of symmetry. The centers of the arcs adjoining the corner must lie

on the arc opposite the corner. These centers, also, must be corners of the rotor. The centers of the arcs adjoining these corners must lie on the lines joining the corners to the ends of the arcs which adjoin the first corner on the axis.

If the rotor is held fixed and the regular polygon is rotated, the envelopes of the sides of the polygon are the boundaries of the rotor. Place the rotor so that its axis of symmetry coincides with a main diagonal of an even polygon. Then a pair of symmetric corners must lie on two adjacent sides of the polygon. As the polygon is moved either way from this position, the envelopes of the sides of the polygon are circular arcs whose centers lie on a circle through these corners and the vertex of the polygon between them as was demonstrated in Section 2. For the odd polygons, all the centers lie on either of two equal circles. For the even polygons, all the centers lie on either of two unequal circles. Call these circles the generating circles of the rotor.

4. Notation. It will be convenient to employ the following notation.

$n \equiv$ the number of sides of the given regular polygon (n-gon).

$n \equiv 2m$ when the number of sides of the given n-gon is even.

$\alpha \equiv \pi/n \equiv \pi/2m$.

$w \equiv$ the diameter of the inscribed circle of the given regular n-gon. Also, this is the width of a rotor of the given n-gon when n is even.

$4\theta \equiv$ the angular measure of the arc on the smaller generating circle intercepted by the larger generating circle.

$r \equiv$ the radius of the smaller generating circle (through the apex).

$R \equiv$ the radius of the larger generating circle.

$c\{t\} \equiv$ the length of the chord subtending an arc of angular measure t on the smaller generating circle of radius r.

$C\{t\} \equiv$ the length of the chord subtending an arc of angular measure t on the larger generating circle of radius R.

$h\{u\} \equiv$ the height of a segment of the inscribed circle of the given n-gon, where $2u\alpha$ is the angular measure of the arc of the segment.

$a\{u\} \equiv$ the altitude of a segment of the given n-gon bounded by u consecutive edges of the n-gon and the diagonal as base.

5. Conditions on the circular-arc rotor for the $2m$-gon. Consider the case for n even. The relative location of the centers is shown in Figure 2. Then, by equating the radii at the ends of a circular arc, a condition on the variables is obtained. When all such conditions are satisfied, the circular arcs form a continuous boundary. The following equations may then be written:

$$(1)\qquad \begin{cases} c\{t\} = 2r \sin t/2, \\ C\{t\} = 2R \sin t/2, \\ h\{u\} = \frac{1}{2}w[1 - \cos u\alpha], \\ a\{u\} = \frac{1}{2}w(1 - \cos u\alpha/\cos \alpha) \qquad \text{for } u \text{ odd,} \end{cases}$$

$$(2)\qquad R = w \sin \theta/\sin 2\alpha.$$

The following equations are derived from the centers on the larger generating circle:

$$
\begin{aligned}
(3)\qquad & c\{\pi - 4k\alpha + 2\theta\} + C\{\pi - 4\alpha - 4k\alpha + 2\theta\} + h\{2k\} \\
&\qquad = c\{\pi - 4\alpha - 4k\alpha - 2\theta\} + C\{\pi - 4k\alpha - 2\theta\} + h\{2k + 2\},
\end{aligned}
$$

where

$$k = 0, 1, 2, 3, \cdots, (m - 1).$$

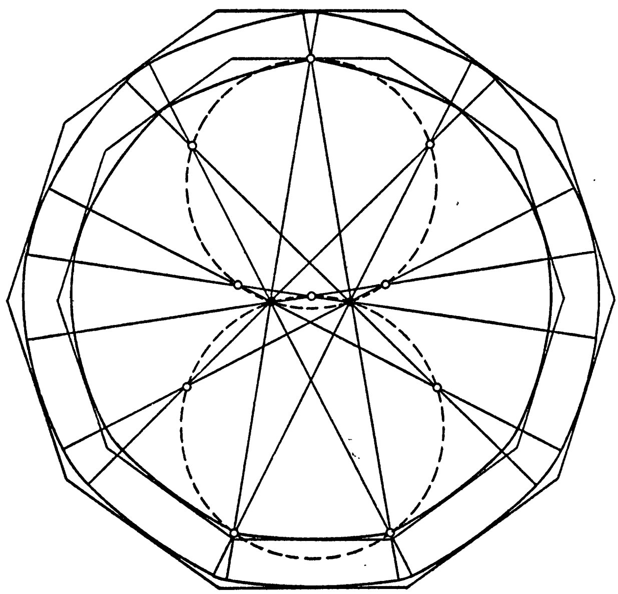

FIG. 2

But from equations (1).

$$
\begin{aligned}
& c\{\pi - 4k\alpha + 2\theta\} - c\{\pi - 4\alpha - 4k\alpha - 2\theta\} \\
(4)\qquad &\qquad = 2r\left[\sin\left(\frac{\pi}{2} - 2k\alpha + \theta\right) - \sin\left(\frac{\pi}{2} - 2\alpha - 2k\alpha - \theta\right)\right] \\
&\qquad = 4r \sin(\alpha + \theta) \sin(\alpha + 2k\alpha).
\end{aligned}
$$

From equations (1) and (2),

$$
\begin{aligned}
& C\{\pi - 4k\alpha - 2\theta\} - C\{\pi - 4\alpha - 4k\alpha + 2\theta\} \\
(5)\qquad &\qquad = 2R\left[\sin\left(\frac{\pi}{2} - 2k\alpha - \theta\right) - \sin\left(\frac{\pi}{2} - 2\alpha - 2k\alpha + \theta\right)\right] \\
&\qquad = 4w \sin\theta \sin(\alpha - \theta) \sin(\alpha + 2k\alpha)/\sin 2\alpha.
\end{aligned}
$$

From equations (1) again,

$$h\{2k+2\} - h\{2k\} = \tfrac{1}{2}w[\cos 2k\alpha - \cos (2\alpha + 2k\alpha)]. \tag{6}$$

Substituting equations (4), (5) and (6) in equations (3) and dividing by the common factor $\sin(\alpha+2k\alpha)$, we obtain

$$4r \sin (\alpha + \theta) = 4w \sin \theta \sin (\alpha - \theta)/\sin 2\alpha + w \sin \alpha. \tag{7}$$

Note that this equation is independent of k.

The following equations are derived from the centers on the smaller generating circle:

$$\begin{aligned} &c\{\pi - 4k\alpha + 2\theta\} + C\{\pi - 4\alpha - 4k\alpha + 2\theta\} + a\{2k+1\} \\ &\qquad = c\{\pi - 4k\alpha - 2\theta\} + C\{\pi + 4\alpha - 4k\alpha - 2\theta\} + a\{2k-1\}, \end{aligned} \tag{8}$$

where

$$k = 0, 1, 2, 3, \cdots, (m-1).$$

But from equations (1),

$$\begin{aligned} &c\{\pi - 4k\alpha + 2\theta\} - c\{\pi - 4k\alpha - 2\theta\} \\ &\qquad = 2r\left[\sin\left(\frac{\pi}{2} - 2k\alpha + \theta\right) - \sin\left(\frac{\pi}{2} - 2k\alpha - \theta\right)\right] \\ &\qquad = 4r \sin\theta \sin 2k\alpha. \end{aligned} \tag{9}$$

From equations (1) and (2),

$$\begin{aligned} &C\{\pi + 4\alpha - 4k\alpha - 2\theta\} - C\{\pi - 4\alpha - 4k\alpha + 2\theta\} \\ &\quad = 2R\left[\sin\left(\frac{\pi}{2} + 2\alpha - 2k\alpha - \theta\right) - \sin\left(\frac{\pi}{2} - 2\alpha - 2k\alpha + \theta\right)\right] \\ &\quad = 4w \sin\theta \sin (2\alpha - \theta) \sin 2k\alpha/\sin 2\alpha. \end{aligned} \tag{10}$$

From equations (1) again,

$$\begin{aligned} a\{2k+1\} - a\{2k-1\} &= w[\cos (2k\alpha - \alpha) - \cos (2k\alpha + \alpha)]/2 \cos \alpha \\ &= w \tan \alpha \sin 2k\alpha. \end{aligned} \tag{11}$$

Substituting equations (9), (10) and (11) in equations (8) and dividing by the common factor $\sin 2k\alpha$, we have

$$\begin{aligned} 4r \sin \theta &= 4w \sin\theta \sin (2\alpha - \theta)/\sin 2\alpha - w \tan \alpha \\ &= [4w \sin\theta \sin (2\alpha - \theta) - 2w \sin^2 \alpha]/\sin 2\alpha. \end{aligned} \tag{12}$$

Note that this equation is independent of k.

Equating both values of $(4r \sin 2\alpha)/w$ derived from equations (7) and (12),

$$\frac{4 \sin\theta \sin (\alpha - \theta) + \sin\alpha \sin 2\alpha}{\sin (\alpha + \theta)} = \frac{4 \sin\theta \sin (2\alpha - \theta) - 2 \sin^2 \alpha}{\sin \theta}. \tag{13}$$

The following series of expansions and transformations leads to equation (14).

$$2 \sin \theta[\cos (\alpha - 2\theta) - \cos \alpha] + \sin \alpha \sin 2\alpha \sin \theta$$
$$= 2 \sin \theta[\cos (\alpha - 2\theta) - \cos 3\alpha] - 2 \sin^2 \alpha \sin (\alpha + \theta),$$
$$\tfrac{3}{2} \sin \theta(\cos 3\alpha - \cos \alpha) = 2 \sin^2 \alpha \sin (\alpha + \theta),$$
$$6 \sin \theta \cos \alpha \sin^2 \alpha = 2 \sin^2 \alpha \sin (\alpha + \theta),$$
$$3 \sin \theta \cos \alpha = \sin \alpha \cos \theta + \cos \alpha \sin \theta,$$
$$\tan \theta = \tfrac{1}{2} \tan \alpha. \tag{14}$$

Note that the lines joining the centers of successive arcs of the rotor pass alternately through one and then through the other of the two intersections of the two generating circles. These two intersections may be called focal points of the circular-arc rotor.

The foregoing construction is applicable for $n > 7$. For $n = 6$, equation (14) does not hold since there are three generating circles instead of two. The radii of the arcs are w and $w/2$. The angular measures of the arcs are 60°, 30° and 15°. For $n = 4$, the familiar Euler triangular curve, frequently called the Reuleaux circular triangle, composed of the three equal 60° arcs, may be taken as the corresponding rotor.

6. Conditions on the circular-arc rotor for the odd n-gon. When n is odd, then

$$r = R = a\{n\} \sin \frac{\alpha}{2} \Big/ \sin 2\alpha = \frac{w}{2}\left(1 + \frac{1}{\cos \alpha}\right)\frac{\sin \frac{\alpha}{2}}{\sin 2\alpha} = \frac{w \cos \frac{\alpha}{2}}{4 \cos^2 \alpha}$$

and

$$c\{t\} = C\{t\}.$$

In this case, the focal points coincide with centers of arcs. The continuity of the arcs can be verified by the following equations, based on equations (1) and (2):

$$\begin{aligned} C\{\pi - 3\alpha - 4k\alpha\} - C\{\pi - 5\alpha - 4k\alpha\} & \\ = 2R\left[\sin\left(\frac{\pi}{2} - \frac{3\alpha}{2} - 2k\alpha\right) - \sin\left(\frac{\pi}{2} - \frac{5\alpha}{2} - 2k\alpha\right)\right] & \\ = 4R \sin \frac{\alpha}{2} \sin (2\alpha + 2k\alpha). & \end{aligned} \tag{15}$$

$$\begin{aligned} C\{\pi - \alpha - 4k\alpha\} - C\{\pi - 7\alpha - 4k\alpha\} & \\ = 2R\left[\sin\left(\frac{\pi}{2} - \frac{\alpha}{2} - 2k\alpha\right) - \sin\left(\frac{\pi}{2} - \frac{7\alpha}{2} - 2k\alpha\right)\right] & \\ = 4R \sin \frac{3\alpha}{2} \sin (2\alpha + 2k\alpha). & \end{aligned} \tag{16}$$

$$(17)\quad \begin{aligned} a\{2k+3\} - a\{2k+1\} &= w[\cos(2k\alpha+\alpha) - \cos(2k\alpha+3\alpha)]/2\cos\alpha \\ &= w\sin(2\alpha+2k\alpha)\tan\alpha \\ &= 8R\cos\alpha\sin\frac{\alpha}{2}\sin(2\alpha+2k\alpha) \\ &= 4R\left[\sin\frac{3\alpha}{2} - \sin\frac{\alpha}{2}\right]\sin(2\alpha+2k\alpha) \end{aligned}$$

Subtracting equations (15) and (17) from (16), we obtain

$$(18)\quad \begin{aligned} &C\{\pi-5\alpha-4k\alpha\} + C\{\pi-\alpha-4k\alpha\} + a\{2k+1\} \\ &\qquad = C\{\pi-3\alpha-4k\alpha\} + C\{\pi-7\alpha-4k\alpha\} + a\{2k+3\}. \end{aligned}$$

This equation holds for $k=0, 1, 2, 3, \cdots, (n-1)$. Applying equation (18) to Figure 3 shows that the circular arcs are contiguous.

The foregoing construction is applicable to all polygons of an odd number of sides. In addition, for $n=3$, Fujiwara [5] has described other types of rotors composed of two, four and five arcs.

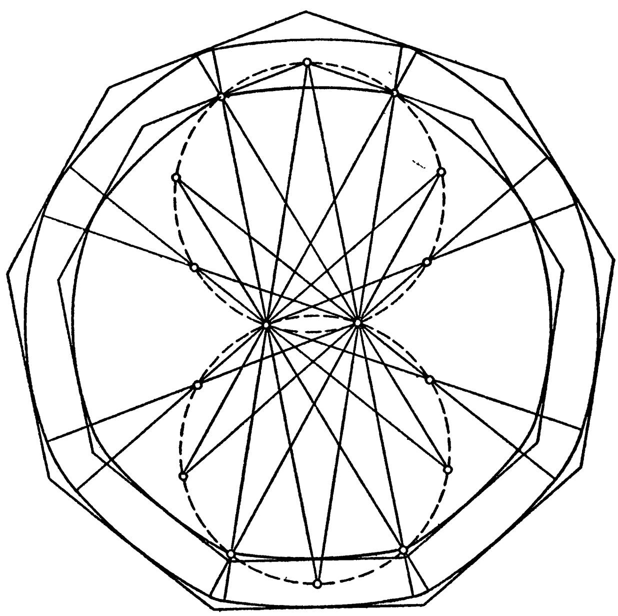

FIG. 3

7. The least number of arcs bounding a circular-arc rotor. A parallel curve to any of the rotors here described is a circular-arc rotor with no corners. (See Figures 2 and 3.) The number of arcs is $2n$ in the case of the even polygons (for n greater than 4) since each center on the generating circles serves as the center

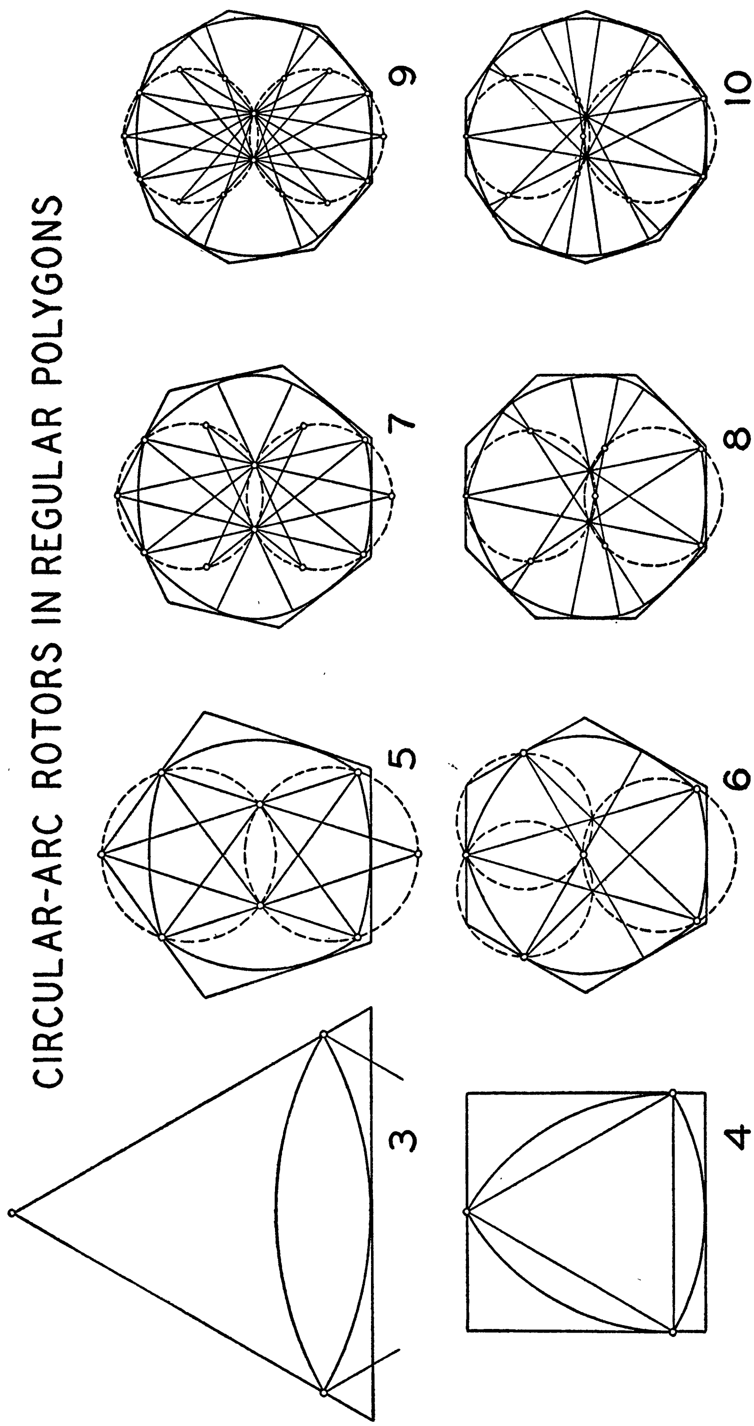

FIG. 4

of two arcs. In the case of the odd polygons, each generating circle carries n center points which serve as centers of single arcs. However, two centers on one generating circle coincide with centers on the other generating circle. Therefore, for odd polygons, the number of arcs bounding the rotor is $2n-2$.

Since at least two generating circles are necessary to produce closure, the rotors described here have the least possible number of arcs if no corners are permitted. When corners are permitted, the number of arcs for even polygons becomes $2n-3$. The equality of the generating circles for the odd polygons permits a rectangular arrangement of four corners resulting in $2n-4$ arcs as the least possible number of circular arcs.

Several examples of rotors with the least number of circular arcs are shown in Figure 4. The dotted circles are the generating circles. The small open circles enclose the centers of the boundary arcs. The small black circles enclose the focal points. Straight lines join the centers of successive arcs and show the sectors determined by the arcs.

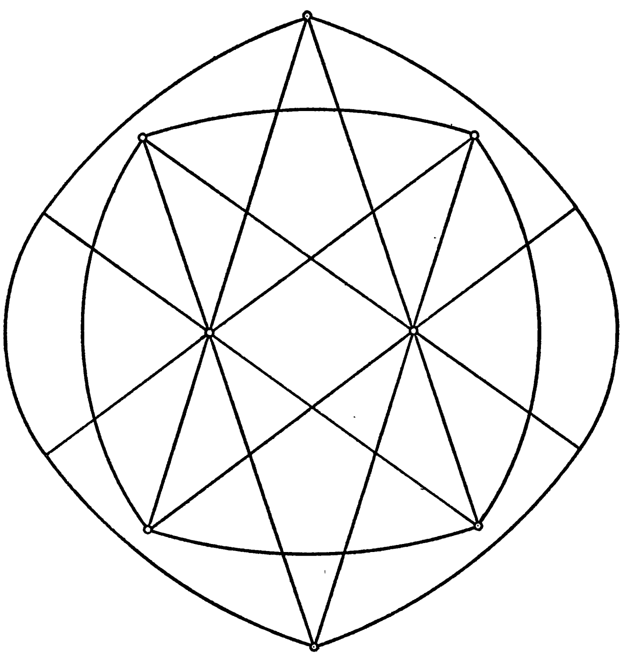

FIG. 5

8. Dual rotors. To every rotor for an odd polygon there corresponds another rotor which plays the rôle of a dual. Just as a parallel rotor is an orthogonal trajectory of the external normals to a given rotor, a dual rotor can be obtained as the orthogonal trajectory of the internal normals if the trajectory is taken at a sufficient distance from the given rotor. The nearby internal trajectories are not simple convex curves. One of these trajectories is a convex rotor with corners;

those on one side are non-convex curves (therefore, they are not rotors), while those on the other side are rotors without corners.

The duals of the circular-arc rotors for the odd regular polygons, described in Section 6, have the same centers and the same number of arcs. However, the duals with corners have only two corners (instead of four) corresponding to the two arcs of maximum radius. The radii of the arcs are always less than the height of the circumscribed polygon. An example of a pair of dual rotors for the pentagon is shown in Figure 5.

The general existence of the dual rotors seems to have been overlooked in the previous literature although one of them was known. The dual composed of two 120° arcs corresponding to the rotor of two 60° arcs, shown in Figure 4, was given by Reuleaux and was discussed by Fujiwara [3].

9. Minimal area of rotors. Fujiwara and Kakeya proposed the problem of determining the rotors of least area for the regular polygons [8]. They showed that for the equilateral triangle the minimal rotor is bounded by two equal 60° circular arcs. It was first shown geometrically by Blaschke [6], and later analytically by Fujiwara [7], that the minimal rotor for the square is bounded by three equal 60° circular arcs. Fujiwara remarked that the method which was successfully employed in the solution of this extremal problem for the triangle and the square was no longer applicable in testing the circular-arc rotor he found for the pentagon even though it could conceivably be the sought minimal rotor. It should be noted that these three rotors are each bounded by the least number of circular arcs and that each includes arcs whose radius is the height of the polygon. These properties are shared by each of the infinite series of rotors, which includes the foregoing three cases, whose constructions are given in this paper. Therefore, they are worthy of consideration as possible solutions of the minimal rotor problem.

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THE LINKING OF MATHEMATICS AND PHYSICS IN UNIVERSITY COURSES

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1. Introduction. The past thirty years have seen a great intensification of the application of mathematical methods to the study of natural phenomena. Associated with this there has been a widening and deepening of our understanding of the nature of matter and radiation, and an increase in the power of designers in many branches of technology. The prospects of further technological developments resulting from our increased understanding are very great and many universities propose to equip some of their students with the techniques necessary to enable them to play an increasing part in the new scientific and technological advances. This, of course, means a linking of the instruction given in mathematics and physics, even closer than that which has been obtained in the past, and it also means consequential changes in some of the instruction given within faculties of engineering.

2. Mathematics and physics. The formal linking of mathematics and physics in English universities is entering upon an interesting phase. In the past there have been few or no attempts by departments of physics to teach their "own mathematics," or by departments of engineering to insist upon giving instruction in mathematics within the walls of their own faculties. Between professors and teachers of all departments and faculties, there has been good-natured disagreement and also much friendly agreement on educational matters. This has led to a cooperative outlook and a willingness to adapt and modify existing courses of instruction.

There are, of course, difficulties inherent in the attempt to cooperate, and to some extent these are reflected in the names which are used to denote various branches of instruction; examples of these are pure mathematics, applied mathematics, mathematical physics, theoretical physics, chemical physics, physical chemistry, metal physics, physics, and so on. Of these, "pure mathematics" alone seems to have retained its time-honored significance, though the content of the instruction given under this heading has changed considerably. "Applied mathematics" used to mean "applications of mathematics," and involved drastic idealization of much of the physical conditions of reality. Surprisingly accurate and surprisingly inaccurate predictions resulted from this, and the uncertainty of the validity of the conclusions arising from the combination led to applied mathematics being regarded with some suspicion.

With the passage of time, however, as mathematicians applied their techniques to problems in which the physical assumptions bore a greater resemblance to reality than previously, and as engineers and physicists became more and more expert in the techniques of pure mathematics, a reasonably happy com-

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promise was effected. "Applied mathematics" came to be known as covering in the main "engineering mathematics"; that is, it included mechanics, hydrodynamics, aerodynamics, elasticity, magnetism and electricity, and so on. Astronomy and gravitational theory do not fit in with this rough sub-division, but the subjects were loosely grouped under applied mathematics. "Physics" was understood to cover, in the main, experimental investigations, but it came to use more and more the terms and techniques of pure mathematics in order to explain its outlook and to convey its results. "Mathematical physics" was understood to mean a linking of mathematics and physics, but with even greater emphasis on the methods of pure mathematics. "Theoretical physics" was accepted as a variant of mathematical physics, in which mathematics and physics were combined with assumptions whose foundations might be rather difficult to formulate in precise terms. These sub-divisions are, of course, extremely rough and there is considerable overlap between them. Change and development continue, and many of us, as members of universities, are trying to lay the foundations for developments likely to bear fruit during the next quarter of a century. Much thought is therefore being given to the proper place of mathematics and physics in the general scheme. We are concerned with mathematical physics, both as a part of a general education, and as an aid to detailed understanding of physical science and technological advance. Education in mathematical physics ought to, and generally does, begin at school.

3. Instruction in mathematics and physics. In many schools pure mathematics is taught as a series of interesting and stimulating puzzles, often necessitating the important mental pastime of deciding what is, and what is not, relevant. This is usually done without an attempt to give a comprehensive picture of what mathematics is or what it purports to be. Then, often under the title of applied mathematics, the elements of physics are introduced and mathematical methods are applied to what is simply a new field of activity introduced by means of bewildering, and often ill-considered, definitions. These may well have the effect, later, of repelling many keen young minds from pure mathematics, mathematical engineering, or mathematical physics. At an early stage the pupil develops crude ideas of physical quantities such as force, mass, and so on, and it is generally only in the universities that an attempt is made to reduce these ideas to some semblance of order. This attempt should, we feel, be made at a much earlier stage. Teachers might take classes into their confidence when pupils are at quite an early age, say even at eleven years, and talk to them on the genesis of mathematics. First of all, it should be presented as essentially a practical subject. Its historical development should be described with special reference to its relation to the problems of primitive societies, such as the counting of possessions and the measuring of land. Then might be explained the fumbling, halting investigations which formulated the properties of numbers, irrespective of the particular objects to which the numbers were attached. At a slightly later stage, at the beginning of the study of mathematical engineering

and mathematical physics, the ideas and definitions of mechanics might be introduced, including the use of machines and the mathematical problems arising in the precision of their design, the effects of matter in motion, the problems which the engineer has to solve, and so on. These might all serve to arouse the interest of the pupil. Definitions might be regarded as statements formulated and gradually made more precise, possibly by the pupils themselves; their display on a large, accessible blackboard would perhaps keep them in mind. Above all, certain types of "definitions" must be avoided, at any rate as the first words on the particular topics to which they refer. The following, for example, are the first words offered by well-known books on the topics which they purport to describe: "The moment of a force about a point is the product of the force and the perpendicular drawn from the point on the line of action of the force," and again "Matter is that which has mass," and still further, "The mass of a body is the quantity of matter in the body." This method of approach must be shunned!

It is very unwise to attempt to achieve apparent academic success by brevity of discussion and of definition when it is accompanied by neglect of the all-important scientific background required for the definition or discussion. Indeed, while too much detail may be confusing, and a compromise between amount of detail and reasonable clarity must be sought, it has to be remembered that too much willingness to abstract and isolate physical phenomena may well develop an attitude toward general experience which may make its possessor a menace to the community in which he lives.

Even at this early stage the method of mathematical physics should be emphasized—the sequence of experiment, thought about the circumstances which are relevant to the phenomenon studied, the tentative (and not always reliable) rejection of circumstances which seem not to be really significant and whose inclusion impedes mathematical formulation, mathematical formulation and solution, final comparison with experiment, and, very often, the decision to repeat the whole sequence with different assumptions about relevance.

The qualities and defects of the instruction in mathematics and physics given in schools are carried over to the universities. In many universities in this country there has been an exaggerated division between those studying the mathematical and experimental aspects of physical science. The enormous development of the field of study has, of course, been the main cause of this. Formerly there was a general tradition of "natural philosophy" in teaching. This involved detailed acquaintance with both experimental and mathematical physics as complementary aspects of one subject. Nowadays it is difficult in an undergraduate course to develop both aspects to provide sufficiently powerful equipment. The policy has been (1) to combine training in mathematical physics, under the name and limitations of applied mathematics, with a full course in pure mathematics including branches of pure mathematics of very specialized interest although admittedly of great aesthetic value, such as branches of pure geometry, parts of the theory of numbers, and so on, or (2) to combine training in experimental physics with an inadequate training in mathe-

matical physics. Of these (1) means limitation to a very narrow field of study and (2) involves giving mathematical training of an almost entirely "set" nature with little or no problem work.

It is clear that the mathematical physicist, in an undergraduate course, cannot be expected to become really proficient in the modern technique of elaborate experiment, but he has a right to demand that his mathematical training will give him reasonable equipment in the mathematical techniques which he really needs, in the most economical manner. He must have an understanding of mathematical method and, at the same time, a clear grasp of physical principles. This must not suggest that the mathematical physicist is an unfortunate hybrid whose conclusions are likely to be unsatisfactory or inadequate. The union of mathematical technique with appropriate understanding and knowledge of physical principles often produces results of as great significance as does the wider equipment in either field alone. For example, in many types of physical theory, and certainly in atomic and nuclear physics, it is not possible to have simple and easily interpreted fundamental experiments on which to build a theory. The experimental verifications of the results to be expected from any physical "model" have to be made at a considerable experimental and mathematical distance from the fundamental mathematical and physical hypotheses. Then mathematical intuition may play the decisive part in formulating quantitative hypotheses. Examples abound in the use of mathematical properties of invariance under Lorentz and other transformations in the theories of elementary particles, the symmetrical and anti-symmetrical wave-functions of quantum mechanics, and even in the displacement current of classical electrodynamics. Indeed, it is wrong perhaps to speak here of mathematical intuition. The mathematical principles used are often simply the precise formulation of conceptions of space, time, and so on, which we have acquired physically, not in a brief laboratory experiment, but by generations of slow endeavor.

In the attack on any problem in natural philosophy it is important that the mathematical physicist should use a technique of adequate power. That uneconomical training should limit him to use crude mathematical methods when more powerful methods are readily available and assimilable with a little effort, is no less culpable than that the experimenter should be limited to antiquated and unreliable techniques in experiment.

Mathematical physicists will hope for the full collaboration of the pure mathematician in the development and teaching of satisfactory and powerful techniques. In much of modern, and of classical, mathematical physics the really relevant and powerful mathematical methods have sufficient intrinsic mathematical elegance to provide that attraction of interest which will reward the pure mathematician for his help and will in itself attract the student. At present, in this country, much could be done for the mathematical physicist who is anxious to become proficient in methods of algebra and group-theory in order to attack problems in molecular, atomic and nuclear physics; this is merely one example.

4. A glance at the future. What then may we hope for in the future? The need for subdivision of natural philosophy into separate schools of experimental and mathematical physics will remain, although the two schools must to a certain extent overlap in their studies. For perhaps three of the usual four years of an undergraduate honors course, it is advisable for the mathematical physicist to take adequate courses in experimental and descriptive physics while, at the same time, taking intensive and specially planned courses in pure mathematics and mathematical physics. On account of the great width of the subject there will certainly have to be a careful choice of the branches of physics which are to be studied; it is probably wise to limit the width of the field of study by demanding an effective and precise quantitative understanding of what is studied, assuming that the function of the mathematical physicist is to offer reasonably precise opinions on his field of study rather than nebulous opinions on a very wide field. In subjects of basic importance such as classical mechanics and electrodynamics the aim should be to give a firm and precise grasp of the fundamental physical principles and the most important mathematical techniques; it is not necessary to discuss almost every type of problem which has ever been of importance in these subjects. The need for cooperation and even self-denial from pure mathematicians has been mentioned before; very often they will have to omit topics of great importance for them and to prove or, occasionally in difficult matters, even merely to make plausible, certain theorems under much more restrictive conditions than are really necessary. That their reward will be a rich one, in interest and respect, is quite certain.

In the last year of the course practically all of the student's time will have to be given to the study of experimental results and the development of mathematical and mathematical-physical techniques. It is advisable that he should see certain important types of experimental arrangement and perhaps take some part in the performance of a few carefully selected experiments. However, he will not in general be able to spare time for detailed study of experimental techniques.

It is perhaps unavoidable that, at this stage, there should be a considerable segregation of students of various branches such as molecular, atomic, and nuclear physics, hydro- and aerodynamics, and so on. One can only hope that it will not be too extreme.

5. Present courses of study. In the University of Liverpool we have made an attempt to formulate suitable and practicable courses leading to a degree with honors in mathematical physics. Quotations from parts of our scheme follow. The actual syllabuses given refer to some of the basic subjects in pure mathematics and mathematical physics. On account of the absence of so many of our physicists on war work, the courses in descriptive and experimental physics could not be adequately discussed and detailed syllabuses for these are therefore not included.

EXAMINATIONS AND COURSES OF STUDY

The period of study normally required for the degree of B.Sc. with Honors in Mathematical Physics will be four years. In the first three of these the courses will be: *Intermediate year*, Introductory Pure Mathematics, Applied Mathematics, Physics; *Part I year*, Courses in Pure Mathematics, Applied Mathematics and Physics; *Part II year*, Courses in Pure Mathematics, Applied Mathematics and Physics. Detailed syllabuses for these, and for some of the Honors courses, are given below, together with notes on admission to the courses.

In the Honors Examination, at the end of the fourth year of study, students will be expected to offer papers in three groups. Those of Groups I and II, seven in number, will be compulsory and will cover certain basic equipment. Those of Group III, two in number, will allow choice in accordance with the candidate's interests. They will be chosen by the candidate in consultation with the Heads of the Departments of Mathematics and Physics.

For these students, there will be no Honors Examination in Experimental Physics. They should, however, be able to spend some time in a Physics Laboratory in the honors year, without unduly overloading their courses, as they will have taken suitable courses and examinations in Experimental Physics in the earlier years of study.

The papers in Group I are:
Methods of Mathematical Physics (one paper),
Statistics and Numerical methods (one paper),
Classical Mechanics and Vector Field Theory (one paper).

The papers in Group II are:
Quantum Theory of Matter and Radiation, Molecular, Atomic and Nuclear Physics (three papers),
Essay paper on the Principles of Physics (one paper).

The papers in Group III are:
Advanced methods in Mathematical Physics including the Theory of Groups, in relation to problems of Mathematical Physics (one paper),
Thermodynamics and Statistical Mechanics (one paper),
Relativity and special problems in Electromagnetic Theory, e.g. Optics, etc. (one paper).

The subjects to be studied in any session will be announced at the beginning of that session.

PROPOSED SYLLABUSES

The following are suggested syllabuses for some of the lecture courses; they will be modified in the light of experience.

INTRODUCTORY COURSES (similar to those given at present):

PART I COURSES:

(A) Pure Mathematics Course (5 lectures per week).
Functions, graphs, continuity. Differential calculus. Expansions in power series. Orders of magnitude. Integration of elementary functions. Solution of ordinary differential equations, including solution in series. Fourier series. Beginning of a calculus of many variables. Functions of many variables. Maxima and Minima, Taylor Series, Euler's Theorem and repeated Cartesian and Polar Integrals. $B(p, q)$ and $\Gamma(x)$. Complex numbers. Determinants (excluding multiplication). Linear equations. Plane differential geometry. Elements of theory of equations. Language of matrices. Quadratic forms. Linear geometry in two and three dimensions. Vectors (brief revision from Applied Mathematics). Elements of conics and quadrics.
(B) Applied Mathematics Course (4 lectures per week).
Theory of vectors with illustrations up to and including scalar and vector products and the two triple products; differentiation of a vector function of a single variable, with illustrations. Foundations of mechanics (elementary). Statics: coplanar forces, friction, theory of mass systems, and stresses in beams and flexure of beams. Dynamics: kinematics of a particle and kinematics of a rigid body for the simpler types of motion. Dynamics of a particle: simple harmonic motion, damped oscillations, forced oscillations, projectiles, central orbits (inverse square law mainly). Work and energy. Kinematics and dynamics for two particles: normal modes for two particles (elementary). Kinematics and dynamics of a system of particles and of a rigid body: principles of momentum and energy, illustrations mainly from two-dimensional motion. Mechanics of continua. Hydrostatics: perfect fluid, resultant thrust, center of pressure, floating bodies. Gases. Elementary elasticity.
(C) Physics Course (5 lectures and 5 hours laboratory work per week).
Similar to the present Part I Course: 2 lectures per week on properties of matter and heat, 3 lectures per week on electricity and magnetism and elementary electronics. 5 hours Experimental Physics.

PART II COURSES:

(A) Pure Mathematics Course (3 lectures per week).
Descriptive account of bounds, limits, convergence of easy real and complex series, especially radius of convergence of power series. Theory of integration. Infinite integrals. Multiple integrals. Green, Gauss, and Stokes theorems. Partial differential equations of Physics. Normal solutions, boundary conditions, first account of eigen-functions and orthogonal series. Study of P_n, J_n, *etc.* Determinants, elementary matrices. Language of groups. Introduction to differential geometry of curves and surfaces. Introduction to calculus of variations. Transformations, Jacobians. Functions of complex variable. Easy contour integration.

(B) Applied Mathematics Course (3 lectures per week).
Mechanics: Moments of inertia and principal axes of inertia for three-dimensional distributions. More general motion of a rigid body. Lagrange's equations. Gyrostatic problems. Normal coördinates. Principle of Least Action (elementary). Electricity and Magnetism: with accompanying theory of vector fields. The course will cover the fundamental parts of the subject as far as and including Maxwell's equations. General: Simple examples of diffusion and wave phenomena. An introduction to the special theory of relativity.
(C) Physics Course (4 lectures and 5 hours laboratory work per week).
3 lectures per week on Optics and Atomic Physics and 1 per week on Electronics. 5 hours Experimental Physics.

HONORS COURSES:

(A) Methods of Mathematical Physics (60 lectures per session).
Complex function theory. Contour integration and conformal representation. Theory of linear differential equations leading to Sturm-Liouville theory. Asymptotic Expansions. Fourier integrals. Differential geometry. Calculus of variations. Tensor calculus (introduction). Optional topics chosen from Lebesgue integration, Fourier series, and General Linear Algebras.
(B) Classical Mechanics and Vector Field Theory (30 lectures per session).
Three dimensional statics (briefly). Three dimensional kinematics. General Dynamics including Lagrange's equations (continued). Theory of vibrations. Hamilton's equations, Hamilton-Jacobi equation. Variational principles. Transformation theory of dynamics. Maxwell's equations and electromagnetic waves. Scalar and vector potentials: retarded and advanced potentials. Electron theory: Lorentz force, *etc.* Principles of energy and momentum. The special theory of relativity. Equations of continuity, motion, and pressure of non-viscous fluids. Rotational and irrotational motion. Velocity potential. Stream function. Complex potentials. Images. Vortices.
(C) Statistics and Numerical Methods. (30 lectures per session and time for practical numerical work.)
Statistics with special reference to the analysis of experimental results. Interpolation. Integration. Solution of algebraic, transcendental and differential equations. Harmonic analysis.

ADMISSION TO COURSES

The admission of students to Part I courses, Part II courses, and honors courses would be decided by the Professors of Pure Mathematics, Applied Mathematics and Physics. They would adopt suitable criteria; for example, entry into Part I courses would be given by a suitable mark at the Introductory Standard or in the Higher School Certificate in the subjects concerned. It might for a few years be advisable, as at present, to allow very good qualifications in H.S.C. (or other equivalent examination) in Pure Mathematics and Physics to

compensate for somewhat inadequate qualifications in Applied Mathematics. Such a deficiency could be helped by the provision of a tutorial class in the second year of study.

In addition, an effort should be made to help students who decide rather late in their courses, say at the end of the Part I year, that they wish to qualify as mathematical physicists. They might be admitted to the necessary courses, provided that they have shown suitable ability, and if they rectify the omissions in their earlier courses by private reading.

ON INDECOMPOSABLE POLYHEDRA

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1. Introduction. We shall prove the following

THEOREM. *If n is an integer not less than 6, then there exists a polyhedron, π_n, with n vertices and the following properties:*

(I) *π_n is simple, and every one of its faces is a triangle.*

(II) *If τ is a tetrahedron, each of whose vertices is a vertex of π_n, then not every interior point of τ is an interior point of π_n.*

(III) *Every open segment whose endpoints are vertices of π_n, but which is not an edge of π_n, lies wholly exterior to π_n.*

It is well known that every *convex* polyhedron can be decomposed into a set of tetrahedra whose vertices are all vertices of the given polyhedron [**1**, p. 280 or **2**, p. 57]; and *every* simple polygon can be decomposed into a set of triangles whose vertices are all vertices of the given polygon [**1**, p. 246 or **2**, p. 46]. Lennes, however, proved [**2**, p. 55] the existence of indecomposable polyhedra by constructing a polyhedron which has properties (I) and (II). His polyhedron, which possesses seven vertices, does not satisfy (III). Schönhardt [**3**] subsequently gave an example of a polyhedron having six vertices and all three of the above properties. He showed, moreover, that there is no indecomposable polyhedron with less than six vertices.

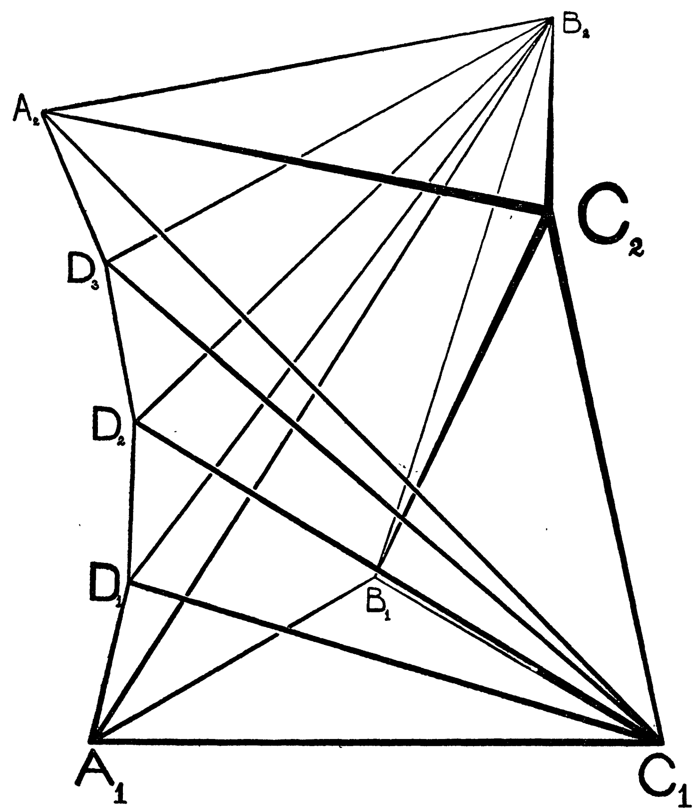
We shall take π_6 to be a Schönhardt polyhedron, and base our construction of π_n, for $n>6$, on π_6. It will be apparent first that, for every $n\geqq 6$, π_n satisfies (I) and the following condition:

(IV) *Every triangle whose sides are edges of π_n is a face of π_n.*

Then we shall demonstrate that (III) holds. Finally, (II) follows from (I), (III), and (IV). For suppose that, on the contrary, there existed a tetrahedron, τ, with its vertices and interior points consisting exclusively of vertices and interior points, respectively, of π_n. Then, because of (III), every edge of τ would be an edge of π_n, otherwise some interior point of τ would be exterior

to π_n; by (IV), every face of τ would be a face of π_n; and, according to (I), π_n is simple, so that π_n would have to be identical with τ. But this would contradict our assumption that $n \geqq 6$.

2. Description of π_6. Let A_1, B_1, C_1 be the vertices of an equilateral triangle, each of whose sides has length 1; and let A_2, B_2, C_2 be the vertices of the triangle obtainable from triangle $A_1B_1C_1$ by first rotating the latter about its center, and in its plane, through 30° in the direction $A_1B_1C_1$, and then translating it one unit in the direction perpendicular to the plane $A_1B_1C_1$, as in the figure, where



π_9

A_2, B_2, C_2 correspond, respectively, to A_1, B_1, C_1 under this transformation. Then π_6 consists of

6 vertices: A_1, B_1, C_1; A_2, B_2, C_2;

12 edges: A_1B_1, B_1C_1, C_1A_1; A_2B_2, B_2C_2, C_2A_2; A_1A_2, B_1B_2, C_1C_2; A_1B_2, B_1C_2, C_1A_2;

8 triangular faces: $A_1B_1C_1$, $A_2B_2C_2$; $A_1A_2C_1$, $B_1B_2A_1$, $C_1C_2B_1$; $A_1A_2B_2$, $B_1B_2C_2$, $C_1C_2A_2$.

It is evident that π_6 possesses properties (I) and (IV); and (III) also holds, because the only (open) segments whose endpoints are vertices of π_6, but which are not edges of π_6, are A_1C_2, B_1A_2, C_1B_2, and these are obviously wholly exterior to π_6.

3. Construction of π_n, $n>6$. Let $n=6+k$. In the interior of π_6, take an open circular arc $\overset{\smile}{A_1A_2}$, with endpoints A_1, A_2, and with a radius so large, that every point of $\overset{\smile}{A_1A_2}$ is on the same side of the plane $C_1A_2C_2$ as A_1, and on the same side of the plane $B_1A_1B_2$ as A_2. (See the figure.) On $\overset{\smile}{A_1A_2}$, choose k distinct points, $D_1, D_2, \cdots, D_k$, in the order $A_1D_1D_2 \cdots D_{k-1}D_kA_2$. Then π_n shall consist of

n vertices: $A_1, B_1, C_1; A_2, B_2, C_2; D_1, D_2, \cdots, D_k$;

$3k+12$ edges: $A_1B_1, B_1C_1, C_1A_1; A_2B_2, B_2C_2, C_2A_2; B_1B_2, C_1C_2; A_1B_2, B_1C_2, C_1A_2; A_1D_1, D_1D_2, D_2D_3, \cdots, D_{k-1}D_k, D_kA_2; C_1D_1, C_1D_2, \cdots, C_1D_{k-1}, C_1D_k; B_2D_1, B_2D_2, \cdots, B_2D_{k-1}, B_2D_k$;

$2k+8$ triangular faces: $A_1B_1C_1, A_2B_2C_2; B_1C_2B_2, C_1B_1C_2, A_1B_2B_1, A_2C_1C_2; C_1A_1D_1, C_1D_1D_2, C_1D_2D_3, \cdots, C_1D_{k-1}D_k, C_1D_kA_2; B_2A_1D_1, B_2D_1D_2, B_2D_2D_3, \cdots, B_2D_{k-1}D_k, B_2D_kA_2$.

Obviously π_n satisfies (I); and it is easy to see that (IV) also holds, by simply comparing the various triangles in question, with the list of faces of π_n.

The open segments whose endpoints are vertices of π_n, but which are not edges of π_n, are

$A_1C_2, B_1A_2, C_1B_2; D_1C_2, D_2C_2, \cdots, D_kC_2; D_1B_1, D_2B_1, \cdots, D_kB_1$; and all segments connecting pairs of nonadjacent vertices of π_n on $\overset{\smile}{A_1A_2}$, its endpoints included. It is clear that every segment of this last class is wholly outside π_n, because $\overset{\smile}{A_1A_2}$ is a circular arc. A_1C_2, B_1A_2, C_1B_2 were seen to be completely exterior to π_6; and since $\overset{\smile}{A_1A_2}$ was chosen to lie *within* π_6, these segments are also entirely exterior to π_n. Every one of the segments $D_1C_2, D_2C_2, \cdots, D_kC_2$ is wholly outside π_n. For, each intersects the (open) triangle $A_1A_2C_1$, because $\overset{\smile}{A_1A_2}$ was taken to lie inside π_6 and, at the same time, on the same side of the plane $C_1A_2C_2$ as A_1. Consequently, it is obvious that the only faces of π_n which any of these segments could possibly intersect, are the faces $C_1A_1D_1, C_1D_1D_2, \cdots, C_1D_{k-1}D_k, C_1D_kA_2$. But, since $\overset{\smile}{A_1A_2}$ is a *circular* arc, any open segment joining a point of this arc to a point of the open triangle $A_1A_2C_1$ cannot contain a point of any one of the faces in question. An analogous argument shows that every one of the segments $D_1B_1, D_2B_1, \cdots, D_{k-1}B_1, D_kB_1$ is completely exterior to π_n. Hence, π_n satisfies (III).

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MATHEMATICAL NOTES

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Material for this department should be sent directly to E. F. Beckenbach, University of California, Los Angeles 24, California.

THE NUMBER OF TERMS IN THE EXPANSION OF AN INFINITE DETERMINANT

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We shall prove the following intuitively obvious result.

THEOREM. *The number of terms in the expansion of an infinite determinant has the power of the continuum.*

In other words, the group of permutations of the integers has the power of the continuum.

N. J. Lennes* settled this problem some time ago; however his proof is based on a definition of the expansion of an infinite determinant which in itself is open to comment. Let

$$D = \begin{vmatrix} a_{11} & a_{12} \cdots a_{1n} \cdots \\ a_{21} & a_{22} \cdots a_{2n} \cdots \\ \cdot & \cdot \;\cdots\cdots \;\cdots \\ a_{n1} & a_{n2} \cdots a_{nn} \cdots \\ \cdot & \cdot \;\cdots\cdots \;\cdots \end{vmatrix} = |a_{i,j}|, \qquad i, j = 1, 2, \cdots, n, \cdots, \tag{1}$$

represent both the determinant and the set of terms obtained on expansion. Let

$$P = \prod_{k=1}^{\infty} a_{i_k j_k}. \tag{2}$$

Definition 1. P is a term of D if in the infinite product there is at most one element from each row, and at most one element from each column.

Definition 2. P is a term of D if in the infinite product there is exactly one element from each row and exactly one element from each column.

For example $a_{12}a_{24} \cdots a_{n2n} \cdots$ is a term of D by Definition 1, but is not a term of D by Definition 2. The Lennes proof is based on the first definition and provides a clever method of mapping the terms of D in a one-to-one manner on the points of an interval. We prove the theorem based on the second definition as follows.

Since we are not concerned with the convergence of the infinite product (2), we may rearrange the order of the elements according to their columns, thus

$$P = \prod_{k=1}^{\infty} a_{n_k k}, \tag{3}$$

* A Direct Proof of the Theorem that the Number of Terms in the Expansion of an Infinite Determinant is of the Same Potency as the Continuum, Bull. Amer. Math. Soc. 18 (1911) pp. 22–24.

and our definition assures us that P is an element of D if and only if the sequence $n_1, n_2, \cdots, n_k, \cdots$ is a permutation of the integers $1, 2, \cdots, k, \cdots$. We make the correspondence $P \to \xi$,

$$\xi = \sum_{m=1}^{\infty} \frac{f_m}{2^m}, \tag{4}$$

where $f_m = 1$, if $m = n_1, n_1+n_2, n_1+n_2+n_3, \cdots$, and $f_m = 0$ otherwise. This provides a one-to-one mapping of the elements of D onto a subset of the interval $0<\xi<1$.

Secondly, we observe that every ξ, in the open interval $0<\xi<1$, can be written in the form (4), and uniquely so if the requirement is made that there be an infinite number of f_m different from zero. The correspondence $\xi \to P$ is made thus. For each k, the set

$$1, 2, \cdots, k, k+1 \tag{5}$$

will have two elements $\alpha_k < \beta_k$ which are not in the set

$$n_1, n_2, \cdots, n_{k-1}. \tag{6}$$

Define

$$n_k = \begin{cases} \alpha_k, & \text{if } f_k = 1; \\ \beta_k, & \text{if } f_k = 0. \end{cases}$$

Then $\xi \to P$ is a one-to-one mapping of the interval $0<\xi<1$ onto a subset of the elements of D.

Now apply Bernstein's Theorem which states that if there is a one-to-one mapping of A onto a subset of B, and a one-to-one mapping of B onto a subset of A, then there exists a one-to-one mapping of A onto B. Thus D and the open interval $0<\xi<1$ have the same power.

This result is trivial, using Bernstein's Theorem, and the proof could have been given in many ways. The problem of constructing an effective one-to-one mapping between the set D and the open interval $0<\xi<1$ appears to be difficult and, as far as the writer is aware, is still unsolved.

A GEOMETRICAL FORM OF THE MULTIPLICATIVE PROPERTY OF THE DIVISOR-SUM FUNCTION

ABDUR RAHMAN NASIR, Talim-ul-Islam College, Qadian, India

1. **Introduction.** Let $\sigma(x) = \sum_{d|x} d$, so that $\sigma(x)$ denotes the sum of the positive divisors of x including 1 and x. A very simple and instructive method is investigated in the following lines to prove the multiplicative property of this function with an appeal to elementary notions of plane geometry.

We shall establish the following result.

THEOREM. *If* $(m, n) = 1$, *then* $\sigma(mn) = \sigma(m)\sigma(n)$.

Unlike the proof in classical books of the above result, the present one does not make use of the value of the function itself, which on the other hand may now be deduced from this property.

2. **Proof.** Let a_i/m, where $1 \leqq a_i \leqq m$ and $1 \leqq i \leqq p$ (say) and b_j/n, where $1 \leqq b_j \leqq n$ and $1 \leqq j \leqq q$.

Since $(m, n) = 1$, it follows that $(a_i, b_j) = 1$ for every i and j.

Hence a divisor of mn can be uniquely written in the form a_ib_j, $1 \leqq a_ib_j \leqq mn$ where i and j run through all the integers from 1 to p and 1 to q respectively.

Now take two straight lines OX and OY at right angles to each other. Cut off segments equal to divisors of m and n along OX and OY respectively, starting from O, both in the ascending order of magnitude. Clearly we can write all the divisors of mn as shown in the following scheme.

O	a_1	a_2	a_3	$\cdots$	a_i	$\cdots$	a_p X
b_1	a_1b_1	a_2b_1	a_3b_1	$\cdots$	a_ib_1	$\cdots$	a_pb_1
b_2	a_1b_2	a_2b_2	a_3b_2	$\cdots$	a_ib_2	$\cdots$	a_pb_2
b_3	a_1b_3	a_2b_3	a_3b_3	$\cdots$	a_ib_3	$\cdots$	a_pb_3
	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
b_j	a_1b_j	a_2b_j	a_3b_j	$\cdots$	a_ib_j	$\cdots$	a_pb_j
	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
b_q	a_1b_q	a_2b_q	a_3b_q	$\cdots$	a_ib_q	$\cdots$	a_pb_q
Y							Z

Clearly all figures are rectangles except the figured designated a_1b_1 which is a square. Therefore the area of any figure is of the form a_ib_j.

In other words a divisor of mn represents the area of some figure in the above scheme.

We calculate in two different ways the area of the whole figure $OXZY$.

In the first place the area $= \sum a_ib_j = \sum_{d|mn} d$.

In the second place the area $= OX \cdot OY = \sum_{d|m} d \cdot \sum_{d|n} d$.

Hence the theorem holds.

ON A THEOREM OF GRIFFITHS

R. GOORMAGHTIGH, Bruges, Belgium

In his note, this MONTHLY, vol. 54, 1947, p. 538, L. Droussent recalls that Griffiths has established that the Euler circle, circumcircle, orthocentroidal circle, and the orthoptic circle of the Steiner ellipse, of a triangle have the orthic axis as common radical axis; he further remarks, p. 540, that Griffiths' theorem

equally well could have included also the polar circle.

But Gallatly, *The modern geometry of the triangle*, p. 41, had already mentioned that the polar circle and also the circumcircle of the tangential triangle belong to the "Griffiths pencil."

So, in Droussent's theorem, the orthocentroidal circles may also be replaced by the circumcircles of the tangential triangles.

On Griffiths' theorem and the curious properties of the intersection points of the circles of the Griffiths pencil, besides the already mentioned references, see: R. Goomaghtigh, *Mathesis*, 1923, p. 475, 1933, p. 393, 1945–1946, p. 217; M. Mineur, *ibid.* 1933, p. 408, 1945–1946, p. 260; R. Deaux, *ibid.*, 1945–1946, p. 200; V. Thébault, *ibid.*, 1945–1946, p. 250.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College and Institute for Advanced Study

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AN ALGORITHM FOR THE DETERMINATION OF THE CONSTANTS IN THE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS

D. W. HALL, University of Maryland and David W. Taylor Model Basin

Suppose we are given the problem of finding the particular solution of the linear differential equation

$$G(D)x(t) = (g_0D^n + g_1D^{n-1} + \cdots + g_n)x(t) = 0, \tag{1}$$

where the coefficients are constant ($g_0 \neq 0$) and the solution is required to satisfy the conditions $x^{(i)}(b) = a_i$, $(i = 0, 1, 2, \cdots, n-1)$, $x^{(i)}(t)$ denoting the ith derivative of $x(t)$.

This problem necessitates the solution of the following system of linear equations:

$$\begin{aligned} C_1e^{r_1b} + \quad C_2e^{r_2b} + \cdots + \quad C_nr^{r_nb} &= a_0 \\ r_1C_1e^{r_1b} + \quad r_2C_2e^{r_2b} + \cdots + \quad r_nC_ne^{r_nb} &= a_1 \\ &\vdots \\ r_1^{n-1}C_1e^{r_1b} + r_2^{n-1}C_2e^{r_2b} + \cdots + r_n^{n-1}C_ne^{r_nb} &= a_{n-1}, \end{aligned} \tag{2}$$

where $r_1, r_2, \cdots, r_n$ are the roots (assumed to be distinct) of the characteristic equation

$$G(r) = g_0(r - r_1)(r - r_2) \cdots (r - r_n) = 0. \tag{3}$$

The system of equations (2) may be easily and quickly solved by means of the following algorithm. Neglecting the constant term of $G(r)$ we write its remaining coefficients in a row and the initial conditions in a column to the right. Columns of products are then formed and added to give the sums $f_0, f_1, \cdots, f_{n-1}$.

$$\begin{array}{rrrr|l}
g_0+ & g_1+ & g_2+\cdots+ & g_{n-1} & \\
\hline
a_0g_0+ & a_0g_1+ & a_0g_2+\cdots+ & a_0g_{n-1} & a_0 \\
 & +a_1g_0+ & a_1g_1+\cdots+ & a_1g_{n-2} & a_1 \\
 & & +a_2g_0+\cdots+ & a_2g_{n-3} & a_2 \\
 & & \vdots & & \\
 & & & +a_{n-1}g_0 & a_{n-1} \\
\hline
f_0+ & f_1+ & f_2+\cdots+ & f_{n-1} &
\end{array}$$

This defines a polynomial

$$F(r) = f_0r^{n-1} + f_1r^{n-2} + \cdots + f_{n-1}, \tag{4}$$

and the solutions of (2) are written down at once as

$$C_h = e^{-r_hb}F(r_h)/G'(r_h),\ (h=1, 2, \cdots, n). \tag{5}$$

Before proving (5) we give two examples to illustrate the use of the algorithm.

EXAMPLE I: Solve $(D^4-17D^3+101D^2-247D+210)x(t)=0$, where $x(0)=1$, $x'(0)=2$, $x''(0)=3$, $x^{(3)}(0)=4$.

We have at once

$$G(r) = r^4 - 17r^3 + 101r^2 - 247r + 210 = (r-2)(r-3)(r-5)(r-7),$$

$$G'(2) = (2-3)(2-5)(2-7) = -15, G'(3) = +8, G'(5) = -12, G'(7) = +40.$$

$$\begin{array}{rrrr|l}
1 & -17 & +101 & -247 & \\
\hline
1 & -17 & +101 & -247 & 1 \\
 & +2 & -34 & +202 & 2 \\
 & & +3 & -51 & 3 \\
 & & & +4 & 4 \\
\hline
1 & -15 & +70 & -92 & 2 \\
 & +2 & -26 & +88 & \\
\hline
1 & -13 & +44 & -4 &
\end{array}
\qquad
\begin{array}{rrrr|l}
1 & -15 & +70 & -92 & 3 \\
 & +3 & -36 & +102 & \\
\hline
1 & -12 & +34 & +10 & \\
 & & & & \\
1 & -15 & +70 & -92 & 5 \\
 & +5 & -50 & +100 & \\
\hline
1 & -10 & +20 & +8 &
\end{array}$$

$$\begin{array}{rrrr|l}
1 & -15 & +70 & -92 & 7 \\
 & +7 & -56 & +98 & \\
\hline
1 & -8 & +14 & +6 &
\end{array}$$

$$x(t) = 4e^{2t}/15 + 5e^{3t}/4 + 3e^{7t}/20 - 2e^{5t}/3.$$

EXAMPLE II: Solve

$$G(D)x(t) = (D^n + g_1D^{n-1} + \cdots + g_n)x(t) = f(t),$$

subject to the conditions

$$x^{(i)}(0) = a_i, \qquad (i = 0, 1, 2, \cdots, n-1).$$

We first apply our algorithm to obtain a particular solution $x_1(t)$ of the homogeneous equation satisfying the initial conditions $x_1^{(i)}(0) = 0, (i = 0, 1, 2, \cdots, n-2)$ $x_1^{(n-1)}(0) = 1$. This is trivially easy from the algorithm since $F(r) = 1$. Thus

$$x_1(t) = \sum_{i=1}^{n} \frac{e^{r_i t}}{G'(r_i)}\cdot$$

It is then a known theorem that the function

$$x_2(t) = \sum_{i=1}^{n} \int_0^t \frac{f(\sigma)e^{r_i(t-\sigma)}}{G'(r_i)} d\sigma$$

is a particular solution of the given differential equation, having the property that $x_2^{(i)}(0) = 0$, $(i = 0, 1, 2, \cdots, n-1)$. Thus to obtain the desired solution to our problem we need only add to $x_2(t)$ the solution $x_3(t)$ of the homogeneous equation satisfying the given initial conditions. This can be written down at once by the algorithm and the sum of $x_2(t)$ and $x_3(t)$ will be recognized immediately as that given by the Heaviside method. Appropriate modifications may of course be made if the integrand "goes bad" at the point $\sigma = 0$.

We now turn to the justification of the algorithm. It is evidently sufficient to show that it holds for C_1, by the symmetry of the problem. The function $F(r_1)$ may be written in the form

$$F(r_1) = a_0\overset{1}{A}_0 + a_1\overset{1}{A}_1 + a_2\overset{1}{A}_2 + \cdots + a_{n-1}\overset{1}{A}_{n-1} \tag{6}$$

where

$$\begin{aligned} \overset{1}{A}_0 &= g_0 r_1^{n-1} + g_1 r_1^{n-2} + \cdots + g_{n-1} \\ \overset{1}{A}_1 &= g_0 r_1^{n-2} + g_1 r_1^{n-3} + \cdots + g_{n-2} \\ \overset{1}{A}_2 &= g_0 r_1^{n-3} + g_1 r_1^{n-4} + \cdots + g_{n-3} \\ &\vdots \\ \overset{1}{A}_{n-1} &= g_0. \end{aligned} \tag{7}$$

These quantities are recognized at once as the coefficients of the polynomial $G(r)/(r-r_1)$. Thus we may write

$$G(r)/(r - r_1) = \overset{1}{A}_{n-1}r^{n-1} + \overset{1}{A}_{n-2}r^{n-2} + \cdots + \overset{1}{A}_0,$$

or, since r_1 is a root of $G(r)=0$,

$$G'(r_1) = A^{1}_{n-1}r_1^{n-1} + A^{1}_{n-2}r_1^{n-2} + \cdots + A^{1}_{0}. \tag{8}$$

In solving the system (2) by Cramer's rule we note that the denominator is a determinant having the value

$$H = \prod_{i<j,i,j=1}^{n} (r_i - r_j) = G'(r_1) \prod_{i<j,i,j=2}^{n} (r_i - r_j). \tag{9}$$

The numerator, when we solve for $C_1e^{r_1b}$ will be obtained by replacing the first column of H by the right members of (2). This means that the coefficient of a_j, $(j=0, 1, 2, \cdots, n-1)$ will be precisely the product of the coefficient of r_1^j in $G'(r_1)$, namely A_j^1, and the troublesome looking product

$$\prod_{i<j,i,j=2}^{n} (r_i - r_j).$$

This product, however, is a factor of H, by (9), and divides out when we take the quotient by Cramer's rule to obtain the exact expression given by the algorithm.

DOUBLE FACTORIALS

B. E. Meserve, University of Illinois

The double factorial notation

$$(2n)!! = 2\cdot 4\cdot 6 \cdots (2n-2)(2n) = 2^n n!$$

$$(2n+1)!! = 1\cdot 3\cdot 5 \cdots (2n-1)(2n+1) = \frac{(2n+1)!}{2^n n!}$$

may be considered as a generalization of $n! = 1\cdot 2\cdot 3 \cdots n$. It was devised to simplify the expression of the Wallis Formulas:

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \frac{(n-1)!!}{n!!}\cdot\frac{\pi}{2},$$

if n is even;

$$= \frac{(n-1)!!}{n!!},$$

if n is odd;

$$\int_0^{\pi/2} \sin^n x \cos^m x\,dx = \frac{(n-1)!!(m-1)!!}{(m+n)!!}\cdot\frac{\pi}{2},$$

if m, n are both even;

$$= \frac{(n-1)!!(m-1)!!}{(m+n)!!}$$

otherwise. These formulas are easily remembered and taken with the usual trigonometric substitutions are valuable time savers in many calculus problems.

Double factorials are also useful along with other multiple factorials in expressing the general terms of many series. In particular, double factorials may be used in the expression for $\Gamma(2k+1/2)$ where k is any integer and, therefore, in the Bessel functions of the first kind $J_{k+1/2}(x)$.

A REJOINDER

A. B. FARNELL, University of Colorado

A note by the author on the teaching of differential equations (this MONTHLY, vol. 54 (1947), pp. 160–162) was attacked by R. A. Johnson (this MONTHLY, vol. 54 (1947), pp. 410–411).

In my note I attempted to unify the treatment of the non-homogeneous linear differential equation with constant coefficients, using the theorem (Ince, *Ordinary Differential Equations*, Dover, pp. 73–75) that the linear differential equation

$$L\{y\} \equiv p_0(x)\frac{d^ny}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}(x)\frac{dy}{dx} + p_n(x)y = r(x)$$

admits, under the assumption that $p_0(x), p_1(x), \cdots, p_n(x)$ and $r(x)$ are continuous functions of x in the interval $a \leqq x \leqq b$ and $p_0(x)$ does not vanish at any point of that interval, a unique solution which, together with its first $(n-1)$ derivatives, is continuous in (a, b) and satisfies the following initial conditions:

$$y(x_0) = y_0,\ y'(x_0) = y_0',\ \cdots,\ y^{(n-1)}(x_0) = y_0^{(n-1)},$$

where x_0 is a point of (a, b). If the coefficients depend upon a real parameter λ, and are continuous for all values of x in (a, b) where λ ranges between λ_1 and λ_2, then $y(x)$ can be proved to depend continuously on λ when λ lies within a closed interval interior to (λ_1, λ_2).

Against this the following objections were raised: first, that the general solution is put into a peculiar form, and, second, that the latter part of the theorem mentioned above has not been proved at the time linear equations are first encountered.

I restrict myself to the following justification of my suggestion: The first part of the theorem mentioned above is certainly basic in any course in differential equations whether proven or not. Further, it is necessary to use it at an early stage in such a course whether or not we like to do it. Other cases of omission of proof can usually be cited. (The same practice is of course followed in almost all mathematical subjects. It is obligatory where some idea of methods of solving problems is part of the goal, and is certainly justified at times in all courses except perhaps in advanced graduate courses.) For example, in the case where the $p_j(x)$'s are constants and we obtain by one scheme or another a solution containing n arbitrary constants, it is usually relegated to the "obvious" that we have obtained a general solution. Lastly, this existence theorem dictates a special form for the solution whether or not we choose to write it in that form.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 826. *Proposed by C. S. Ogilvy, Trinity College*

Find the equation of the ellipse with foci at $(-1, 0)$ and $(1, 0)$ and with semiperimeter equal to the length of one arch of the sine curve, $y=\sin x$.

E 827. *Proposed by Leo Moser, University of Manitoba*

Show that the reciprocal of every integer greater than 1 is the sum of a finite number of consecutive terms of the infinite series

$$\sum_{j=1}^{\infty} 1/j(j+1).$$

E 828. *Proposed by Fritz Herzog, Michigan State College*

For positive integral x, let $P(x)$ denote the number of distinct primitive Pythagorean triangles of perimeter x. Show that $P(x)$ is unbounded; in fact, for any given non-negative integer r, the equation $P(x)=r$ has an infinitude of solutions. (Compare Problem E 812 [1948, 248].)

E 829. *Proposed by S. H. Gould, Purdue University*

Let $f(x)$ be defined in a closed interval $[a, b]$ and have the property of assuming, in any subinterval $[c, d]$, every value between $f(c)$ and $f(d)$. Prove that $f(x)$ is continuous if no rational value is infinitely often assumed.

E 830. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The six planes bisecting the adjacent dihedral angles around the base of a tetrahedron, taken four by four, form fifteen tetrahedra circumscribed about a common paraboloid of revolution.

SOLUTIONS

The Centers of Three Circles

E 539 [1942, 546]. *Proposed by Howard Eves, Oregon State College*

Give a ruler construction for finding the centers of three given linearly independent circles, no two of which are intersecting, tangent, or concentric.

(For an elucidation by the proposer, including references, see [1943, 388].)

Solution by P. H. Daus, University of California. A solution of the problem can be made to depend upon the following two ruler constructions.

CONSTRUCTION I. It is possible to construct pointwise the circle of a coaxal system through a given point.

The circles of the system determine on an arbitrary line through the point an involution. The point corresponding in the involution to the given point lies on the required circle. The involution is determined by the pairs of points in which the arbitrary line cuts the given circles and can be constructed by ruler alone. The difficulty introduced by the fact that there might be no line through the given point which cuts both circles may be avoided by first constructing pointwise a circle of the system through a point inside one circle.

CONSTRUCTION II. If one point of intersection of two circles is known, it is possible to construct pointwise a circle passing through the other (known or unknown) point of intersection.

This is accomplished by considering a variable line through the given point, and projecting its intersections with the given circles through two fixed points, one on each circle, respectively. The resulting figure for one such line is the Miquel configuration.

In terms of these constructions, the Cauer-Mierendorff solution can be reduced to the case of two intersecting circles by means of five circles constructed pointwise. A similar solution, for which there are a number of variants, can be made by means of the following four circles constructed pointwise.

Let the given non-interesting circles be c_1, c_2, c_3.

(1) Through any point A on c_1 construct the circle a coaxal with c_2 and c_3 (by I). Let the unknown intersection of a and c_1 be X. (Since the three given circles are linearly independent, a is distinct from c_1.)

(2) Through a second point B on c_1 construct the circle b coaxal with c_2 and c_3 (by I). Let the unknown intersection of b and c_1 be Y.

(3) Through any point C on b construct the circle ACX (by I).

(4) Construct the circle ACY (by II), using the pencil of lines at B. These last two circles have two known points in common.

It may be remarked here that the details of the construction (see H. Steinhaus, *Mathematical Snapshots*) when given two intersecting circles may be simplified by first applying Construction II.

An entirely different type of construction for this case than that given in the above reference is as follows. If K is a point on the radical axis of two intersecting circles c_1 and c_2, the polars of K with respect to these circles cut them, respectively, in four points on a circle k of their conjugate coaxal system. The polars of one such point with respect to c_1 and c_2 meet in a fifth point, diametrically opposite this point. The circle k is thus determined pointwise. The tangents to k (constructed by Pascal's theorem) at these four points determine the required centers.

An attempt to use these ideas directly for the case of three nonintersecting circles has not been successful.

Editorial Note. The result of this problem, coupled with the Poncelet-Steiner theorem, gives us the theorem: Any euclidean construction may be accomplished with ruler alone in the presence of three linearly independent circles; if two of the circles intersect, are tangent, or are concentric, then the third circle is redundant.

We state here, without proof, some other theorems of this character.

Any euclidean construction may be accomplished with ruler alone in the presence of (1) two circles and a point on their radical axis, (2) two circles and a center of similitude, (3) one circle and a parallelogram.

The Mathematician and the Jester

E 791 [1947, 545]. *Proposed by G. W. Walker, Buffalo, N. Y.*

The court mathematician once received his salary for a year's service all at one time, and all in silver "dollars," which he proceeded to arrange in nine unequal piles, making a magic square. The king looked, and admired, but complained that there was not a single prime number in any of the piles. "If I had but nine coins more," said the mathematician, "I could add one coin to each pile and make a magic square with every number prime." They investigated, and found that this was indeed true. The king was about to give him nine "dollars" more, when the court jester said, "Wait!". Then the jester subtracted one coin from each pile instead; and they found in this case also a magic square with every element a prime number. The jester kept the nine "dollars." How much salary must the mathematician have been receiving?

Solution by C. W. Trigg, Los Angeles City College. It is desired to form a magic square of order three from the set of primes, p_i, having the property that p_i+2 is also prime. There are 61 members of the set less than 2,000. The search is narrowed by use of the following properties:

(1) There are four conjugate pairs having the same sum.

(2) The central element, E, equals one-half each conjugate sum, *i.e.*, one-third the constant.

(3) Unless 3 or 5 appears as an element, the p_i must terminate with 1, 7, or 9.

(4) The sums of the unit's digits of the elements in each column, row, and diagonal must have the same unit's digit.

(5) Properties (3) and (4) require that all elements have the same unit's digit, or that their unit's digits form the configuration

$$\begin{array}{ccc} 1 & 7 & 9 \\ 7 & 9 & 1 \\ 9 & 1 & 7 \end{array}$$

or a rotation or reflection thereof.

Within the set there are ten values of $2E$, the conjugate sum, which meet

these properties, namely: 298, 838, 1318, 1618, 1762, 2038 2098, 2122, 2182, and 2638. The first of these is the only one for which the sum of every outside column and row is $3E$. Thus the three magic squares involved are:

191	17	239	192	18	240	193	19	241
197	149	101	198	150	102	199	151	103
59	281	107	60	282	108	61	283	109

The mathematician's salary was $9E+9=9(149+1)$ or 1350 "dollars."

Remarks by Leo Moser, University of Manitoba. I would conjecture that the number of solutions is infinite, and the number of solutions with largest entry under n is $O(n/(\log n)^{18})$. This estimate is based on certain conjectural formulae of Hardy, Littlewood, and Lord Cherwell (Quarterly Journal of Math., Vol. 17, pp. 47–62, *Note on the Distribution of Intervals between PrimeNumbers*). I would also conjecture that there exist magic squares of this type of any order not less than three. An example of such a square of order four is

29	1061	179	227
269	137	1019	71
1049	101	239	107
149	197	59	1091

Of course, proving these conjectures would definitely not be an "elementary" problem, as they would imply the existence of an infinity of prime pairs, proof of which is one of the outstanding unsolved problems of number theory.

Also solved by Josephine and Richard Andree, A. S. Anema, W. E. Buker, Joseph Clare, Monte Dernham, R. L. Greene, Frank Herlihy, Roger Lessard, Leo Moser, Bart Park, S. T. Parker, and the proposer.

Buker called attention to the discussion of magic squares, whose elements are primes, on p. 125 of Dudney's *Amusements in Mathematics.*

Lessard claimed that the next solution to the given problem is

6198	5502	3030
2142	5010	7878
6690	4518	3822

In this case the mathematician's salary would have been 45090 "dollars"!

Shuffling Cards

E 792 [1947, 545]. *Proposed by N. S. Mendelsohn, University of Manitoba*

Show that a pack of $2n$ cards is brought back to its original position in at most $2n-2$ perfect riffle shuffles. (A perfect riffle shuffle is one which sends the cards from the arrangement $1, 2, 3, 4, \cdots, 2n-1, 2n$ to the arrangement $1, n+1, 2, n+2, \cdots, n, 2n$.)

Solution by the Proposer. Let $m=2n-1$. It is obvious that a perfect riffle shuffle sends the card in position x to position y where $y\equiv 2x-1 \pmod m$. After k shuffles the card in position originally x moves to position $2^k x-(2^k-1) \pmod m$. If after k shuffles each card is in its original position, then $2^k x-(2^k-1) \equiv x \pmod m$ for all x. Hence $(2^k-1)(x-1)\equiv 0 \pmod m$ for all x or $2^k-1\equiv 0 \pmod m$. Hence the least number of shuffles needed to bring the pack to its original position is the minimum $k>0$ for which $2^k-1\equiv 0 \pmod m$. Since m is odd, $(2\ m)=1$, whence $2^{\phi(m)}-1\equiv 0 \pmod m$. Therefore, the minimum $k<\phi(m) \leq m-1=2n-2$. Note that for a pack of 52 cards we have $k=8$, a remarkably small number.

Also solved by J. B. Kelly, Roger Lessard, and Leo Moser. Moser pointed out that this problem is essentially the same as that solved on pp. 244–245 of *Elementary Theory of Numbers* by Uspensky and Heaslet. The riffle shuffle on $2n$ cards leaves the top and bottom cards fixed and shuffles the remaining cards in the manner there described. The number of shuffles required to regain the original is therefore the exponent to which 2 belongs mod $(2n-1)$. This is a divisor of $\phi(2n-1)$ and hence not greater than $2n-2$.

Irving Kaplansky stated that the problem is treated in *Math. Gazette*, vol. 15 (1930), pp. 17–20.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4305. *Proposed by H. F. Sandham, Trinity College, Ireland*

Prove that

$$1+\left(\frac{1+\frac{1}{2}}{2}\right)^2+\left(\frac{1+\frac{1}{2}+\frac{1}{3}}{3}\right)^2+\left(\frac{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{4}\right)^2+\cdots=\frac{17\pi^4}{360}.$$

4306. *Proposed by Paul Erdös, Syracuse University*

Let there be given eight arbitrary points in space. Prove that one can always select three of them which do not form an acute-angled triangle.

4307. *Proposed by Emma Lehmer, Berkeley, California*

Let p be a prime number of the form $p=3\cdot 2^{\alpha}k+1$, and let n be any number not divisible by 3 or 2^{α}. Show that there are numbers a, b, independent of n, such that a, b, $a+1$, $b+1$ are nth power residues modulo p.

4308. *Proposed by H. S. Wall, University of Texas*

Let $f(z)$ be a function of the complex variable z, such that

$$f(-w^2) = -\frac{F(w)}{w},$$

where $F(w)$ has the properties: (a) $F(w)$ is an odd function of w, (b) $F(w)$ is analytic and $I[F(w)]\leqq 0$ for $I(w)>0$. Show that $I[f(z)]\leqq 0$ for $I(z)>0$.

4309. *Proposed by R. Goormaghtigh, Bruges, Belgium*

Let ABC be a triangle, D, E, F the contact points of one of the tritangent circles with BC, CA, AB, respectively, and let A', B', C' and D', E', F' be the projections of a point M of that circle on BC, CA, AB and EF, FD, DE. Show that the lines joining the projections of M on $A'E'$ and $A'F'$, $B'F'$ and $B'D'$, $C'D'$ and $C'E'$ are concurrent.

SOLUTIONS

Five Parallel Forces in Equilibrium

4080 [1943, 264]. *Proposed by Victor Thêbault, Tennie, Sarthe, France*

Given five points in space, if five parallel forces applied at these points are in equilibrium, then equilibrium results also when each force is translated to the center of the circumsphere of the tetrahedron which has for vertices the points of application of the four other forces.

Note. The similar problem for four points in a plane was proposed by J. Neuberg, *Educational Times, Reprints*, 1891, v. 55, p. 82.

*Solution by R. Blanchard, Le Havre, France.** Let us demonstrate first the following theorem: *In a tetrahedron the barycentric coördinates of a point with respect to the pedal tetrahedron of this point are the same as the barycentric coördinates of its isogonal conjugate with respect to the given tetrahedron.*

Consider a tetrahedron $ABCD$, two points P_1 and P_2 isogonally conjugate with respect to it, and $A_1B_1C_1D_1$ the pedal tetrahedron of P_1, and let x, y, z, t be the normal coördinates of P_1 with respect to $ABCD$. The barycentric coördinates of P_1 with respect to $A_1B_1C_1D_1$ are proportional to $yzt\,\sigma_a, \cdots$, where $\sigma_a, \cdots$, designate the sines [1] of trihedral angles supplementary to the trihedral angles of $ABCD$. The normal coördinates of P_2 with respect to $ABCD$ are proportional to $yzt, \cdots$, and its barycentric coördinates are proportional to $S_a yzt, \cdots$, where $S_a, \cdots$, designate the areas of the faces of $ABCD$. The

* Translated and checked by W. E. Byrne, Virginia Military Institute.

theorem follows from the proportionality of $\sigma_a, \cdots$, and $S_a, \cdots$, (Dostor's relation).

Now let us designate by A, B, C, D, E the five given points and by $0, 0_a, 0_b, 0_c, 0_d$ the circumcenters of the tetrahedra $ABCD$, $EBCD$, $ECDA$, $EDAB$, $EABC$. The problem is equivalent to that of proving that E has the same barycentric coördinates with respect to $ABCD$, that O has with respect to $O_aO_bO_cO_d$ [2]. Let $A'B'C'D'$ be the antipedal tetrahedron [3] of E with respect to $ABCD$ and E' the isogonal conjugate of E with respect to $A'B'C'D'$ [4]. On the one hand E' has the same barycentric coördinates with respect to $A'B'C'D'$ that E has with respect to $ABCD$. On the other hand, O_a, O_b, O_c, O_d, O are the transforms of A', B', C', D', E' under the homothety $(E, \frac{1}{2})$. We conclude that O has the same barycentric coördinates with respect to $O_aO_bO_cO_d$ as E has with respect to $ABCD$.

J. Neuberg gave the same generalization of his proposition concerning four coplanar points. See *Educational Times, Reprints*, v. 54, 1891, p. 30.

Notes by the Translator: 1. The sine of a trihedral angle is the triple scalar product of the three unit vectors laid off from the vertex along the three edges.

2. For an explanation of the theorem of statistics attached to the barycentric coördinates of Möbius, see Darboux, *Principes de Géométrie Analytique*, Ch. II.

3. The antipedal tetrahedron of E with respect to $ABCD$ is the tetrahedron formed by planes drawn through A, B, C, D perpendicular to EA, EB, EC, ED, respectively.

4. If two points are isogonally conjugate with respect to a tetrahedron, they have the same pedal sphere, whose center bisects the line segment joining them. Hence O is the midpoint of EE'. See Coolidge, *A Treatise on the Circle and the Sphere*, p. 233.

An Inequality Connected with n!

4226 [1946, 594]. *Proposed by Paul Erdös, Syracuse University*

Let $n! = \prod_{p \leqq n} p^{k_p}$ (p, primes). Assume that $p_1 > p_2$ and $k_{p_1} < k_{p_2}$. Prove that $p_1^{k_{p_1}} < p_2^{k_{p_2}}$.

Solution by W. J. Harrington, Pennsylvania State College. The exponent of the highest power of a prime p contained in $n!$ is given by

$$\nu(n) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots, \tag{1}$$

where $[Q]$ denotes the greatest integer $\leqq Q$.* The following well known properties of $\nu(n)$ will facilitate the solution

$$\nu(n) < n \sum_{j=1}^{\infty} 1/p^j = n/(p-1). \tag{2}$$

$$\nu(n_1 + n_2) \geqq \nu(n_1) + \nu(n_2). \tag{3}$$

* Uspensky and Heaslet, Elementary Number Theory, pp. 99–100.

For positive integers m and c,

$$\nu(mp) = m + \nu(m), \tag{4}$$

and

$$\nu(cn) \geqq c\nu(n). \tag{5}$$

If m is defined by

$$mp \leqq n < (m+1)p,$$

m being a positive integer, it follows from (4) that

$$\nu(n) \geqq \nu(mp) = m + \nu(m).$$

If $m \geqq p$, then $\nu(m) \geqq 1$ and $\nu(n) \geqq m+1$. If $1 \leqq m \leqq p-1$, then $\nu(n) \geqq m$. Since in each case $n \leqq (m+1)p-1$, we have

$$\nu(n)/n > 1/p, \qquad n \geqq p^2, \tag{6}$$

$$\nu(n)/n \geqq m/[(m+1)p-1] \geqq 1/(2p-1), \qquad n < p^2. \tag{7}$$

To handle the proposed problem, let

$$p_1 = cp_2 + b, \tag{8}$$

where $c \geqq 1$ and $1 \leqq b \leqq p_2 - 1$. We shall write k_i for k_{p_i} and $\nu_i(n)$ for $\nu(n)$ relative to the prime p_i. From (2) we have

$$n > k_1(p_1 - 1), \tag{9}$$

and thus from (3), (4) and (8) we have

$$k_2 = \nu_2(n) \geqq \nu_2(k_1cp_2 + k_1b - k_1) \geqq k_1c + \nu_2(k_1c) + \nu_2(k_1b - k_1). \tag{10}$$

Consider separately the two cases $c \geqq 2$ and $c = 1$.

Case I. $c \geqq 2$.

From (10) we have $k_2 > k_1c$, and thus $p_2^{k_2} \geqq (p_2^c)^{k_1}$, whence our desired result is established if $p_2^c > p_1$. From (8), $p_1 \leqq (c+1)p_2 - 1$, and hence we consider the inequality.

$$p_2^c > (c+1)p_2 - 1. \tag{11}$$

For all $p_2 \geqq 3$ with $c \geqq 2$ and for all $p_2 \geqq 2$ with $c \geqq 3$, (11) holds and the problem is solved. The special case $p_2 = c_2 = 2$ requires $p_1 = 5$ by (8). From (9) and (10),

$$k_2 \geqq 2k_1 + \nu_2(2k_1) \geqq 2k_1 + k_1 = 3k_1$$

and thus

$$p_2^{k_2} \geqq (p_2^3)^{k_1} = 8^{k_1} > p_1^{k_1}.$$

Case II. $c = 1$.

We notice that the problem is equivalent to that of showing

$$\log p_1/\log p_2 < k_2/k_1. \tag{12}$$

From (8) with $c=1$, and (12) it follows that it will be sufficient to show

$$k_2/k_1 \geqq 1 + b/p_2 \log p_2, \tag{13}$$

which, with the aid of (10) and (5), becomes

$$k_2/k_1 \geqq 1 + b\nu_2(k_1)/k_1. \tag{14}$$

The investigation is continued by considering separately $k_1 \geqq p_2^2$, $p_2 \leqq k_1 < p_2^2$, and $k_1 < p_2$.

Case II(a). $c=1$, $k_1 \geqq p_2^2$.

From (6) and (14), (13) can be established for $p_2 \geqq 3$. For the remaining case, $p_2=2$, write $k_1 = 2h_1 + h_0$ where $h_1 \geqq 2$ and $h_0 = 0$ or 1. The proof of (12) for the case $h_1=2$, $h_1=3$, $h_1 \geqq 4$ presents no difficulty.

Case II(b). $c=1$, $p_2 \leqq k_1 < p_2^2$.

Take $mp_2 \leqq k_1 < (m+1)p_2$ with $1 \leqq m \leqq p_2 - 1$. From (7) and (14),

$$k_2/k_1 \geqq 1 + mb/[(m+1)p_2 - 1] \geqq 1 + b/(2p_2 - 1).$$

This establishes (13) for $p_2 \geqq 7$ and for $p_2 \geqq 5$ with $m \geqq 2$. Special cases remaining are

$$p_2 = 5, \quad m = 1; \qquad p_2 = 3, \quad m = 1 \text{ or } 2; \qquad p_2 = 2, \quad m = 1,$$

for each of which there is no difficulty in establishing (12).

Case II(c). $c=1$, $k_1 < p_2$.

Here $n \geqq k_1 p_1 = k_1 p_2 + k_1 b$, and hence

$$\nu_2(n) \geqq k_1 + \nu_2(k_1 b).$$

For $k_1 b \geqq p_2$, we have by (7)

$$k_2/k_1 \geqq 1 + b\nu_2(k_1 b)/k_1 b \geqq 1 + b/(2p_2 - 1)$$

and (13) is established for $p_2 \geqq 7$. As in case II(b), (12) can readily be established for the special cases $p_2 = 5$, 3, and 2. Also for $k_1 b < p_2$ there is no difficulty in establishing (12).

Developables of a Congruence

4238 [1947, 112] *Proposed by C. E. Springer, University of Oklahoma*

At each point of the revolute

$$x = u \cos v, \qquad y = u \sin v, \qquad z = \phi(u)$$

a straight line generator of a congruence is taken with direction cosines proportional to

$$\cos v, \qquad \sin v, \qquad \phi' - u\phi'',$$

where primes indicate differentiation with respect to u. Show that the developables of the congruence intersect the revolute in its lines of curvature.

Solution by V. G. Grove, Michigan State College. The problem as stated is a special case of a more general one. Let the generators of the congruence have direction cosines proportional to X, Y, Z defined by expressions

$$X = \psi(u, v) \cos v, \qquad Y = \psi(u, v) \sin v, \qquad Z = \theta(u, v).$$

The pairs of functions (X, x), (Y, y), (Z, z) may be shown by elementary methods to be solutions of the following system of differential equations

$$\begin{aligned} X_u &= \overset{1}{M}_1 x_u + \overset{2}{M}_1 x_v + E_1 X, \\ X_v &= \overset{1}{M}_2 x_u + \overset{2}{M}_2 x_v + E_2 X, \end{aligned}$$

wherein M_β^α are defined as follows and E_1, E_2 are not material for our purposes:

$$\begin{aligned} \Delta \overset{1}{M}_1 &= u(\theta\psi_u - \theta_u\psi), \qquad & \Delta \overset{2}{M}_1 &= 0 \\ \Delta \overset{1}{M}_2 &= u(\theta\psi_v - \psi\theta_v), \qquad & \Delta \overset{2}{M}_2 &= +\psi(\theta - \psi\phi') \\ \Delta &= u(\theta - \phi'\psi). \end{aligned}$$

The meridians ($v = \text{const}$) and parallels ($u = \text{const}$) of the revolute are known to be the lines of curvature. Moreover the developables of the congruence intersect the revolute in the parametric curves if and only if $M_2^1 = M_1^2 = 0$. (See V. G. Grove, *A general theory of surfaces and conjugate nets*. Trans. Am. Math. Soc. V. 57 (1945) p. 106.) Hence the congruence we are discussing has the desired property if and only if $M_2^1 = 0$, $\Delta \neq 0$, that is if and only if $\theta = \psi U(u)$. The direction numbers of the generators of the congruence may be written in the form

$$\cos v, \qquad \sin v, \qquad U(u), \qquad U \neq \text{const}, \qquad U \neq \phi',$$

$U(u)$ being otherwise an arbitrary function of u. The function $U = \phi' - u\phi''$ in the problem is a special case.

Also solved by P. D. Thomas, and the Proposer.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Elements of Mathematical Statistics. By C. V. L. Charlier. Translated and published by J. A. Greenwood (25 Winthrop Street, Brooklyn 25, New York), 1947, 120 pages. $3.00.

This translation follows essentially the 1920 edition. Part One, which consists of eight chapters, deals with homograde statistics; that is, with data in which each individual who has been measured with respect to an attribute either possesses (or is assumed to possess) the attribute to the same degree of intensity. Successive chapters treat the arithmetic mean, the dispersion, the mean error, the theorems of Bernoulli, Poisson, and Lexis, and observed statistical data.

Part Two, which consists of seven chapters, is concerned with heterograde statistics; that is, data in which the degree of intensity of a measured attribute is of concern. There are chapters on the Normal, the Type A, and the Type B frequency curves; on correlation; and on certain abridged methods of calculation.

There is an appendix which consists of four sets of tables, a bibliography, and an index. The reader should consult the translator's preface concerning both the tables in the appendix and those in the text.

This reviewer believes the translator should be commended not only for making this monograph more available to beginning students, but also for his work on the tables.

A. T. CRAIG.

College Algebra. By E. R. Heineman. New York, The Macmillan Company, 1947. 9+359 pages. $3.25.

The reviewer holds the opinion that, in undergraduate courses in mathematics, very little, if any, classroom time should be spent in discussion of new material, the student being required to "dig it out" for himself. Especially in elementary courses, such a plan requires the use of a text which is clear in its presentation, and which contains good illustrative examples and apt warnings about common errors. The text under consideration is especially good in this respect.

It is of the standard type of college algebra text, devoting 22 chapters to the topics usually classified as college algebra. Texts covering a variety of topics are often well-done in parts and quite the opposite in other portions. The reviewer was interested to note how consistently he found this text to treat topics in the fashion his experience had taught him to prefer. In only two instances was there disappointment in this regard: (1) In presenting the factoring method of solving equations the following principle *alone* is stated: "If the product of two or more quantities is zero, then at least one of the factors must be zero." In examining sixteen recent texts, the reviewer found three which also confuse the issue by not stating this principle in reverse. (2) In treating multiplication of fractions, illustrations of products are given in which a factor of the numerator of one fraction is cancelled against the factor in the denominator of another, no justification for this being given. Although the text is superior in its treatment of incorrect cancelling in fraction simplification, the student having trouble in this matter will not be helped by these illustrations.

The presentation of conditional equations and identities is the best the reviewer has seen. The treatment of mathematical induction is also outstanding. There are instances throughout the book of helpful cautionings and suggestions which are not often found in texts. Examples of these: The student is warned not to rule out a number as a possible rational root of a depressed equation just because it was found to be a root of the original equation. There is an illustration of the importance of indicating clearly the main division line in certain complex fractions.

Few errors of a typographical nature were noted, the only one of any consequence being the omission of the words "of the logarithm" in the theorem giving the characteristic of a logarithm.

This book will rank along with the author's *Plane Trigonometry* (McGraw-Hill, 1942) as a successful text.

K. W. WEGNER

NEW BOOKS RECEIVED

Mathematical Table Makers (Scripta Mathematica Studies, No. 3). By R. C. Archibald. New York, Scripta Mathematica, 1948. 82 pages. $2.00.

Modern-School Geometry. New edition. By R. Schorling, J. R. Clark, and R. R. Smith. Yonkers, New York, World Book Co., 1948. 12+436 pages. $1.88.

Tables of the Bessel Functions of the First Kind of Orders Sixteen through Twenty-seven. Prepared by the Staff of the Computation Laboratory of Harvard University. Cambridge, Harvard University Press, 1948. 764 pages. $10.00.

The Works of the Mind. By R. B. Heywood. Chicago, University of Chicago Press, 1947. 11+246 pages. $4.00.

Methods of Algebraic Geometry. Vol. I. By W. V. D. Hodge and D. Pedoe. Cambridge, at the University Press; New York, The Macmillan Company, 1947. 8+440 pages. $6.50.

Basic Mathematics: A Workbook. By M. W. Keller and J. H. Zant. Boston, Houghton-Mifflin Co., 1948. 4+253 pages. $1.50.

Algebre et Analyse Lineaires. By A. Lichnerowicz. Paris, Masson, 1947. 316 pages. 800 fr.

Analytic Geometry. Fourth Edition. By C. E. Love. New York, The Macmillan Co., 1948. 11+306 pages. $3.50.

Fundamentals of Statistics. By J. B. Scarborough and R. W. Wagner. Boston, Ginn and Co., 1948. 7+145 pages. $2.40.

Projective and Analytical Geometry. By J. A. Todd. New York, Pitman Publishing Co., 1948. 10+289 pages. $4.50.

Analytic Geometry. Revised Edition. By Roscoe Woods. New York, The Macmillan Co., 1948. 14+322 pages. $3.50.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

Summary of Papers Reported by Clubs

A study was made of the papers presented at mathematics club meetings, as reported in this MONTHLY, for the period from January, 1943, to December, 1947, inclusive. A total of 652 papers was reported for that period with 183 somewhat different titles. It was deemed inadvisable to record here all of the different titles reported, so that, for summary purposes, many were grouped together under related headings. The number of papers reported on some of the most popular topics include:

Men and women in mathematics	55
Mathematical recreations	54
Astronomy, meteorology, navigation and applications	46
History and teaching of mathematics	36
Mathematical systems	35
Application of mathematics to war	35
Calculating machines and other aids	30
Topics related to probability and finite differences	29
Various geometries	23
Geometric figures and constructions	23
Discussion of various curves	22
Statistical and financial applications	21
Greek contributions to mathematics	21
Topological aspects	20
Modern algebra	17
Theory of numbers	16
Cryptography	15
Mathematics in arts and sciences	15
Foundations and logical concepts	14
Fourier and other series	12

Club Reports, 1946–47

Pi Mu Epsilon, University of Arkansas

The *Pi Mu Epsilon* Chapter of the University of Arkansas reports that initiation ceremonies were held for both fall and spring semesters of 1946–47. Two formal programs were given; Melvin Lieberstein talked on *Mathematics and logical systems* and D. P. Richardson talked on *The history of the calendar.*

Officers for the year 1946–47 were: President, James L. Fischer; Secretary, Lillia Dewees; Treasurer, Hartman Hotz.

Officers elected for the year 1947–48 are: President, M. A. Lilly; Vice-President, Maclyn McKeehan; Secretary, H. P. Hotz; Treasurer, Webb Henderson.

Professor D. P. Richardson was elected permanent secretary.

Mathematics Club, Connecticut College

The *Mathematics Club* of Connecticut College held four meetings during the school year 1946–47, some of which were recreational as well as educational.

The officers for 1946–47 were: President, Shirley Bodie; Secretary-Treasurer, Edith Techner; Committee Chairmen, Roberta Richards, Virginia Doyle, and Jacqueline Greenblatt.

Officers elected for 1947–48 are: President, Roberta Richards; Secretary-Treasurer, Virginia Doyle; Committee Chairmen, Eleanor Penfield, Jean Webber, and Janet Johnson.

Kappa Mu Epsilon, Kansas (Pittsburg) State Teachers College

The *Kansas Alpha* Chapter of *Kappa Mu Epsilon* conducts open meetings once a month, to which anyone interested is invited. Two of these meetings were picnics, one in October and the other in May. Ten new members were initiated in July, 1946, and twenty-three in February, 1947. This makes a total of 541 members initiated into *Kansas Alpha* Chapter. Topics presented by members were:

Lost and rediscovered mathematics
Unsolved mathematics
Addition-subtraction logarithms
Mathematics in the Navy
Curves used in civil engineering
Ancient methods of calculating the value of π
Modern methods of calculating π
Scientific and mathematical satire in Gulliver's Travels
The number 2 in mathematics
Report of K. M. E. National Convention.

The chapter awards Kappa Mu Epsilon keys annually to the two seniors making the highest grade point average in their mathematics courses. James Standley and Donna Stewart of the class of 1947 were the recipients of these awards.

Officers for 1946–47 are: President, James Standley; Vice-President, Jack Lambert; Secretary, Billie Schultz; Treasurer, Donna Stewart; Corresponding Secretary, Professor J. A. G. Shirk. Professor R. G. Smith, as the new head of the Department of Mathematics, becomes sponsor of the chapter. He succeeds Professor Shirk, who retired as Head of the Department after serving thirty-two years.

Pi Mu Epsilon, Oklahoma A. & M. College

The *Oklahoma Beta* Chapter of *Pi Mu Epsilon* made a very successful attempt in restoring the mathematical interests of students to prewar level.

The annual spring banquet and initiation was held on May 15, at which time fifty-two students and faculty members were initiated. The group was addressed by K. S. Chester, Head of the Botany and Pathology Department and

Director of the College's Research Foundation. His topic was *Measurements, estimates, and guesses.*

The annual *Pi Mu Epsilon* scholarship award to the outstanding freshman in Mathematics was presented at the Honors Day Convocation to Larry Gene Sigler.

During the year the following lectures were given:

Fixed point theorems, by Dr. O. H. Hamilton

Integral distances, by Robert R. Reynolds

Magic squares, by Raymond L. Caskey.

The officers for 1945–47 were: President, Roscoe Jones; Vice-President, Margaret Ann Reiff; Faculty Adviser, Professor J. C. C. McKinsey.

The officers elected for 1947–48 are: President, Robert D. Morrison; Vice-President, Philip J. Miller; Secretary-Treasurer, Helen Dayton; Faculty Adviser, Professor J. C. C. McKinsey.

Graduate Mathematics Club, Indiana University

The *Graduate Mathematics Club* was organized at Indiana University in the fall of 1946. Members of the Mathematics Department gave talks on various fields of mathematics. The topics were:

Mathematical quotations from non-mathematical sources, by M. A. Zorn

Antipodal theorems, by S. Eilenberg

The problem of Kakeya, by J. W. T. Youngs

The billiard ball problem, by D. Gilbarg

The Bernoullian numbers, by H. S. Vandiver

The motion of the perihelion of Mercury, by K. P. Williams.

A picnic was held in a nearby State Park at the beginning of June.

The Executive Committee for the year was composed of Mrs. Lee Pruett, Gordon Overholtzer, and I. M. Herstein.

Delta-X, University of the City of Toledo

In the past year, *Delta*-X has maintained a membership of sixty active members. The annual get-acquainted roast opened the social activities of the club early in September. A Christmas party was combined with the regular December meeting and a pot luck supper was held before the March business meeting. The annual *Delta*-X banquet was held in May at which Professor Edward Ebert spoke on *Inequalities. Delta*-X yearbooks were presented to members at the banquet. The last function of the club was the annual spring picnic which was held in June.

The following papers of interest to students of mathematics were presented at regular monthly meetings:

The geometry of paper folding, by Jane McFillen

The development of the calendar, by Wayne McQuillin

The geometry of soap bubbles, by George Messomore

The theory of probability, by Sam Part

Nomograms, by Richard Reisbach.

Officers for the past year were: President, Lois Zeigler and Mary Novick; Vice-President, Sam Part; Secretary-Treasurer, Shirley Remmert.

Newly elected officers are: President, Fred Miller; Vice-President, Wayne McQuillin; Secretary-Treasurer, Martha Goodwin. Professor Wayne Dancer is adviser of the club.

Pi Mu Epsilon, University of California

At the fall initiation banquet in December 1946, sixteen new members were initiated; all sixteen were students and teaching assistants of the mathematics department.

The spring initiation, held in April, 1947, consisted of a picnic at the John Hinkle Park and evening entertainment in the clubhouse.

During the year, the following mathematical papers were presented at the various *Pi Mu Epsilon* meetings:

Extension of the definition of exponents to irrational numbers, by Bjarni Jonnson

The large computing units of the war, by D. H. Lehmer

Trilinear coordinates of modern geometry, by F. Harary

Survey of elementary potential theory, by Fred Newstadter

Symbolic logic in real variables, by F. Wolf

Cantor's definition of the real numbers in terms of fundamental sequences of rationals, by Alan Davis

Suspension bridge flutter, by Edmund Pinney

Identities in the theory of partitions, by Henry Alder.

The officers for the academic year, 1946–47, were: Director, Henry Alder; Vice-Director, Mrs. Alan Davis; Secretary, Mrs. Jack Gysbers; Treasurer, Hewitt Kenyon; Librarian, Miss Sarah Hallam.

The officers for the year 1947–48 are: Director, Miss Barbara Vernon; Vice-Director, Hewitt Kenyon; Secretary, Miss Edith Wetzel; Treasurer, Ralph Willoughby; Librarian, Miss Sarah Hallam.

Pi Mu Epsilon, Hunter College

The topics considered by the *New York Beta* Chapter of *Pi Mu Epsilon* during the year, 1946–47, were: the theory of higher plane curves, and certain functions of a complex variable and the mapping of the complex plane determined by these functions.

The students who presented papers were Sally Rothstein, Florence Salom, Helvie Thors Impola, Clara Rogovin, Jean Ruderman, Ina Pinchuck, Cecile Salwen, Joyce Marrits, Fay Kleiner, Annette Drucker, Blanche Marten, Mildred Vogt, Anna Ferrara, Florence Myres, Wilhelmina Fluhr, Esther Silver Chad, Roslyn Tarnell, Dorothy Beck, and Marianne Weisz.

Nineteen new members were elected to the Chapter during the year.

At the fall initiation the speaker was Professor Lester Hill of the Hunter College Department of Mathematics who discussed *Teaching mathematics for the United States Army in the Biarritz experiment.*

The guest speaker at the spring initiation was Professor E. R. Lorch of Columbia University whose topic was *The foundations of Euclidian geometry.*

Chapter members participated in the Second Annual Intercollegiate Mathematics convention sponsored by the Brooklyn College, Hunter College, and New York University Chapters of the fraternity. This affair was held on April 19 in Boylan Hall of Brooklyn College. The guest speakers of the evening were P. A. Smith of Columbia University who discussed *Topics in topology* and E. C. Molina of the Newark College of Engineering who spoke on *Applications of probability.*

Prizes for the best papers presented to the society were given to Mary Greene, Cecile Salwen, and Mildred Vogt.

The officers for the year, 1946–47, were: President, Cecile Salwen; Corresponding Secretary, Ruth Friedman; Recording Secretary, Annette Drucker; Treasurer, Mildred Vogt; Director, R. Lucile Anderson.

Newly elected officers are: President, Efrosene Josephides; Corresponding Secretary, Alexa Drexler; Recording Secretary, Marian Boykan; Treasurer, Anna Lemont; Director, Marguerite Darkow.

Mathematics Club, Kansas State College

In addition to two social events, the members of the *Mathematics Club* of Kansas State College heard the following papers discussed:

Methods of graphing, by James Smith

Nomographs, by Robert Kromhout

The operational aspect as the foundation of mathematics, by Elizabeth Button

Topology, by Thirza Mossman and Robert Reinking

Vector methods as applied to trigonometry, by F. B. Sloat

Probability, by Prof. H. C. Fryer.

Officers for the year 1946–47 were: President, Joe Ludlum, Jr.; Vice-President, C. S. Clay; Secretary, Elizabeth Button; Treasurer, Robert Reinking; Program Chairman, Marn Johnson; Sponsor, Prof. C. F. Lewis.

Zeno Club, Alfred University

Among the most interesting papers presented to the *Zeno Club* of Alfred University for 1946–47 were:

Probability in Gin Rummy, a very successful and original paper by John Freund

Some special methods of solving equations, by Dr. A. E. Whitford

Fermat's last theorem, by Marion Miller

Mathematical series, by Frank Olson.

The 1946–47 officers were: President, Marion Miller; Vice-President, Juli-

anne Sanford; Secretary-Treasurer, Joan Berkman.

Those elected for 1947–48 are: President, Ralph Jordan; Vice-President, Joan Berkman; Secretary-Treasurer, Mary van Norman.

Mathematics Club, Butler University

Two social events, a Christmas party and the Annual Spring picnic, were held by the *Mathematics Club* of Butler University during 1946–47. Papers presented include:

Mathematical fallacies, by Marianne Buschmann and Audrey Klein

What is mathematics? by Reba Marshall, leader of round-table discussion.

Mathematical instruments, by Edward O'Nan, Mrs. Mary Bilby, and William Farmer

Short cuts in computing, by Mr. W. R. Krickenberger of Technical High School

Mathematics in meteorology, by Mr. Williamson of U. S. Weather Bureau

Theory of relativity, by Mrs. Juna L. Beal

The art of map making, by Mr. Walter Gingery of Washington High School

Anti-aircraft, by Miss Elna J. Hilliard

Atomic fission, by Mr. Donald G. Wittig

Men of mathematics, by Clifford Wagoner and William Fuller.

Officers for 1946–47 were: President, Betty Beck; Vice-President, Audrey Klein; Secretary, Elsie Popplewell; Treasurer, Reba Marshall; Sponsor, Mrs, Juna L. Beal.

Officers for 1947–48 are: President, Dorothy Reinacker; Vice-President, William Fuller; Secretary, Ethel Jackson; Treasurer, Clifford Wagoner.

Mathematics Club, Ball State Teachers College

The material presented to the *Mathematics Club* of Ball State Teachers College during 1946–47 dealt with the role of mathematics in modern warfare as observed by the veterans presenting the papers:

The general problems encountered in preparation for and during a flight, by Roy Tussey

Weather observing, by Ernest McClain

The assembly of data and the drawing of maps, by Clay Babcock

Problems encountered as chief forecaster on a Pacific island base, by Ed Shreve

Use of the maneuvering board in fleet operations, by Fred Deal

Experiences at the Bikini atom bomb tests, by Clarence Oswald

Mathematics in field artillery, by Clarence Buesking and Kenneth Paucher

Radar, by William Miner, a former club member.

Officers elected for 1947–48 are: President, Joe Scherrer; Vice-president, Kenneth Paucher; Secretary, Ed Shreve; Treasurer, Iraida Reed; Program Chairman, Joe Jackson.

Mathematics Club, New York State College for Teachers

The meetings of the club were held monthly, featuring one or two speakers each meeting. The topics discussed were:

Bridges and unicursal curves, by E. A. Butler, Instructor of Mathematics

The curve of the devil, by Herbert Weiner

Puzzles, by Louise Dodge

Clerical and mechanical aspects of teaching, by C. J. Haughey, Instructor and Supervisor of Mathematics

Derivation of trigonometric formulas by geometry, by Marilyn Burnup

History of verbal problems, by Doris Quinn

Magic squares, by Marie Balfoort

The oddities of numbers, by Eleanor Merritt.

The officers for the year 1946–47 were: President, Ruth Seelbach; Vice-president, Betty Whitney; Secretary, Florence Mace; Treasurer, Elsa Moberg. The officers for the year 1947–48 are: President, Eleanor Merritt; Vice-president, Betty Whitney; Secretary, Elsa Moberg; Treasurer, Ruth Marschner.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

RESOLUTIONS OF THE NATIONAL COUNCIL

At the Annual Meeting of the National Council of Teachers of Mathematics on April 2–3, 1948 the following resolution was adopted: The membership of the N.C.T.M. commends the Mathematical Association of America in its recent adoption of resolutions stating their desire for close cooperation with the N.C.T.M. in the efforts of both organizations to improve the coordination of high school and college mathematics.

THE PHILOSOPHY OF SCIENCE ASSOCIATION

The Philosophy of Science Association has been reorganized with Philipp Frank of Harvard University as president; C. W. Churchman of Wayne University is secretary-treasurer. Applications for membership may be sent to Professor Churchman. Dues are $5.00 per year.

THE INTERNATIONAL CONGRESS OF MATHEMATICIANS

An International Congress of Mathematicians will be held in Cambridge, Massachusetts, in 1950 under the auspices of the American Mathematical

Society. The Society originally planned to act as host for a Congress in September, 1940, which was also scheduled to meet in Cambridge. At the 1936 Congress in Oslo, Norway, the invitation for the 1940 Congress was issued by the American delegation in the name of the American Mathematical Society. Plans for the 1940 Congress were practically completed when the outbreak of World War II in September, 1939, made it necessary for the Society to postpone the Congress to a more favorable date. An Emergency Committee was established to carry on in the interim and, on recommendation of this Committee, the Council of the Society voted to hold the Congress in 1950.

The 1950 Congress will be the third International Congress of Mathematicians to be held on the continent of North America. The first was held at Northwestern University in 1893 and the second at the University of Toronto in 1924. International Congresses were held at intervals of approximately four years, except when war intervened, until 1936. There has been no international gathering of mathematicians since that time and it is the sincere hope of the Organizing Committee that the gathering in 1950 will be a truly international one, that the American mathematicians will attend in large numbers, and that all other countries will be well represented. The Council of the American Mathematical Society has voted unanimously to hold a Congress which will be open to mathematicians of all national and geographical groups.

Time and Place. The dates for the Congress have been fixed as August 30–September 6, 1950. Harvard University will be the principal host institution. A number of other institutions in metropolitan Boston will join in the entertainment of Congress visitors by arranging special features on their campuses.

Type of Congress. In recent years mathematicians have been much impressed by the success of the conference method for presenting recent research in fields where vigorous advances have just been made or are in progress. In view of the success of mathematical conferences on special topics which have been held in Russia, France and Switzerland and, more recently, at the Princeton Bicentennial Celebration, the 1950 Congress will include Conferences in several fields. For the 1940 Congress, Conferences in four fields had been planned. The number of Conferences was thus restricted lest the introduction of a promising and novel feature result in failure through the dissipation of interest and energy. A subcommittee of the Organizing Committee, under the chairmanship of Professor A. A. Albert, is now studying the question of the number and the fields of the Conferences to be included in the 1950 Congress and the results of the committee's deliberations will be reported at a later date.

Following the established custom, the Organizing Committee plans to have a number of invited hour addresses by outstanding mathematicians. In addition, sectional meetings for the presentation of contributed papers not included in Conference programs will be held in the following fields: I, Algebra and Theory of Numbers; II, Analysis; III, Geometry and Topology; IV, Probability and Statistics, Actuarial Science, Economics; V, Mathematical Physics and Applied Mathematics; VI, Logic and Philosophy, History and Education.

The official languages of the 1950 Congress will be English, French, German, Italian, and Russian.

Organization. The plans for the Congress are under the supervision of an Organizing Committee which was elected by the Council of the American Mathematical Society in February, 1948. The Chairman is Professor Garrett Birkhoff of Harvard University and the Vice Chairman is Professor W. T. Martin of Massachusetts Institute of Technology. Other members of the committee are: Professors J. L. Doob, G. C. Evans, J. R. Kline, Solomon Lefschetz, Saunders MacLane, Dean R. G. D. Richardson, Professors J. L. Synge, Oswald Veblen, J. L. Walsh, D. V. Widder, Norbert Wiener, and R. L. Wilder.

Many of the subventions promised for the 1940 Congress are still available. A Financial Committee under the chairmanship of Professor John von Neumann is endeavoring to secure additional funds. Besides support from Harvard University and Massachusetts Institute of Technology, generous subventions have been subscribed for the Congress by the Carnegie Corporation, the Institute for Advanced Study, the National Research Council, and the Rockefeller Foundation.

An Editorial Committee under the chairmanship of Professor Salomon Bochner will assume responsibility for the publication of the Proceedings of the Congress.

Professor J. R. Kline of the University of Pennsylvania has been named Secretary of the Congress and Dr. R. P. Boas, Executive Editor of Mathematical Reviews, has been designated Associate Secretary.

Entertainment. Harvard University has offered the use of its dormitories and dining rooms for mathematicians and their guests for the period of the Congress. The Organizing Committee hopes that it will be possible to furnish room and board without charge to all mathematicians from outside continental North America who are members of the Congress. Congress membership fees and rates for room and board will be announced well in advance of the opening of the Congress.

The Entertainment Committee, of which Professor L. H. Loomis of Harvard University is Chairman, is planning many interesting features, including a reception, garden party, symphony concert, and banquet. It is hoped that American mathematicians will be able to assist in the entertainment by putting their automobiles at the disposal of the Entertainment Committee for trips to be made out of Cambridge.

Every effort will be made to facilitate the travel at reasonable cost of foreign participants while in the United States. Previous to the Congress, opportunity will be given them to see New York City under the guidance of some mathematicians.

Information. Detailed information will be sent in due course to individual members of the American Mathematical Society and to foreign mathematical societies and academies. Others interested in receiving information may file their names in the Office of the Society, and such persons will receive from time

to time information regarding the program and arrangements.

Communications should be addressed to the American Mathematical Society, 531 West 116th Street, New York City 27, U.S.A.

STANFORD UNIVERSITY COMPETITIVE EXAMINATION

The third Stanford University Competitive Examination in Mathematics (see this MONTHLY, vol. LIII, 1946, pp. 406–409) was held April 10, 1948, in 53 high schools in California, with 206 students taking part. The following problems were proposed:

1. Consider the table:

$$\begin{aligned} 1 &= 1 \\ 2+3+4 &= 1+8 \\ 5+6+7+8+9 &= 8+27 \\ 10+11+12+13+14+15+16 &= 27+64 \end{aligned}$$

Guess the general law suggested by these examples, express it in suitable mathematical notation, and prove it.

2. Three numbers are in arithmetic progression, three other numbers in geometric progression. Adding the corresponding terms of these two progressions successively, we obtain

$$85, \ 76, \text{ and } 84$$

respectively, adding all three terms of the arithmetic progression, we obtain 126. Find the terms of both progressions.

3. From the peak of a mountain, you see two points, A and B, in the plain. The lines of vision, directed to these points, include the angle γ. The inclination of the first line of vision to a horizontal plane is α, that of the second line β. It is known that the points A and B are on the same level and that the distance between them is c.

Express the elevation x of the peak above the common level of A and B in terms of the angles α, β, γ and the distance c.

4. A first sphere has the radius r_1. About this sphere circumscribe a regular tetrahedron. About this tetrahedron circumscribe a second sphere with radius r_2. About this second sphere circumscribe a cube. About this cube circumscribe a third sphere with radius r_3.

Find the ratios $r_1:r_2:r_3$ (which should be, according to Kepler, the ratios of the mean distances of the planets Mars, Jupiter, and Saturn from the Sun, but which are, in fact, rather different from the true ratios).

The writer of the best paper, C. L. Moller, Jr., Glendora, California, student at Citrus Union High School, Azusa, California, received a $500 scholarship at Stanford University. R. C. Hill, Los Angeles, California, student at Harvard School, North Hollywood, California, received honorable mention.

PERSONAL ITEMS

Professor C. V. Newsom, chairman of the Department of Mathematics of Oberlin College, has accepted the position of Assistant Commissioner for Higher Education for the State of New York. Editorial correspondence in regard to the MONTHLY should now be addressed to Dr. C. V. Newsom, State Education Building, Albany 1, New York.

Professor C. J. Coe represented the Association at the inauguration of President White of the University of Toledo, Ohio on May 11, 1948.

Professor L. R. Ford was a representative of the Association at the meeting of the American Council on Education, Chicago, Illinois on May 7–8, 1948.

Professor C. V. Newsom represented the Association at the inauguration of the president of Case Institute of Technology, Cleveland, Ohio on May 21, 1948.

Professor Charles Wexler was appointed to represent the Association at the inauguration of Dr. J. B. McCormick as president of the University of Arizona, Tucson, Arizona on May 5, 1948.

Professors George Pólya, E. B. Roessler, and R. M. Winger served as representatives of the Northern California Section of the Association at the Pacific Regional Conference on UNESCO on May 13–15, 1948.

Professor C. C. Chevalley of Princeton University, Assistant Professor Irving Kaplansky of the University of Chicago, and Associate Professor Norman Levinson of Massachusetts Institute of Technology have been awarded Guggenheim fellowships.

Dr. Churchill Eisenhart, chief of the Statistical Engineering Laboratory, National Bureau of Standards, has been elected a fellow of the Royal Statistical Society of Great Britain.

Associate Professor Maurice Ewing of Columbia University and Professor E. J. McShane of the University of Virginia have been elected to membership in the National Academy of Sciences.

Professor Gillie A. Larew, head of the Department of Mathematics at Randolph-Macon Woman's College, has been honored by the establishment of the Gillie A. Larew Chair of Mathematics. The fund for the endowment of the Chair was contributed by alumnae of the College.

Dean R. G. D. Richardson of Brown University has been awarded the honorary degree of Doctor of Laws by Brown University.

President W. M Whyburn of Texas Technological College has been awarded an honorary degree by Texas Technological College

Brown University announces the following promotions: Assistant Professors Herbert Federer and G. W. Whitehead to associate professorships, Lecturer Wouter van der Kulk to an assistant professorship, Dr. Bjarni Jonsson to an assistant professorship. Lectures were given at the Mathematics Colloquium of Brown University during the past year by the following mathematicians from abroad: Professor H. A. Bohr of the University of Copenhagen, Professor Henri Cartan of the University of Paris, and Professor Natan Aronszajn of the Centre National de la Recherche Scientifique.

Princeton University makes the following announcements: Associate Professor Alonzo Church has been promoted to a professorship; Assistant Professors R. H. Fox and J. W. Tukey have been promoted to associate professorships; Professor L. K. Hua of Tsing Hua University and Professor J. P. LaSalle have been appointed Lecturers; Associate Professor Witold Hurewicz of Massachusetts Institute of Technology has been appointed Visiting Professor.

Purdue University announces the following: Dr. Ralph Hull of Boeing Aircraft Corporation has been appointed Professor of Mathematics and Head of the Department of Mathematics replacing Dean W. L. Ayres, who will devote his full time to the School of Science; Dr. Olav Reiersol, now on a fellowship at the University of California, has been appointed Lecturer in Statistics; Assistant Professors M. E. Shanks and M. S. Webster have been promoted to associate professorships; Mr. Stanley Bolks, Mr. K. W. Crain, Mr. M. W. DeJonge, Mr. L. G. Black, Mr. P. W. Overman, and Dr. W. P. Reid have been promoted to assistant professorships.

Queens College announces the following promotions: Assistant Professor Arthur Sard to an associate professorship, and Associate Professors R. G. Archibald and John Williamson to professorships.

Rensselaer Polytechnic Institute announces the following promotions: Associate Professors R. E. Huston and W. G. Warnock to professorships, Assistant Professors J. D. Campbell, A. W. Jones, and Dis Maly to associate professorships, Mr. G. F. Guilford to an assistant professorship.

Syracuse University announces the appointments of Associate Professors M. F. Rosskopf and Atle Selberg and Assistant Professor H. E. Goheen. Assistant Professor D. E. Kibbey has been promoted to an associate professorship, Dr. P. W. Gilbert and Dr. Erik Hemmingsen have been promoted to assistant professorships.

The University of Alberta makes the following announcements: Professor E. W. Sheldon has retired with the title of Professor Emeritus; Professor J. W. Campbell has succeeded Dr. Sheldon in the position of Head of the Department of Mathematics; Mr. Edgar Phibbs and Mr. E. T. Sheffield have been appointed Assistant Professors; Mr. R. C. Jacka and Mr. Thorlief Fostvedt have been appointed Sessional Lecturers.

The University of Illinois announces the following appointments and promotions: Professor A. H. Taub has been appointed to a research professorship in applied mathematics; Mr. A. G. T. Carlton has been appointed to an assistant professorship; Associate Professors P. W. Ketchum and H. J. Miles have been promoted to professorships; Assistant Professors Harry Levy, C. W. Mendel and J. W. Peters have been promoted to associate professorships; Dr. V. R. Nuess has been promoted to an assistant professorship.

The University of Utah announces the following appointments: Professor C. R. Wylie Jr., chairman of the Department of Mathematics, Air Institute of Technology, Wright Field, has been appointed Professor and Chairman of the

Department of Mathematics; Dr. R. E. Chamberlin and Dr. James Wolfe have been appointed to assistant professorships.

Washington University makes the following announcements: Mr. Eric Kristensen of the University of Copenhagen has been appointed Lecturer; Dr. Zeev Nehari of Hebrew University has been appointed to an associate professorship; Assistant Professors H. M. Schaerf and G. B. Van Schaack have been promoted to associate professorships; Assistant Professor J. Y. Stephens has been promoted to an associate professorship of astronomy; Dr. Franklin Haimo and Mr. Marlow Sholander have been promoted to assistant professorships.

Wayne University announces the resignation of Associate Professor J. A. Pierce and the appointments of Associate Professor Benjamin Epstein and Assistant Professors Harvey Cohn and Gerald Harrison.

Dr. P. H. Anderson has been appointed Professor of Marketing at Loyola University.

Professor H. T. R. Aude of Colgate University has retired.

Dr. R. E. Basye, Agricultural and Mechanical College of Texas, has been promoted to an assistant professorship.

Associate Professor Z. W. Birnbaum of the University of Washington has accepted an appointment as director of the Laboratory of Statistical Research.

Assistant Professor Robert Breusch of Amherst College has been promoted to an associate professorship.

Professor W. H. Bussey of the University of Minnesota has retired.

Professor S. S. Cairns of Syracuse University has been appointed Head of the Department of Mathematics of the University of Illinois.

Professor B. H. Camp of Wesleyan University, Middleton, Connecticut, has retired.

Assistant Professor D. G. Chapman of the University of British Columbia has accepted a position as research assistant at the University of California.

Assistant Professor I. S. Cohen of the University of Pennsylvania has been appointed to an assistant professorship at Massachusetts Institute of Technology.

Assistant Professor Lester Dawson of Michigan College of Mining and Technology has been appointed to an associate professorship at Adams State College.

Dr. Wade Ellis has been appointed to an assistant professorship at Oberlin College.

Associate Professor H. F. Fehr of Montclair State Teachers College, New Jersey, has accepted a position as professor of mathematics at Teachers College, Columbia University.

Associate Professor W. W. Flexner of Cornell University has received an appointment as senior statistician in charge of transport statistics in the Department of Economic Affairs of the United Nations.

Professor F. C. Gentry of Louisiana Polytechnic Institute has been appointed to an associate professorship at Arizona State College.

Associate Professor S. G. Hacker of Washington State College has been promoted to a professorship.

Associate Professor D. A. Hatch of Lafayette College has retired.

Professor G. A. Hedlund of the University of Virginia has been appointed to a professorship at Yale University.

Assistant Professor D. M. Hester of Northwest Missouri State Teachers College has accepted a position at Baker University.

Associate Professor P. M. Hummel of the University of Alabama has been promoted to a professorship.

Mr. Sidney Kravitz of the Newark College of Engineering has received an appointment as mathematician with the Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland.

Professor D. C. Lewis of the University of Maryland has been appointed to a professorship at Johns Hopkins University.

Associate Professor E. R. Lorch of Columbia University has been promoted to a professorship.

Mr. R. C. Luippold of West Virginia Institute of Technology has been appointed to an assistant professorship at Antioch College.

Associate Professor C. T. McCormick of Illinois State Normal University has been promoted to a professorship.

Dr. N. W. McLachlan of London served as visiting professor of mathematics at Carnegie Institute of Technology during the second semester of 1947–1948.

Associate Professor A. K. Mitchell of the University of Maryland has accepted a position as aerodynamicist at the Applied Physics Laboratory of Johns Hopkins University.

Dr. Deane Montgomery has been appointed to permanent membership in the Institute for Advanced Study.

Mr. Dewey Moore has received an appointment as physicist with the National Advisory Committee for Aeronautics, Langley Field, Virginia.

Professor F. C. Ogg has been appointed Head of the Department of Mathematics of Bowling Green State University.

Mr. Richard Otter of Princeton University has been appointed to an assistant professorship at the University of Notre Dame.

Mr. Eugene Park, graduate assistant at the University of Wisconsin, has been appointed Assistant Professor at Clemson College.

Assistant Professor H. V. Price of the State University of Iowa has been promoted to an associate professorship.

Mr. A. O. Qualley of Lehigh University has accepted an appointment as assistant professor at Drake University.

Professor J. F. Randolph of Oberlin College has been appointed Head of the Department of Mathematics at the University of Rochester.

Professor W. D. Reeve of Teachers College, Columbia University has retired.

Dean R. G. D. Richardson of Brown University has retired.

Dr. Klaus Schocken, formerly of Siena College, is now a member of the Medical Department, Field Research Laboratory, Fort Knox, Kentucky.

Associate Professor K. C. Schraut has been promoted to a professorship at the University of Dayton.

Assistant Professor Ruth G. Simond of Morningside College has been appointed to an assistant professorship at the University of Vermont.

Associate Professor M. F. Smiley of Northwestern University has accepted an appointment as professor of mathematics at the State University of Iowa.

Assistant Professor E. R. Stabler has been promoted to an associate professorship at Hofstra College.

Assistant Professor D. W. Western of Brown University has been appointed to an associate professorship at Franklin and Marshall College.

Professor Anna Pell Wheeler of Bryn Mawr College has retired.

President W. M. Whyburn of Texas Technological College has been appointed to a professorship at the University of North Carolina.

Dr. Albert Wilansky of Brown University has been appointed Assistant Professor of Mathematics at Lehigh University.

Assistant Professor H. P. Wirth of the College of the City of New York has been promoted to an associate professorship.

The following appointments to instructorships are announced:

Case Institute of Applied Science: Mr. Charles Saltzer

Purdue University: Miss Marianne Bernstein, Mr. L. D. Kovach

Syracuse University: Mr. Helmut Aulback, Mr. S. G. Campbell, Mr. Eckford Cohen, Mr. Daniel Resch, Mr. C. J. Titus.

United States Naval Academy: Mr. F. P. Kowalewski, Jr.

University of Illinois: Dr. M. D. Springer

University of Utah: Mr. F. E. Haupt

Wayne University: Mr. Willard Clatworthy.

Connecticut College announces the appointment of Miss Elizabeth Hahnemann as teaching assistant.

Dr. Clara L. Bacon, professor emeritus of mathematics, Goucher College, died April 14, 1948. She was a charter member of the Association.

Dr. C. E. Comstock, professor emeritus of mathematics at Bradley University and a charter member of the Association, died on April 3, 1948.

Professor W. L. Johnson, head of the Department of Mathematics of Mississippi Southern College died March 14, 1948. He was Chairman of the Louisiana-Mississippi Section of the Association.

Professor Emeritus F. R. Sharpe of Cornell University died on May 18, 1948 at the age of seventy-eight years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following eighty-seven persons have been elected to membership on applications duly certified:

R. H. ACKERSON, A.M.(Columbia) Asst. Professor, Catawba College, Salisbury, N. C.

R. V. ANDERSON, A.M.(Colorado) Instructor, Colorado Agricultural & Mechanical College, Fort Collins, Colo.

P. M. BAILEY, B.S.(Wyoming) Student, State University of Iowa, Iowa City, Iowa

MARY M. BAXTER, M.A.(Columbia) Instructor, Junior College of Kansas City, Mo.

LULU BECHTOLSHEIM, Ph.D.(Zurich) Asst. Professor, University of Redlands, Calif.

MARTIN BERMAN, M.S.(Virginia Polytechnic Institute) Instructor, University of Cincinnati, Ohio

L. J. BERNER, B.A.(St. Norbert College) Instructor, St. Norbert College, West De Pere, Wis.

W. J. BRUNS, Ph.D.(Strassburg) Visiting Professor, Syracuse University, N. Y.

BREDIE E. (Mrs. Buron) BUFFKIN, A.B.(Coker) Instructor, University of South Carolina, Columbia, S. C.

P. B. BURCHAM, Ph.D.(Northwestern) Asst. Professor, University of Missouri, Columbia, Mo.

MARY P. BURKART, B.S.(Nazareth College) Graduate Asst., University of Detroit, Mich.

C. D. CALHOON, M.A.(Illinois) Asst. Professor, University of Toledo, Ohio

P. A. CLEMENT, M.A.(State College of Washington) Teaching Asst., University of California at Los Angeles, Calif.

W. J. CONNER, M.A.(Texas) Asst. Professor, The Citadel, Charleston, S. C.

GRACE CUTLER, M.A.(Toledo) Asst. Professor, University of Toledo, Ohio

R. C. DAVIS, B.S.(Akron) Instructor, University of Akron, Ohio

DOUGLAS DERRY, Ph.D.(Gottingen) Asso. Professor, University of British Columbia, Vancouver, B. C.

J. G. DIAZ, Agricultural Engineer, Alarcon 27, Madrid, Spain

I. A. DODES, Ph.D.(New York University) Teacher, Stuyvesant High School, 345 East 15th St., New York, N. Y.

MARY C. DOREMUS, M.A.(Columbia) Teacher, North High School, Denver, Colo.

HELEN ENGEBRETSON, M.A.(Minnesota) Instructor, State College, Brookings, S. D.

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J. L. SLECHTICKY, M.S.(Washington University) Asst. Professor, Toledo University, Ohio

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W. D. WOOD, B.S.(Case Institute of Technology) Graduate Student, Oberlin College, Ohio

A. G. WOOTTON, B.S.(Rutgers) Asst. Professor, Mohawk College, Utica, N. Y.

V. A. ZORA, B.S.(Pittsburgh) Chemical Process Engineer, Blaw-Knox Const. Co., Chemical Plants Division, Pittsburgh, Pa.

THE EIGHTH CARUS MATHEMATICAL MONOGRAPH "RINGS AND IDEALS" BY N. H. McCOY

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It is expected that the prices of all Carus Monographs will be increased as of December 1, 1948, and the above-mentioned prices will hold only until that date.

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for a three-year term beginning July 1, 1948 by vote of the membership of the Association in the Sections indicated:

Allegheny Mountain	H. L. Dorwart, Washington and Jefferson College
Indiana	P. D. Edwards, Ball State Teachers College
Kentucky	M. C. Brown, University of Kentucky
Metropolitan New York	F. H. Miller, Cooper Union
Nebraska	C. C. Camp, University of Nebraska
Northern California	F. R. Morris, Fresno State College
Oklahoma	N. A. Court, University of Oklahoma
Rocky Mountain	A. J. Lewis, University of Denver
Wisconsin	R. H. Bardell, University of Wisconsin in Milwaukee

Since the system of Regional Governors is being replaced by a system of

Sectional Governors, no elections have been held to replace the Regional Governors whose terms expired on July 1, 1948.

BACK NUMBERS WANTED

One dollar a copy will be paid (or credited on members' dues) for any of the following issues of this MONTHLY until fifty copies of each number are received:
1946—January, May, June-July, August-September
1947—January, December
1948—January, February, May

Copies should be mailed to the Mathematical Association of America, University of Buffalo, Buffalo 14, New York with the name and address of the sender marked clearly on the package.

JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The tenth annual meeting of the Northern California Section of the Mathematical Association of America was held at the University of California on Saturday, January 24, 1948. Professor George Pólya, Chairman of the Section, and Professor G. C. Evans, Vice-Chairman, presided at the morning session, and Mr. S. A. Francis presided at the afternoon session.

The attendance was ninety-three including the following twenty-nine members of the Association: H. M. Bacon, G. A. Baker, T. J. Bass, E. M. Beesley, M. T. Bird, A. C. Burdette, Louise Chin, M. A. Dernham, G. C. Evans, S. A. Francis, C. M. Fulton, J. M. Good, W. R. Hanson, Sophia McDonald, E. D. Miller, F. R. Morris, W. H. Myers, C. D. Olds, George Pólya, Edris P. Rahn, R. M. Robinson, E. B. Roessler, Kathryn Rolfe, Abraham Seidenberg, Pauline Sperry, Ruth Sumner, Gabor Szegö, L. A. Walker, A. R. Williams.

At the business meeting the following officers were elected for the coming year: Chairman, G. C. Evans, University of California; Vice-Chairman, H. M. Bacon, Stanford University; Secretary-Treasurer, E. B. Roessler, University of California at Davis; Representative on the *California Journal of Secondary Education*, Ruth G. Sumner, Oakland High School.

By invitation of the Section, Professor Harold Davenport of University College, London, and Visiting Professor at Stanford University, gave an address during the morning session.

The following papers were read:

1. *A matrix arising in correlation theory*, by Professor H. M. Bacon, Stanford University.

Consider a set of n variables $x_1, x_2, x_3, \cdots, x_n$, representing observations taken in n successive years, with correlation coefficients r_{ij} between x_i and x_j. If the correlation coefficients are assumed to diminish in such a way that $r_{ij}=r_{ij}=1-|i-j|p$, where $0<p<2/(n-1)$, then the determinant $|r_{ij}|=R$ can be expressed in terms of n and p. It was shown that

$$R = 2^{n-2}p^{n-1}[2-(n-1)p].$$

Expressions for the cofactors R_{ij} of the elements r_{ij} of R were given. Assuming the x_i to be normally distributed, an expression for the frequency function was obtained in terms of n and p.

2. *Treatment of trigonometric functions in calculus*, by Dr. C. M. Fulton, University of California at Davis.

Instead of introducing the number e as a limit, one can define $\log x = \int_1^x (1/t)dt$ for $x>0$, or taking its existence for granted, the function $\log x$ can be characterized by means of the properties $D_x \log x = 1/x$, $\log 1 = 0$. The addition theorem is proved by observing that $D_x \log ax = D_x \log x$. Further properties of the logarithm and its inverse function follow easily. A graphical construction of the exponential curve using pieces of tangent lines is recommended.

In an analogous way let $\tan^{-1} x = \int_0^x [1/(1+t^2)]\,dt$. From $D_x \tan^{-1}[(a+x)/(1-ax)] = D_x \tan^{-1} x$ the addition formula of the inverse tangent is found. Some difficulties arise with respect to the intervals. The inverse function and the other trigonometric functions and their basic properties can be derived by simple manipulations. A geometric interpretation of the defining equation for $\tan^{-1} x$ is given. Finally it is pointed out that $\tanh^{-1} x$ can be handled in like fashion.

3. *The geometry of numbers*, by Professor Harold Davenport, University College, London, and Visiting Professor at Stanford University, introduced by the Chairman.

In the geometry of numbers, we treat a general class of problems in the theory of numbers by methods which are suggested by a geometrical interpretation. The problems in question relate to "Diophantine inequalities," *i.e.* inequalities which are to be satisfied by integral values of the variables. Questions of this kind arise quite naturally. For example, if we wish to prove the theorem of Fermat that every prime p of the form $4n+1$ is representable as x^2+y^2, we easily find a general formula which represents only multiples of p as x^2+y^2, and the proof can be completed by showing that the formula also represents a number less than $2p$ (and not zero).

The geometrical approach is that of conceiving an inequality $f(x_1, \cdots, x_n) < \lambda$ as representing a certain region in n-dimensional space. We then ask for conditions which will ensure that this region contains a point with integral coordinates. If the inequality is trivially satisfied by $x_1 = \cdots = x_n = 0$, we then need a point other than the origin.

In the special example where the inequality represents an ellipse in the plane, it is not difficult to prove that if the area of the ellipse is greater than $2\pi/\sqrt{3}$, there is always a point with integral coordinates in it, other than the origin. But the argument used is too special to be applied effectively in n dimensional space, or to more general regions.

A simple train of reasoning was discovered by Minkowski (about 1891) which led him to this very general theorem: Any body in n dimensional space, which is convex and symmetrical about the origin, and which has a volume greater than 2^n, must contain a point other than the origin with integral coordinates. This result has many applications in the theory of numbers.

In the last few years, new methods have been developed by several English mathematicians for dealing with problems in which the region represented by the inequality is not convex.

4. *How to solve it*, (Report on the New Haven Symposium of the Association) by Professor George Pólya, Stanford University.

Reports of the meeting appeared in this MONTHLY, vol. 54, p. 612, and vol. 55, p. 22.

5. *A problem in calculus*, by Professor M. T. Bird, San Jose State College.

Given a continuous curve $y=f(x)$ which intersects the line $y=mx+b$ in just two points, namely the points for which $x=a$ and $x=c$. It is required to find the volume generated when the area between the curve and the line is revolved about the line. An erroneous analysis with elements of volume which are circular plates is seen to lead to the correct value for the volume. An analysis with elements of volume which are conical shells is seen to be appropriate.

6. *The variance of the proportions of samples falling within a fixed interval for a normal population*, by Professor G. A. Baker, University of California at Davis.

The usual variance of the proportions of a sample falling in a fixed interval given in elementary textbooks is the binomial variance $p(1-p)/N$, and is the same for all parent populations. If a normal population is sampled, estimates of $\bar{x}$ and s^2 being formed in the usual way, and if p is estimated by putting $\bar{x}$ and s^2 for m and σ^2 in the integral defining p, we get an estimate of p that has a considerably smaller variance than the corresponding binomial variance.

7. *The Banach-Tarski paradox*, by Professor R. M. Robinson, University of California.

It is possible to cut a solid sphere into five pieces and reassemble then by rigid motions to form two solid spheres of the same size. No fewer than five pieces may be used, but one of the pieces may consist of just one point. The proof of this result will appear in *Fundamenta Mathematicae*. A similar result with an unspecified finite number of pieces was proved by Banach and Tarski in 1924. It is evident from the Banach-Tarski paradox that there cannot be any completely general method of defining volume so that it will be additive and invariant under motion.

E. B. ROESSLER, *Secretary*

FEBRUARY MEETING OF THE OKLAHOMA SECTION

The annual meeting of the Oklahoma Section of the Mathematical Association of America was held in connection with the convention of the Oklahoma Education Association in Tulsa on Friday, February 13, 1948. Professor D. R. Shreve, Chairman of the Section, presided.

Fifty-three persons attended the meeting, including the following sixteen members of the Association: A. Bernhart, J. C. Brixey, H. N. Carter, N. A. Court, N. A. Eisen, J. O. Hassler, O. H. Hamilton, J. E. LaFon, Eunice Lewis, R. D. McDole, Josephine Mitchell, E. P. Northrop, C. C. Pruitt, D. R. Shreve, R. W. Veatch, Marion West.

At the business session the following officers were elected: Chairman, J. E. LaFon, University of Oklahoma; Vice-Chairman, H. V. Huneke, Northwestern State College; Secretary, J. C. Brixey, University of Oklahoma.

The program consisted of the following nine papers:

1. *Professor W. T. Short Memorial*, by Professor N. A. Court, University of Oklahoma.

2. *How to use the history of mathematics in teaching*, by Professor J. O. Hassler, University of Oklahoma.

Professor Hassler recounted briefly the most significant items in the history of the development of algebra, trigonometry, analytics, and calculus, suggesting how and where the material could be included with the instruction in the subject matter without the necessity of a separate course in the history of mathematics. He also made a plea for the inclusion of more historical material in mathematical instruction in order to give the students the proper appreciation of the place of mathematics in the development of civilization.

3. *Use of trigonometric and hyperbolic functions in empirical equations*, by N. A. Eisen, Mid-Continent Petroleum Corporation.

A procedure for fitting an equation of the form $y=a+bf(x)$, where $f(x)$ is a trigonometric or hyperbolic function, was developed. This included the determination of the range of the angle to be used. The procedure involved the determination of four constants.

4. *Mathematics of the drafting board,* by Mr. S. S. Orman, Tulsa Industrial Arts High School, introduced by Professor D. R. Shreve.

The speaker described the geometrical constructions and calculations pertaining to drafting, with emphasis on the need to visualize three-dimensional figures. The comprehension and accuracy necessary to do detail design work was discussed.

5. *On asymptotes,* by Professor D. R. Shreve, University of Tulsa.

The vertical asymptotes of the locus of an irreducible integral rational equation of the form $a_0x^n+a_1x^{n-1}+\cdots+a_n=0$ are known to be the locus of $a_0=0$. (Here a_0 is understood to be a polynomial in y of degree $m\geqq 1$.) An extension to the determination of asymptotic planes and cylinders of surfaces was indicated.

6. *Summability of double orthogonal series whose coefficients satisfy certain conditions,* by Professor Josephine Mitchell, Oklahoma A. and M. College.

A discussion of the known results on convergence and $(C, 1)$ summability of simple orthogonal series was given. A theorem on the connection between $(C, 1, 1)$ summability and the convergence of the double orthogonal series which has been proved by the author was described.

7. *An upper bound for the number of irreducible factors of a polynomial with coefficients in a Gaussian domain,* by Mr. R. R. Gordon, introduced by Professor J. C. Brixey.

Given a Gaussian domain G, and k distinct elements $b_j\in G$, $j=1, 2, \cdots, k$, and a polynomial of degree n

$$P(x) \equiv \prod_{i=1}^{f} p_i(x) \tag{1}$$

where the coefficients of $P(x)$ and each $p_i(x)$ are in G, each $p_i(x)$ being of degree greater than zero, and irreducible into polynomials of type (1). Then the total number of irreducible factors f in (1) for any positive rational integral k and n is such that

$$f \leqq \frac{un}{k} + \frac{1}{k}\sum_{j=1}^{k} c[P(b_j)] \tag{2}$$

where u is the number of distinct units in G, and for any element $b\in G$, $c[P(b_j)]$ is the total number of primes in G contained by any factored form of b in which each factor is a unit or a prime in G. By assigning certain conditions to (2) and some of its arrangements, certain interesting conclusions can be obtained.

8. *Roots of polynomials,* by Professor A. Bernhart, University of Oklahoma

In applying Descartes' rule of signs, a negative coefficient between two positive coefficients of greater absolute value indicates a pair of conjugate imaginary roots.

The *average* product of roots taken r at a time is the same for a polynomial and its derivative. Representing the roots of a cubic by A, B, C, on a Gauss plane, the quadratic roots L, N may be interpreted geometrically by means of the auxiliary points R, T at the centroids of equilateral triangles with base AB.

Let S be the midpoint of AB and RT, and M the midpoint of LN. Then SR and ST are the geometric means of SL and SN; and conversely ML and MN are the geometric means of MR and MT. Thus L, N, R, T are concyclic, and LN is parallel to an asymptote of the equilateral hyperbola with center S which passes through R.

9. *Some generalizations of Brouwer's fixed point theorem for n-cells,* by Professor O. H. Hamilton, Oklahoma A. and M. College.

Brouwer's fixed point theorem for n-cells was stated. Generalizations of the theorem were classified as those which deal with continua more general than an n-cell subjected to a continuous transformation, and those which pertain to transformations more general than single valued continuous transformations. Examples of applications of fixed point theorems to analysis were given.

J. C. BRIXEY, *Secretary*

MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The twenty-eighth regular meeting of the Southern California Section ofthe Mathematical Association of America was held at the University of Redlands, Redlands, California, on Saturday, March 13, 1948. Professor D. V. Steed Chairman of the Section, presided at the morning and afternoon sessions.

The attendance was one hundred, including the following forty-eight members of the Association: L. J. Adams, O. W. Albert, H. M. Bacon, E. F. Beckenbach, May M. Beenken, Clifford Bell, L. T. Black, Herbert Buseman, F. A. Butter, Jr., W. D. Cairns, Frances L. Campbell, L. M. Coffin, E. L. Crow, D. R. Curtiss, P. H. Daus, R. P. Dilworth, Iva B. Ernsberger, B. K. Gold, J. W. Green, H. J. Hamilton, W. L. Hart, R. B. Herrera, M. R. Hestenes, P. G. Hoel, R. E. Horton, D. H. Hyers, C. G. Jaeger, B. W. Jones, Margaret B. Lehman, Ada McClellan, Sophia L. McDonald, G. F. McEwen, F. R. Morris, P. M. Niersbach, C. L. Olsen, W. T. Puckett, H. R. Pyle, E. C. Rex, J. M. Robb, G. E. F. Sherwood, Ernst Snapper, I. S. Sokolnikoff, R. H. Sorgenfrey, D. V. Steed, C. W. Trigg, S. E. Urner, Morgan Ward, Euphemia R. Worthington.

At the business meeting the following officers were elected for the next academic year: Chairman E. F. Beckenbach, University of California at Los Angeles; Vice-Chairman, H. R. Pyle, Whittier College; Program Committee, Herbert Buseman (Chairman), F. A. Butter Jr., E. J. Hills, and the Secretary, P. H. Daus, ex-officio. The next meeting was scheduled for March 12, 1949 at John Muir Junior College, Pasadena, California.

The following papers were presented:

1. *A college teacher's attitude toward the secondary curriculum in mathematics*, by Professor W. L. Hart, University of Minnesota.

Professor Hart called attention to the fact that uncomplimentary opinions are frequently expressed about the attitudes of college teachers of mathematics toward the curriculum in secondary mathematics. He indicated his belief that many of these adverse opinions, and certain serious evils in the secondary field, are a consequence of attaching the label "college preparatory" to the high school courses containing the most substantial mathematics. Then, for a college teacher of mathematics, Professor Hart advised a viewpoint designed to be helpful to the secondary field as well as to the colleges.

2. *Current practices and problems in secondary mathematics*, by Mr. Dale Carpentar, Supervisor, Mathematics Education Section, Los Angeles City Schools, introduced by Professor P. H. Daus.

Mr. Carpentar reviewed the history of the reorganization or reconstruction of high school mathematics, and the effect that the organization of Junior High Schools has had upon the problem. He suggested a mathematics program which would meet the vocational needs and aptitudes of all major groups of secondary students, and gave reasons why progress toward such a program

has been slow. He cited what the Los Angeles School District is planning to do to meet this problem. This involves the organization of a non-college preparatory sequence in mathematics, and an experimental mathematics program for college preparatory students who are not science majors. This experiment, which is still in the organizational stage, represents a cooperative endeavor of the Los Angeles City and County High Schools and the University of California.

3. *Geometry and physical space*, by Professor H. P. Robertson, California Institute of Technology, introduced by Professor J. W. Green.

This was an invited address given before a joint meeting of this Section and the Southern California Section of the American Association of Physics Teachers.

The speaker discussed the relation between the notion of curvature of a space, with its geometrical implications, and the nature of the device with which distance in the space is measured. This relation was illustrated by consideration of a plane in which exists a steady temperature distribution, and in which distances are measured by a rod which expands with change of temperature according to a linear law. The geometry which one would deduce from such measurements is a non-euclidean geometry, which in the case of a linear temperature distribution is the geometry of the Poincare half-plane.

4. *Report of the joint committee on mathematical education*, by Professor F. R. Morris, Fresno State College.

This was a progress report on the activities of the committee, appointed last year, and representing the Northern California and Southern California Sections of the Association.

5. *Concerning variations on Newton's method*, by Professor H. J. Hamilton, Pomona College.

Let z^* be a regular point for $f(z)$ and $\phi(z)$, and a simple zero for $\phi(z)$. Putting $z_n=f(z_{n-1})$, for $n=1, 2, \cdots$, and taking z_0 sufficiently near to z^*, we shall have $z_n \to z^*$ with $(z_n-z^*) = 0[(z_{n-1}-z^*)^k]$, ($k$ an arbitrary integer $\geqq 2$), if and only if $f(z)$ is expressed in certain ways in terms of $\phi(z)$ and its derivatives. Thus for $k=2$ we have $f(z)=z-[\phi(z)/\phi'(z)]+h(z)[\phi(z)]^2$, where $h(z)$ is regular at z^*. The derivations involve elementary manipulations of power series.

6. *Problems with many or no solutions*, by Professor Herbert Buseman, University of Southern California.

The curves of constant width furnish a solution of a mechanical problem which at first sight seems to have the circle as the only solution. The problem of Kakaya to find a domain of least area in which a segment ab can be moved continuously such that a falls on b, and b on a, was discussed as an example of a seemingly reasonable problem with no solution.

7. *Calculations of some properties of certain supersonic wing sections*, by Dr. F. A. Butter, Jr., Hughes Aircraft Co.

A method is derived for the calculation of the area, the principal and polar moments of inertia, and the squares of the corresponding radii of gyration of a biconvex supersonic airfoil section in the form of a hollow double circular segment with chord c, thickness t, and constant skin thickness τ(= difference between internal and external radii). Let Q denote any one of the seven quantities to be calculated. The calculation of Q is effected by two divisions, three to five multiplications, and the reading of a value from curves. The error due to curve interpolation does not exceed 0.8 percent, which accuracy is more than ample for the purpose intended, that is, the preliminary design phase of an airfoil. If Q has the dimensions L^s, where L is length, and $s=2$ or 4, then Q is expressed in the form

$$Q = N\tau^{\alpha}c^{b}t^{a}H(g,k)$$

where N $(=1$ or $2)$ is dimensionless, $g=2\tau/t=$skin ratio, $k=t/c=$thickness ratio, and α, b, and a are non-negative integers with $\alpha+b+a=s$. The "shape function" $H(g, k)$ has positive lower and upper bounds in the closed unit square $0\leqq g\leqq 1$, $0\leqq k\leqq 1$. Each of the seven functions $H(g, k)$ has been calculated to five significant figures for g and k equal to 0.0, 01., $\cdots$, 1.0, and is presented graphically as a function of g with one curve for each value of k, all plotted on the same pair of axes.

8. *On p-adic numbers*, by Professor B. W. Jones, Cornell University.

The p-adic numbers were defined in terms of formal series, and their fundamental properties derived. Their relation to congruence theory was discussed, and some applications were briefly dealt with.

P. H. DAUS, *Secretary*

MARCH MEETING OF THE PACIFIC NORTHWEST SECTION

The second annual meeting of the Pacific Northwest Section of the Mathematical Association of America was held at the University of Oregon, Eugene, on Friday and Saturday, March 26–27, 1948.

Fifty-nine persons attended, including the following forty-four members of the Association: W. H. Bunch, L. G. Butler, Paul Civin, C. L. Clark, Douglas Derry, N. S. Free, K. S. Ghent, W. M. Gilbert, F. L. Griffin, S. G. Hacker, H. H. Irwin, R. D. James, J. M. Kingston, W. J. Kirkham, Celia E. Klotz, M. S. Knebelman, J. C. R. Li, J. J. Livers, C. F. Luther, L. H. McFarlan, H. C. Mayer, A. S. Merrill, W. E. Milne, A. F. Moursund, D. C. Murdoch, Ivan Niven, Andrewa Noble, T. G. Ostrom, T. S. Peterson, A. R. Poole, H. F. Price, Margaret Ramsey, R. A. Rosenbaum, J. J. Rowland, R. B. Saunders, W. G. Scobert, R. H. Stair, W. M. Stone, G. H. Van Arkel, J. R. Vatnsdal, G. A. Williams, L. B. Williams, R. M. Winger, F. E. Wood.

At the business meeting of the Section the following officers were elected for the year 1948–9: Chairman, W. E. Milne, Oregon State College; Vice-Chairman, R. M. Winger, University of Washington; Secretary-Treasurer, S. G. Hacker, State College of Washington. The section voted to accept the invitation of Oregon State College to hold the next annual meeting in Corvallis in the spring of 1949.

The Friday afternoon session was opened by Professor M. S. Knebelman, Chairman of the Section, and Professors F. L. Griffin and R. M. Winger were invited by him to preside. Professor Ivan Niven, University of Oregon, delivered an invited hour address on *The Density of a Set of Integers*. He gave various definitions of number theoretic density and some of the basic theorems in each case, including a discussion of the recent work of R. C. Buck on measure theoretic density and H. B. Mann's solution of the $\alpha+\beta$ problem.

Following the business meeting on Friday night a conference on the mathematical training of prospective secondary school teachers in the Pacific Northwest was held, with Professor Milne presiding. Professor Knebelman presented a brief report of the studies being made by this Association's Committee for the Coordination of Studies in Mathematical Education, of which he is a member. As a result of the spirited discussion which the conference provoked, it was de-

cided to appoint a committee with representation from the several institutions represented in this section to study this important matter and to make specific recommendations, including the possible specification of minimum requirements in mathematical preparation for certification of prospective high school teachers of mathematics in this region. Professor H. H. Irwin, State College of Washington, was appointed chairman of this committee.

The first five of the following papers as well as Professor Niven's address were presented at the Friday afternoon session. The remainder were read on Saturday morning, Professors A. S. Merrill and R. D. James presiding at the invitation of Professor Milne.

1. *Some theorems on measure*, by Professor R. A. Rosenbaum, Reed College.

A generalization of a theorem of Ostrowski on the Lebesgue measure of the direct sum of certain euclidean sets was proved. This theorem, together with some others, was applied to a discussion of finite solutions of sub-additive functions.

2. *Power series and Laplace transforms with multiply monotonic conditions*, by Mrs. Helen Clucas, University of Oregon, introduced by Professor Paul Civin.

The behavior of a power series with multiple monotonic functions was discussed. In particular, asymptotic evaluations were given for the remainder term of power series with coefficients of monotonic order 3 and of Laplace transforms of functions of monotonic order 4. (In the absence of the author this paper was read by Mr. W. G. Scobert, University of Oregon.)

3. *The solution of linear integral equations by means of Wiener integrals*, by Professor T. G. Ostrom, Montana State University.

An abstract of this paper has appeared in the *Bull. Amer. Math. Soc.*, vol. 53, 1947, p. 490 (Abstract 53-5-222).

4. *The spherical representation of a hypersurface*, by Professor M. S. Knebelman, State College of Washington.

This paper dealt primarily with the total curvature of a hypersurface expressed in terms of the curvature tensor. Allendoerfer gave such an expression as a rational function for a subspace of an even number of dimensions. The expression obtained in this paper is valid for any number of dimensions but it is not rational, and in the case of a space of an odd number of dimensions it may fail to be real.

5. *The June meeting of the American Mathematical Society at the University of British Columbia*, by Professor R. D. James, University of British Columbia.

Professor James gave an interesting account of the extensive scientific and social preparations which have been made for the entertainment of the mathematicians and other scientists meeting in Vancouver during the month of June. A cordial invitation was extended to the members of this section to attend these meetings and also to enjoy the many recreational facilities of Vancouver and its environs.

6. *On the precision of representative sampling*, by Professor Z. W. Birnbaum, University of Washington.

The frequently stated belief that the precision of the mean of a representative sample is always at least as good as that of the mean of a random sample of the same size is shown to be incorrect

for sampling without replacements. This paper was read by title.

7. *A discussion of the basic course in statistics*, by Professor W. J. Dixon, University of Oregon, introduced by Professor A. F. Moursund.

An outline and discussion of a course in statistics for all students which follows the recommendations of the National Research Council were presented.

8. *The quadrilaterals of Pascal's hexagram*, by Mr. W. H. Bunch, Nyssa (Oregon) High School.

This paper has since been published in this MONTHLY, vol. 55, 1948, p. 210.

9. *An intermediate algebra course for third year*, by Professor Douglas Derry, University of British Columbia.

The author criticized the traditional theory of equations course as failing to be in accord with modern algebraic development. An alternative course in linear algebra and analytic geometry was suggested.

10. *The axiomatic method in analysis*, by Professor Paul Civin, University of Oregon.

Certain weaknesses are pointed out in the logical structure of the analysis program as it is typically presented to the undergraduate. The desirability and feasibility of placing analysis on the same foundation as algebra or topology at the beginning graduate level is noted. This paper was read by title.

11. *The Babbage idea and its realization*, by Professor S. G. Hacker, State College of Washington.

Babbage's idea regarding his difference and analytical engines was discussed, with lantern slide illustrations of the respective parts completed by C. and H. P. Babbage. A realization of Babbage's "analytical engine" was first accomplished in the Harvard IBM-ASCC. The relative merits of the ASCC and ENIAC, as balanced machines, were cited. It was suggested that the average mathematics department should not allow interest in the high-speed automatic calculators to overshadow, in instruction and research, the importance of the modern "difference engines," electric and hand desk computers, and certain ledger-posting machines. The work in this connection of L. J. Comrie and others was cited. A brief critical bibliography of the latter, as well as of the Babbage machines and their modern counterparts, was given.

12. *Note on the solution of differential equations by symbolic methods*, by Professors W. H. Gage and R. D. James, University of British Columbia.

In textbooks on differential equations there is usually a fairly complete account of the use of symbolic methods in finding the particular integral of a linear differential equation with constant coefficients. The authors stated that it does not seem to have been pointed out, however, that a symbolic method may also be employed to find the complementary function. This paper described such a method which the authors have used to advantage in classes at the University of British Columbia.

13. *The adjoint of a partial differential equation*, by Professor L. H. McFarlan, University of Washington.

The formation of the adjoints of certain types of partial differential equations was shown to be simplified by use of Euler equations of the calculus of variations.

14. *Alternate digit cancellation*, by Mr. G. M. Peterson, University of British Columbia, introduced by the Secretary.

In certain cases the operation of cancelling alternate digits in numerator and denominator yields a result which coincides with ordinary cancellation (e.g. $16/64=1/4$). All two digit fractions having this property, for radix α, were determined.

15. *Some remarks on indices*, by Mr. G. E. Latta, University of British Columbia, introduced by Professor R. D. James.

The ξ of a prime to the base a, $\xi_a(p)$, is defined to be the exponent to which a belongs mod p, and if $a^{\alpha-1}+^{\alpha-2}+\cdots+a+1$ is the smallest number of this form which p divides, then $\xi_a(p)=\alpha$. Certain interesting sequence of numbers such as the Fermat and Mersenne numbers are such that the ξ_a of all the factors of any one of these numbers is the same. This makes it possible to form a certain diophantine equation, the integral solutions of which lead to the factors of the number in question. By one of the properties of $\xi_a(n)$, the relative primeness of the Mersenne numbers was demonstrated, and a method was obtained for forming any desired number of infinite sequences of relatively prime integers.

16. *The graph of a function, a classroom note*, by Professor S. A. Jennings, University of British Columbia.

When functions of the form $f(x)$, $f(x, y)$, $f(x, y, z)$ are defined as numerical valued mappings of regions in one, two, or three dimensions, they are all of the type $P\rightarrow f(P)$. Methods of presenting this idea at an elementary level are discussed, and certain of its advantages for the teaching of multiple integration are pointed out. This paper was read by title.

S. G. HACKER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-second Annual Meeting, Columbus, Ohio, December 31, 1948.

Thirty-first Summer Meeting, Boulder, Colorado, September, 1949.

The following is a list of the Sections of the Association with dates of future meetings insofar as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, University of Pittsburgh, November 6, 1948
ILLINOIS, Peoria, May 13–14, 1949
INDIANA
IOWA, Drake University, Des Moines, April 15–16, 1949
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, University of Mississippi, Oxford, Spring, 1949
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May, 1949
NORTHERN CALIFORNIA, San Francisco, January 29, 1949
OHIO, Ohio State University, Columbus, April 2, 1949
OKLAHOMA
PACIFIC NORTHWEST, Oregon State University, Corvallis, Spring, 1949
PHILADELPHIA, Philadelphia, November 27, 1948
ROCKY MOUNTAIN, Colorado School of Mines, Golden, April, 1949
SOUTHEASTERN, University of Alabama, University, March 18–19, 1949
SOUTHERN CALIFORNIA, John Muir Junior College, Pasadena, March 12, 1949
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, University of Buffalo, May, 1949
WISCONSIN, Lawrence College, Appleton, May, 1949

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WHAT IS A FUNCTIONAL?

L. M. GRAVES, University of Chicago

1. Introduction. The use of the word "functional" (French—"fonctionnelle") to denote a function whose value depends on the form of an ordinary function of a real variable began with the work of Volterra and Fréchet about a half century ago. An example of such a functional is the length

$$L(y) = \int_a^b \sqrt{1 + y'^2}\, dx$$

of the graph of $y = y(x)$. Here the value of $L(y)$ depends on all the values taken by $y(x)$ on the interval (a, b). The same remark is true in general of the integrals

$$F(y) = \int_a^b \phi(x, y, y')dx$$

of the calculus of variations. The calculus of variations is in fact concerned with the theory of maxima and minima for a somewhat vaguely characterized class of functionals. In the general theory of functionals it is convenient to admit to consideration such special functionals as

$$F_1(y) = y(c),$$
$$F_2(y) = y'(c),$$

where c is a fixed point of the interval (a, b), as well as other more general types which are not considered in the calculus of variations.

The values taken by a functional may also be functions, as, for example,

$$z(x) = \int_a^b K(x, t, y(t))dt. \tag{1}$$

We may write this equation in the form $z = T(y)$, where we use $T(y|x)$ as a symbol for the right hand side of (1). Such functionals are sometimes called "functional transformations." Another example of a functional transformation is given by the familiar operation of taking the derivative of a function.

2. Functional equations. Many types of functional equations which may be used to determine functions are well known, and in fact their study makes up a large part of analysis. For example, we may consider an ordinary differential equation

$$y' = \phi(x, y)$$

with fixed initial condition $y(\xi) = \eta$. Then under suitable conditions on the function ϕ, the solution $y(x)$ is uniquely determined. This solution clearly depends on the form of ϕ, so we may write $y = F(\phi)$, and inquire into the properties of

the functional transformation $F(\phi)$. A theory of "differential corrections" based on such an inquiry was developed by Bliss [**3, 4**]. This theory is useful, for example, in computing the perturbations of the trajectories of projectiles, due to wind or other disturbances.

Another type of functional equation is the linear integral equation

$$y(x) = \int_a^b K(x, t)y(t)dt + z(x). \tag{2}$$

Here the solution $y(x)$ depends on both $z(x)$ and $K(x, t)$, so we are led to consider the functional transformation $y = F(z, K)$ giving the solution of (2) for each z and K for which it exists. In this case the transformation F is linear in the variable z.

3. Development of abstract function theory. The common features of various existence theorems in analysis suggest that a general and abstract theory, embracing many of them under one head, is possible. It is now generally recognized that an abstract theory has many advantages. For example, the existence and embedding theorems for ordinary differential equations become special cases of a general implicit function theorem. The title "functional analysis" is frequently used for an abstract theory of functions, as well as for theories of special functionals such as those described above. But the term "functional" has now become somewhat outmoded, and the term "function" is used to refer to a correspondence between variables of any type whatever.

The most widely used set of postulates for the basis of an abstract function theory characterize what is usually called a Banach space. Memoirs using essentially equivalent sets of postulates were published about the same time by Banach [**1, 2**], Hahn [**7**], and Wiener [**14**]. Another equivalent set may be stated as follows.

I. $\mathfrak{Y}$ is a class of elements, denoted by $y, y_1, y_2, \cdots$, and called points, which is a linear space over the real numbers. That is, (a) Addition of elements of $\mathfrak{Y}$ is defined, and $\mathfrak{Y}$ with addition forms a commutative group; (b) Scalar multiplication (meaning multiplication of elements of $\mathfrak{Y}$ by real numbers) is defined, and is associative with multiplication of real numbers, and distributive with respect to addition of real numbers and to addition of elements of $\mathfrak{Y}$.

II. Corresponding to each point y in $\mathfrak{Y}$ there is determined a real number $\|y\|$, called the norm of y, such that: (a) $\|y_1+y_2\| \leqq \|y_1\| + \|y_2\|$; (b) $\|ay\| = |a| \|y\|$; (c) $\|y\| = 0$ if and only if y is the identity element Θ of the group $(\mathfrak{Y}, +)$.

III. The space $\mathfrak{Y}$ is complete, in the sense that to every sequence (y_n) such that $\lim_{m=\infty, n=\infty} \|y_m - y_n\| = 0$, there corresponds a point y in $\mathfrak{Y}$ such that $\lim_{n=\infty} \|y_n - y\| = 0$. In other words, every Cauchy sequence in the space has a limit in the space.

The following examples of Banach spaces are very frequently useful.

1. The space $\mathfrak{E}_n$, which is the n-dimensional number space consisting of all n-uples $(y^1, \cdots, y^n)$ of real numbers. Here we may take $\|y\| = \max |y^i|$, or $\|y\| = (\sum |y^i|^2)^{1/2}$.

2. Real Hilbert space $\mathfrak{H}$, consisting of all infinite sequences (y^i) of real numbers such that $\|y\| = (\sum |y^i|^2)^{1/2}$ is finite.

3. The space $\mathfrak{L}_p$, corresponding to a fixed measurable set E and a number $p \geqq 1$, and consisting of all functions $y(t)$ which are measurable on E and have $|y(t)|^p$ Lebesgue-integrable over E. Here we set $\|y\| = [\int_E |y(t)|^p dt]^{1/p}$, and it becomes necessary to identify functions which are equal almost everywhere.

4. The space $\mathfrak{C}$, composed of all functions $y(t)$ which are continuous and bounded on a fixed set E. Here we set $\|y\| = \text{l.u.b. } |y(t)|$ on E. The set E is sometimes required to be bounded and closed, and may even be restricted to be an interval.

5. The space $\mathfrak{C}^{(n)}$, composed of all functions $y(t)$ which are continuous and bounded on a set E with suitable properties, and which have continuous and bounded derivatives up to and including those of the nth order. Here we may set $\|y\|$ equal to the least upper bound of $|y(t)|$ and of all its derivatives up to the nth order.

At the present time Hilbert space is also usually characterized postulationally. From this point of view the space $\mathfrak{L}_2$ becomes another realization of Hilbert space, on a par with $\mathfrak{H}$ as described under 2 above. It is clearly possible to admit functions with complex values in all the examples listed above. The scalar multipliers in the postulate system may then be allowed to be complex also.

Examples of functions defined on the spaces $\mathfrak{E}_n$ are familiar. The integrals of the calculus of variations mentioned above are functions defined on the space $\mathfrak{C}^{(1)}$, with real values. If we consider the transformation

$$z(x) = \int_a^b K(x, t)y(t)dt,$$

we see that it defines a function $z = F(y)$ on the space $\mathfrak{L}_2$, with values also in the space $\mathfrak{L}_2$ if $K(x, t)$ is in the space $\mathfrak{L}_2$ for the square $a \leqq x \leqq b$, $a \leqq t \leqq b$. If $K(x, t)$ is in the space $\mathfrak{C}$, then the values of $F(y)$ are also in the space $\mathfrak{C}$.

The generalizations of the fundamental ideas of function theory to abstract spaces, so far as these generalizations have been developed, naturally present a greater variety than is present in the classical function theory. For example, definitions of differentiability which coalesce for functions defined in the space $\mathfrak{E}_n$ turn out not to be equivalent in general. For brevity, we shall state only a definition of the class C' of functions defined on an open set Y_0 in a Banach space $\mathfrak{Y}$. Such a function $F(y)$ is said to be of class C' on Y_0 in case there exists a function $dF(y; \delta y)$ defined for each y in Y_0 and each δy in $\mathfrak{Y}$, with the properties: (a) $dF(y; \delta y)$ is linear and continuous in δy; (b) $dF(y; \delta y)$ is continuous in y uniformly for $\|\delta y\| = 1$; (c) for each y in Y_0.

$$\lim_{\|h\|=0} \frac{\|F(y + h) - F(y) - dF(y; h)\|}{\|h\|} = 0.$$

The function $dF(y; \delta y)$ is called the *differential* of F. When the space $\mathfrak{Y}$ is the Cartesian product of two Banach spaces $\mathfrak{W}$ and $\mathfrak{Z}$, that is, the variable y is a

symbol for the pair of variables (w, z), then the differential $dF(y; \delta y)$ is the sum of the partial differentials $d_w F(w, z; \delta w)$ and $d_z F(w, z; \delta z)$.

The ordinary theorem on the existence of functions defined implicitly has an immediate generalization in Banach spaces. See [8]. Let $F(y)=F(w, z)$ be of class C' on the open set Y_0, with values in the space $\mathfrak{W}$, and suppose that $F(w_0, z_0)=\Theta=$the zero of the space $\mathfrak{W}$. Suppose also that the linear transformation $d_w F(w_0, z_0; \delta w)$ maps the space $\mathfrak{W}$ into itself in one-to-one fashion, that is, $w'=d_w F(w_0, z_0; \delta w)$ has an inverse transformation $\delta w = K_0(w')$. (This condition takes the place of the usual hypothesis on the non-vanishing of the Jacobian.) Then there is a unique function $G(z)$ defined and of class C' near z_0 and taking values near w_0, such that $F(G(z), z)=\Theta$. This theorem may be proved by the classical method of successive approximations applied to the equation

$$w = w - K_0(F(w, z)),$$

which is equivalent to $F(w, z)=\Theta$. It is interesting because such results as existence theorems for differential equations (including boundary value problems) and integral equations may be regarded as special cases of it. Moreover, various apparent extensions of the theorem itself become special cases of it when appropriate new variables are introduced.

As an application of the implicit function theorem, we may consider an integral equation of the form

$$y(x) = \int_a^b K(x, t, y(t))dt + z(x). \tag{3}$$

If we require the functions y and z to be in the class $\mathfrak{C}$, and suppose that $K(x, t, y)$ and its partial derivative $K_y(x, t, y)$ are continuous, and if equation (3) has an initial solution $(y_0(x), z_0(x))$ at which the linear integral equation

$$\eta(x) = \int_a^b K_y(x, t, y_0(t))\eta(t)dt + \zeta(x)$$

has a unique solution for $\eta(x)$, then the nonlinear equation (3) has a unique solution $y(x)$ near $y_0(x)$ for each $z(x)$ near $z_0(x)$. Furthermore, the solution $y=G(z)$ is of class C' as a function of z.

The general implicit function theorem may also be applied in certain cases to show the existence of solutions of differential equations satisfying boundary conditions. Let the equation be of the form

$$y'' = f(t, y, y'), \tag{4}$$

where f is of class C' for $a \leqq t \leqq b$, and for all values of y and y', and suppose the partial derivatives satisfy the inequalities

$$0 \leqq f_y(t, y, y') \leqq M,$$
$$|f_{y'}(t, y, y')| \leqq M,$$

for all (t, y, y'). Then the differential equation (4) has a solution taking arbitrary preassigned values A and B at the ends of the interval (a, b). Other applications include embedding theorems for use in the calculus of variations, in cases where a direct proof would be extremely involved, and theorems on elliptic partial differential equations.

4. Conclusion. The functional calculus has developed in many other directions, and no adequate notion of these developments and their ramifications can be conveyed here. As an additional sample, a theorem on the existence of a minimum for a function may be mentioned. As is well known, a function $f(y)$ defined and lower semi-continuous on a bounded closed set S in the space $\mathfrak{E}_n$ has a minimum, that is, an attained lower bound, on S. If we suppose that S is a set in an abstract Banach space $\mathfrak{Y}$, the condition that S is bounded and closed is no longer sufficient. Instead, we may now require that S is compact on itself, that is, that every infinite subset of S has a point of accumulation in S. But this makes a theorem which is too restricted for applications in the calculus of variations. If we call a sequence (y_n) in S a *minimizing sequence* for f in S in case

$$\lim_{n=\infty} f(y_n) = \text{lower bound of } f \text{ on } S,$$

then we may state the following extension of the theorem on existence of a minimum:

THEOREM. *If $f(y)$ has a minimizing sequence (y_n) in S which is compact on S, and if $f(y)$ is lower semi-continuous relative to compact subsets of S, then f has a minimum on S.*

This theorem is valid in spaces more general than Banach spaces, and is only one of a number of theorems which it is interesting and useful to study under the least restrictive hypotheses.

Several books devoted to various aspects of functional analysis are included in the bibliography below. The number of memoirs on the subject is enormous, and the few works listed can give only an introduction to the field. Some of the numerous books and memoirs of Volterra include applications to hysteresis, phenomena connected with competing or coöperating populations, and phenomena of heredity.

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A CALL FOR REFORM IN HIGH SCHOOL MATHEMATICS*

C. N. SHUSTER, New Jersey State Teachers College

Down through the ages mathematics has been a highly respected subject, but in our present scientific, mechanistic, industrial civilization, mathematics is vastly more important than it ever was in any former age. Furthermore, the need for mathematics and trained mathematicians is constantly increasing, and this seems to be a permanent trend.

At a time, however, when mathematics is becoming constantly more necessary, and the talented student with deficient mathematical training is finding more and more doors leading to interesting professions closed to him, mathematics is being slowly but surely crowded out of our secondary curriculum by far less important subjects and activities. In a large portion of the country a pupil may graduate from high school with one year or less of mathematics.

High school mathematics is in a very unhealthy condition at the present time. When a doctor seeks to cure a patient, it is quite important for him to determine the cause of the malady.

Fifty years ago, mathematics was king of the required subjects; mathematics was supposed to develop, strengthen, and discipline the mind. However, early in the present century, educators and administrators began to question seriously the value of the mathematics then being taught. Teachers with little knowledge of mathematics or rigor were constantly insisting on mathematical rigor. Mortality in mathematics was very high; and as the great influx of pupils of lesser ability began, this became a serious problem. Finally the entire doctrine of formal discipline was swept away, and the mathematics based on this

* The second paper of a Symposium, "College entrance requirements in mathematics," held at the thirty-first annual meeting of the Association, Athens, Georgia, January 1, 1948. The first paper appeared in the August–September issue.

doctrine began to be supplemented by less difficult and more interesting subjects. This trend was augmented by the wave of soft pedagogy that began to sweep the country like a plague.

Unfortunately, leaders in mathematics did not first put their house in order before essaying its defense. As a result we have had a forty-year war between the educator and the mathematician, with mathematics constantly on the losing end. Like most wars, this has been unnecessary. More than fifty years ago the great leaders, Klein, Tannery, Perry, Borel, Eliot, Judd, Young, and Moore, pointed the way to reform that would have given us mathematics so easy to defend that no defense would have been necessary.

We can easily double the value of high school mathematics, make it far more interesting to our pupils and much easier to teach by heeding the advice given by E. H. Moore in 1903 and by dozens of other great leaders since his time. Among these leaders are Myers, Breslich, Smith, Reeve, Schorling, Betz, Clark, Hedrick, and Swenson. But as Hedrick said shortly before his death, "Everybody renders lip service to the reform movement, but nobody does anything about it."

To be brief at the expense of a certain amount of oversimplification, the reforms needed are as follows:

1. We must stop trying to teach mathematics by the stupid "water-tight compartment" method long since abandoned or greatly modified by practically all other countries. The now classic report of the Association's 1923 Committee on Mathematical Requirements says: "The results already achieved by those who are experimenting with the new methods of organization warrant the abandonment of extreme 'water-tight compartment' methods of presentation."

W. D. Reeve says, "Our traditional 'water-tight compartment' method of teaching algebra, then geometry, then intermediate algebra, leads to a great deal of unnecessary repetition of subject matter that results in the loss of time and energy" [1]. B. Bradford repeats the charge with the statement, "Of the many vices characteristic of much mathematical education not the least is the separation of the science into a number of almost water-tight compartments" [2]. W. Betz says, "The prevailing curriculum ignores not only the organic relationships of the various branches of mathematics but also their fruitful application. Instead we still insist on studying each mathematical subject in a compartment by itself and in a manner which is neither psychological nor practical. Moreover, we crowd each of these compartmentalized courses into a single semester or a single year" [3]. And, from D. E. Smith comes the accusation, "Of all the advanced countries of Europe and America, and we may also include Japan, we have been unquestionably the most backward in our introductory work in mathematics" [4].

2. We must remove the large amount of relatively useless material that is found in our best modern texts, and introduce more interesting and more practical mathematics to take the place of the dead wood eliminated.

More than twenty-five years ago a committee of the Association reported:

"The situation that needs to be met may best be illustrated by the case of algebra. Our elementary algebra is, in theory and symbolism, substantially what it was in the seventeenth century. The present standards of drill work, largely on non-essentials, were set up about fifty years ago. A considerable number of teachers, both in the secondary schools and the colleges, believe that the amount of time spent by pupils on abstract work in difficult problems in division, factoring, fractions, simultaneous equations, radicals, *et cetera*, is excessive; that such work leads to nothing important in the science, and adds little to facility in the manipulation of algebraic forms." This was quoted in the recent report of the National Council of Teachers of Mathematics Commission on Post War Plans and is one of the highlights of the report. W. D. Reeve says, "We shall have to get rid of the deadwood, the debris that clutters up the courses in mathematics" [1]. Raleigh Schorling also asserts, "No one should assume that all is well with the traditional courses. In most schools they are woefully out-of-date as regards both subject matter and method" [5].

3. We must have ability grouping in mathematics. There must be at least two and at best three types or levels of mathematics in high schools.

For "track" one we should have four years of the best type of "integrated" mathematics or some form of superior "general mathematics" for our very best students. Unfortunately, the term "general mathematics" has come to mean inferior or non-college mathematics. Other terms used for the original meaning are "fused," "correlated," "integrated," and "related." This is the kind of mathematics taught in all countries outside the United States.

For "track" two we should have three years of highly practical mathematics including at least a year of social and economic mathematics.

For "track" three we should have, where possible, a third track of two years of work similar to that of track two but less difficult.

We have always had ability grouping in athletics. If the results in the teaching of mathematics were as immediately determined as in athletics, we should have ability grouping at once. Ability grouping is the essence of true democracy, that is, each pupil entitled to be developed to his limit without being held back by less capable pupils.

Unfortunately, ability grouping has been tried in many places without much success. We must have texts, courses of study, and methods for each level, or much of the possible efficiency of ability grouping will be lost. We have never had such texts.

In the January *Mathematics Teacher*, Professor Reeve says, "How can the content be so reorganized and taught as to be worth while in the lives of the boys and girls who make up the secondary school? It is time for action." And in a recent letter from which I have received permission to quote, Professor Wren says, "I have a matter I want to present to you for your consideration. I have been thinking for some time that there is a big problem which the National Council and the Association ought to attack together. It is a reconsidera-

tion of the problem of the curriculum and instruction in mathematics, including teacher training, for grades 1 through 14. We need a strong committee, representative of the best thought in the organizations. It should be given money with which to operate, with the thought in mind that it should think through its problem and frame its procedure in such a way that an appeal might be made for support from some foundation such as the General Education Board or the Carnegie Foundation. This is really a major problem of great import to the schools. The personnel of the committee should be composed of individuals who have a sincere interest in mathematics but who are not so steeped in traditionalism that they cannot approach the problem with an open mind. We need to weed out some of the errors and stupidity of traditionalism and shape a program which is directed more nearly toward the interests and welfare of the pupils concerned."

No one can see the problem and its possible solution more clearly than Professor Wren. If the problem can be solved, it will be worth many millions of dollars yearly to the country and will be one of the most important projects in the history of American education. To secure the needed and long overdue reforms in mathematics we must have a committee of experts with adequate financing, so that they may devote their full time to this most important project. This may well take a year or more. The final report should be printed and given wide distribution. Part of this report could be a set of specimen or model textbooks from which textbooks authors could use any material they desired.

Nothing is so important to a nation, school, association, or factory as a plentiful supply of the best possible raw material. We can never expect to get adequate raw material—pupils with the mathematical training the colleges desire—until we have solved the problem Professor Wren poses.

At the present time several state curriculum committees are studying the reorganization of mathematics. There is very little hope that they will materially improve the situation in mathematics. A few tired teachers with heavy teaching schedules cannot do the work that Professor Wren and the author believe to be beyond the financial ability of such strong associations as the Mathematical Association of America and the National Council of Teachers of Mathematics. It is like sending a child with a trowel to do the work of a powerful bulldozer. In fact, considerable harm may be done by such state committees, should they succeed in convincing the teachers in their states that they have solved the problem.

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ON THE PARTITION OF NUMBERS INTO SQUARES

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1. Introduction. Let us denote by $P_k(n)$ the number of partitions of n into k integral squares $\geqq 0$. The term partition implies that we do not reckon as distinct two decompositions of n into k squares in which the squares are merely permuted. For example, since

$$\begin{aligned} 98 &= 9^2 + 4^2 + 1^2 \\ &= 9^2 + 3^2 + 2^2 + 2^2 \\ &= 8^2 + 5^2 + 3^2 \\ &= 8^2 + 4^2 + 3^2 + 3^2 \\ &= 7^2 + 7^2 \\ &= 7^2 + 6^2 + 3^2 + 2^2 \\ &= 6^2 + 6^2 + 5^2 + 1^2 \end{aligned}$$

we have $P_1(98)=0$, $P_2(98)=1$, $P_3(98)=3$ and $P_4(98)=7$.

The function $P_k(n)$ is practically unknown in spite of the extensive literature on sum of squares, and it appears to be a function of great complexity. The simplest question one can think of is that of solving the equation $P_k(n)=0$. This problem is completely solved. Every number, according to Fermat's theorem, is the sum of four squares, so that for $k\geqq 4$, $P_k(n)>0$. For $k=3$, every number, except all those of the form $4^\alpha(8m+7)$, is a sum of three squares. For $k=2$, $P_2(n)=0$ if and only if the highest power of a prime $p=4x+3$ dividing n is odd. For $k=1$, $P_1(n)=0$ only for non-squares.

In this note we attempt the modest question of solving the equation

$$P_k(n) = 1,$$

that is, of finding all integers n which are sums of k squares in essentially one way. We shall see that this problem is almost solvable. It is comparatively easy to write down generating functions for $P_k(n)$. One such is

$$k!\sum_{n=0}^{\infty} P_k(n)x^n = \frac{\partial^k}{\partial z^k}\left\{(1-z)(1-xz)(1-x^4z)(1-x^9z)\cdots\right\}^{-1}\Big]_{z=0}.$$

However this is not a very useful tool for the direct attack on $P_k(n)$ since the development of the right member in powers of x is best accomplished by actually partitioning each number n into squares.

2. Partition into 4 squares. We begin with the case of $k=4$. The question of the number of ways in which every number is the sum of four squares was answered by Jacobi [1] in 1828 by the following remarkable theorem:

JACOBI'S THEOREM. *The number $R_4(n)$ of representations of n as the sum of four squares is given by*

$$R_4(n) = 8[2 + (-1)^n]\sigma_0(n),$$

where $\sigma_0(n)$ denotes the sum of the odd divisors of n.

This beautiful result would seem to leave nothing to be desired. Thus for $n=98$ we have

$$R_4(98) = 8\cdot 3\cdot \sigma_0(98) = 8\cdot 3\cdot (1 + 7 + 49) = 1368.$$

However, as we have seen, $P_4(98)=7$. The explanation of this apparent discrepancy lies, of course, in the distinction between "partition" and "representation." For the problem of 4 squares the latter term implies that all permutations of the summands and both signs of the square roots of the squares >0 are reckoned as distinct solutions of the problem of making up the desired sum. To put the same thing in geometrical language, $R_4(n)$ is the number of lattice points in four-dimensional space on the surface of the hypersphere of radius $n^{1/2}$ about the origin. To the uninitiated student there appears to be only 7 "really" distinct solutions and to him this note is dedicated. To him it is a fair question to ask how one may obtain the reasonable number 7 from the obviously inflated value 1368. This problem is unfortunately one of extreme difficulty. At least no solution has ever been given. The difficulty lies in the unequal number of representations derivable from a single partition. It is not difficult to see that there are eleven different types of partitions as far as we are concerned. These may be listed, each with the number of derivable representations, as follows, where a, b, c, d represent different non-zero integers.

	I	$a^2 + b^2 + c^2 + d^2$	384
	II	$0^2 + a^2 + b^2 + c^2$	192
	III	$a^2 + a^2 + b^2 + c^2$	192
	IV	$a^2 + a^2 + b^2 + b^2$	96
	V	$0^2 + a^2 + a^2 + b^2$	96
(1)	VI	$a^2 + a^2 + a^2 + b^2$	64
	VII	$0^2 + 0^2 + a^2 + b^2$	48
	VIII	$0^2 + a^2 + a^2 + a^2$	32
	IX	$0^2 + 0^2 + a^2 + a^2$	24
	X	$a^2 + a^2 + a^2 + a^2$	16
	XI	$0^2 + 0^2 + 0^2 + a^2$	8

Thus the above 7 partitions of 98 into squares are of the types II, III, II, III, IX, I, III, respectively, and we find that

$$192 + 192 + 192 + 192 + 24 + 384 + 192 = 1368$$

as predicted by Jacobi's Theorem.

A fundamental relation for the function $P_4(n)$ is

$$P_4(2n) = P_4(8n). \tag{2}$$

This follows at once from noting that any partition of $8n$ into 4 squares $\geqq 0$ must involve even squares only. Hence there is a one to one correspondence between the partitions of

$$2n = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

and

$$8n = (2x_1)^2 + (2x_2)^2 + (2x_3)^2 + (2x_4)^2.$$

By a similar argument it is seen that

$$P_4(4n) = P_4(n) + P_4'(4n)$$

where $P_4'(n)$ is the number of partitions of n into 4 odd squares. This relation contains (2) as a special case when n is even. Formula (2) shows that we may confine our problem of evaluating $P_4(n)$ to the case in which n is not divisible by 8.

Inequalities for $P_4(n)$ follow at once from Jacobi's theorem and from (1) which shows that

$$R_4(n)/384 \leqq P_4(n) \leqq R_4(n)/8.$$

Thus if ω is an odd number,

$$P_4(\omega) \geqq \sigma(\omega)/48 \tag{3}$$

$$P_4(2\omega) \geqq \sigma(\omega)/16 \tag{4}$$

$$P_4(4\omega) \geqq \sigma(\omega)/16 \tag{5}$$

where $\sigma(\omega)$ is the sum of all the divisors of ω.

We can prove the following theorems.

THEOREM 1. *The numbers* 1, 3, 5, 7, 11, 15, 23 and $4^\alpha r$, $r=2$, 6, and 14, $\alpha \geqq 0$, *and no others may be partitioned into* 4 *squares* $\geqq 0$ *in only one way.*

THEOREM 2. *The numbers* 9, 13, 17, 21, 29, 31, 35, 39, 47, 71 *and* $5\cdot 2^\alpha$, $11\cdot 2^\alpha$, $M\cdot 4^\alpha (M=1, 3, 30, 46)$, $\alpha > 0$, *and no others may be partitioned into* 4 *squares* $\geqq 0$ *in precisely two ways.*

A recent table of Gino Loria [2] gives partitions of n into four squares $\geqq 0$ for all $n \leqq 100$. This makes the proofs of Theorems 1 and 2 very easy. In fact we may invert this table after correcting its errata [3] and list for each $r \geqq 1$ the values of $n \leqq 100$ for which $P_4(n) = r$ as follows:

r	
1	1, 2, 3, 5, 6, 7, 8, 11, 14, 15, 23, 24, 32, 56, 96
2	4, 9, 10, 12, 13, 16, 17, 19, 20, 21, 22, 29, 30, 31, 35, 39, 40, 44, 46, 47, 48, 64, 71, 80, 88
3	18, 25, 26, 27, 28, 33, 37, 38, 41, 43, 51, 53, 55, 59, 60, 62, 72, 79, 92, 95
4	34, 36, 42, 45, 49, 57, 61, 63, 65, 67, 68, 69, 77, 78, 83, 87, 94
5	50, 52, 54, 58, 70, 73, 74, 75, 76, 84, 85, 86, 89, 91, 93
6	66, 81, 97, 99
7	82, 98, 100
8	–
9	90

To prove Theorem 1 we note that inequalities (3), (4), and (5) imply

$$P_4(n) \geqq \frac{\sigma(n)}{48} > \frac{n}{48} \qquad (n \neq 4m)$$

$$P_4(n) \geqq \frac{\sigma\left(\frac{n}{4}\right)}{16} > \frac{n}{64} \qquad (n = 4\omega).$$

In order that $P_4(n)=1$, it is therefore necessary that $n<64$ if n is not a multiple of 8. Hence, excepting multiples of 8, inspection of the above table shows that $n=1, 2, 3, 5, 6, 7, 11, 14, 15$, and 23. If n is a multiple of 8, write $n=4^\alpha\cdot 2m$, where $2m$ is not a multiple of 8. Then by α applications of (2),

$$P_4(n) = P_4(2m).$$

In order that $P_4(n)=1$ it is therefore necessary and sufficient that $2m=2, 6, 14$. Thus the proof of Theorem 1 is complete.

Theorem 2 may be proved in the same general way.

The case of $k>4$. We now consider the case of partitions into 5 squares. Let n be an integer for which $P_5(n)=1$. Suppose first that $n\geqq 16$. Then each of the integers

$$n - 0^2, \quad n - 1^2, \quad n - 2^2, \quad n - 3^2, \text{ and } n - 4^2$$

is non-negative and hence is the sum of 4 squares. Each of these partitions into 4 squares implies a partition of n itself into 5 squares of which one of the squares implies a partition of n itself into 5 squares of which one of the squares is 0^2, 1^2, 2^2, 3^2, and 4^2, respectively. Since n is supposed to possess but one such partition it follows that

$$n = 0^2 + 1^2 + 2^2 + 3^2 + 4^2 = 30.$$

But also $30=3^2+3^2+2^2+2^2+2^2$ so that $P_5(30)>1$. Hence we must look for n among the first 15 integers. By Theorem 1 we narrow the search to $n=1, 2, 3, 5, 6, 7, 8, 11, 14$ and 15. But 5, 8, 11 and 14 are each sums of 5 non-zero squares. Hence $P_5(n)=1$ if and only if $n=1, 2, 3, 6, 7$ and 15.

Now 6 and 15 are sums of 6 non-zero squares so that $P_6(n)=1$ if and only if $n=1, 2, 3$, and 7. Finally for $k>6$, $P_k(n)=1$ if and only if $n=1, 2$ and 3.

The case of $k<4$. It remains to consider the cases $k=1$, 2 and 3. The first of these is trivial since $P_1(n)=1$ or 0 according as n is a square or not. By a theorem announced by Fermat [4] and proved by Euler it follows that $P_2(n)=1$ if and only if n is of one of the forms

$$n = 2^\alpha Q^2 \quad \text{or} \quad n = 2^\alpha Q^2 p,$$

where in either case $\alpha \geqq 0$, Q is an odd integer having no prime factor of the form $4x+1$, and where p is a prime of the form $4x+1$.

Finally we take up the case $k=3$. It is easy to see that there is a one-to-one correspondence between the solutions of $x^2+y^2+z^2=n$ and of $(2x)^2+(2y)^2+(2z)^2=4n$. Hence we may lay aside those n's which are multiples of 4. Also $n=8x-1$ is not sum of three squares. For all other values of n the number $R_3(n)$ of representations of n as a sum of three squares is expressible in various ways [5] in terms of the number of classes of binary quadratic forms of determinant $-n$. Here we find that in order to get this result one uses only non-negative values of x, y, z in $x^2+y^2+z^2=n$ (as opposed to the case of four squares). All permutations of x, y, z are counted. There are thus only the following three types of partitions with corresponding numbers of representations

I	$a^2+b^2+c^2$	6
II	$a^2+a^2+b^2$	3
III	$a^2+a^2+a^2$	1

where a, b, c, are distinct integers $\geqq 0$. What prevents us from repeating the type of argument used in solving $P_4(n)=1$ is the absence of a good inequality for the class number function similar to $\sigma(n)>n$. Recent researches [6] have shown that the class number function assumes a given value only finitely often, as conjectured by Gauss. Hence the equation $P_3(n)=1$ has only a finite number of solutions which are not multiples of 4.

A recent table of Gupta [7] gives the number of representations $R_3(n)$ for all $n \leqq 10000$. In order that $P_3(n)=1$, it is necessary but not sufficient that $R_3(n)=1$, 3, or 6. By examining all cases where $R_3(n)$ has these values we are led to the following theorem.

THEOREM 3. *The equation* $P_3(n)=1$ *has the following solutions:* $n=4^\alpha m$, *where* $\alpha \geqq 0$, *and* $m=1$, 2, 3, 5, 6, 10, 11, 13, 14, 19, 21, 22, 30, 35, 37, 42, 43, 46, 58, 67, 70, 78, 91, 93, 115, 133, 142, 162, 182, 190, 235, 253, 403, 427. *Any further values of m lie beyond* 10000.

It is extremely likely that the above list of solutions of $P_3(n)=1$ is complete. An actual proof of this fact however must await more precise information about the class number of binary quadratic forms.

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THE ACTUARIAL PROFESSION

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1. Character of actuarial work. This article has been prepared to furnish college students with a brief description of the character of actuarial work and an outline of the opportunities, inasmuch as there is a great shortage of young men with the mathematical ability, knowledge, and qualities of business leadership required in the actuarial profession.

It is the actuary who is responsible for calculating the premiums a life insurance company must charge and it is he who prepares the tables of death rates upon which such calculations are based. In actual practice his duties cover a much wider field than such technical responsibilities. They include the decision as to what benefits shall be contained in life insurance policies and how much money must be set aside from year to year to guarantee the payment of such benefits many years in the future. The actuary must analyze the sources of earnings under policy contracts so that he may determine proper rates of dividends. He investigates the effect on mortality of various physical impairments, hazardous occupations, and other unusual risks, and in collaboration with the medical officer determines the basis for accepting or rejecting applicants for insurance. Because of his broad fundamental training, the actuary of a life insurance company usually has an important part in developing the general executive policies of the company. Although he cannot operate without a thorough knowledge of the mathematical basis of life insurance, he is more of a business man than a mathematician. Not the least of his duties is to explain complicated problems to other business men and to policyholders in language clear enough to be readily understood.

The great changes in our social and economic structure that have taken place in recent years have brought about problems requiring the special attention of the actuary. The severe reduction in the rate of interest realized on new investments has required a general revision of premium rates as well as comprehen-

hospitalization and surgical and medical reimbursement benefits.
(5) Social insurance: fundamental principles; existing social insurance plans in the United States and Canada.

MATHEMATICAL NOTES

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CASSINI OVALS ASSOCIATED WITH A SECOND ORDER MATRIX

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In a recent paper [1], Brauer proved that, if

$$P_k = \sum_{j=1,\, j\neq k}^{n} |a_{kj}|, \qquad k = 1, 2, \cdots, n,$$

then each characteristic root, λ, of a matrix $A=(a_{ij})$ lies in the interior or on the boundary of at least one of the $n(n-1)/2$ Cassini ovals:

$$|z - a_{kk}|\,|z - a_{mm}| = P_k P_m.$$

In case $n=2$, there is only one oval and the characteristic roots are on its boundary. The purpose of this note is to exhibit a relationship between the Cassini oval thus defined for a second order matrix and the field of values of the matrix, used elsewhere as the basis of theorems on limits for the characteristic roots [2].

Consider the set M of matrices which are unitary transforms of a given second order matrix A (i.e., the matrices $UA\overline{U}'$, where $U\overline{U}'=I$). These matrices have the same characteristic roots as A and the same field of values, an ellipse whose foci are the characteristic roots [3]. Since for each matrix of M there is a unique Cassini oval there is generated a set, C, of Cassini ovals associated with the fixed ellipse. Each Cassini oval of C has as its foci the principal diagonal elements of the corresponding matrix of M. Now the principal diagonal elements of a matrix belong to its field of values, and, since

$$\sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i,$$

they have the same centroid as the characteristic roots. Therefore, the foci of any Cassini oval of C are a pair of points within or on the ellipse and sym-

metrically placed with respect to its center. Conversely, each two such points, p and q, are the foci of some curve of C, for, to obtain the corresponding matrix of M, it suffices to take as the first row of U a unit vector u such that $uA\bar{u}'=p$.

The idea of comparing the Cassini ovals of C with the field of values as limits for the characteristic roots led to the following result.

THEOREM. *The set of points covered by the interiors and boundaries of the Cassini ovals of C consists of the interior and circumference of the circle concentric with the ellipse and with radius $[a^2+b^2]^{1/2}$, where a is the semi-major axis, and b the semi-minor axis, of the ellipse.*

The ellipse may be assumed in standard position. Evidently the Cassini ovals which extend farthest from the center are those whose foci are on the circumference of the ellipse. Consider, then, the sub-family C' whose foci are at (u, v) and $(-u, -v)$ for $u^2/a^2+v^2/b^2=1$. Let r and s be the constants of the Cassini oval; thus, r is the distance from the center to a focus, and s^2 is the product of the distances from a point on the oval to the two foci. Then

$$r^2 = u^2 + v^2 = \frac{a^2b^2 + c^2u^2}{a^2},$$

where c is the distance from the center to a focus of the ellipse.

Since the oval passes through the foci of the ellipse, we have

$$s^2 = [(c-u)^2 + v^2]^{1/2}[(c+u)^2 + v^2]^{1/2} = \frac{a^4 - c^2u^2}{a^2}.$$

On adding the above results, we obtain

$$r^2 + s^2 = a^2 + b^2.$$

But $[r^2+s^2]^{1/2}$ is the distance from the origin to the end of the major axis of the Cassini oval, so that this distance, for members of C', is constantly $[a^2+b^2]^{1/2}$. Furthermore, all other points of the oval are closer to the origin. The fact that C covers the circle of the theorem is now evident, as each point in or on the ellipse is a focus for some oval of C, and hence certainly interior to it, and each point outside the ellipse but inside or on the circle lies on the line joining the focus of some member of C' to the end of its major axis.

The diagonals of the circumscribing rectangle of the ellipse with sides parallel to the axes of the ellipse are diameters of the circle of the theorem, and serve to classify the ovals of C' into those with two loops, the lemniscates, and those with one loop, according as the foci are in the smaller angle formed by the diagonals, on a diagonal, or in the larger angle.

Some special cases are of interest. There are some matrices in M whose diagonal elements are equal; the associated Cassini ovals are degenerate, being circles with the line joining the foci of the ellipse as diameter. If $b=0$, the ellipse degenerates to a straight line segment, and the circle of the theorem has this line

segment as diameter. If $c=0$, the ellipse is a circle, and the associated Cassini ovals are all lemniscates.

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A SYMBOLIC FORM OF AN INVERSION FORMULA FOR A LAPLACE TRANSFORM

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In a paper which appeared in this journal [1], W. C. G. Fraser proved the following theorem

THEOREM A. *If* $\alpha(t)$ *is a function of bounded variation in every finite interval* $(1, R)$ *with* $\alpha(1)=0$, *and if*

$$f(s) = \int_1^\infty t^{-s} d\alpha(t), \tag{1}$$

the integral converging for some value of s, *then*

$$\alpha(t) = \lim_{s\to+\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} t^{ks} f(ks) \tag{2}$$

at points t *where* $\alpha(t)$ *is continuous.*

As Fraser pointed out, this is an elaboration of an earlier result of E. Phragmén [2]. If $\alpha(t)$ is the integral of a function $\phi(t)$, a corresponding theorem follows by the same methods. The conclusion can be obtained formally by differentiating both sides of equation (2) with respect to t. Using the more usual form of the Laplace transform, we state it as follows:

THEOREM B. *If* $\phi(t)$ *is continuous and such that the integral*

$$f(s) = \int_0^\infty e^{-st}\phi(t)\,dt \tag{3}$$

converges for some value of s, *then*

$$\phi(t) = \lim_{s\to+\infty} s \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!} e^{kst} f(ks), \qquad (0 < t < \infty). \tag{4}$$

This inversion is of particular interest since it involves only the values of $f(s)$ itself (and not of its derivatives) for large positive values of s. It thus appears, at first sight, to be very different from the well known inversion [3, 288]

$$(5) \qquad \phi(t) = \lim_{k \to +\infty} \frac{(-1)^k}{k!} f^{(k)}\left(\frac{k}{t}\right)\left(\frac{k}{t}\right)^{k+1}, \qquad (0 < t < \infty).$$

The present author has shown [4] that the inversion operator on the right-hand side of equation (5) may be regarded symbolically, after an exponential change of variable, as $1/\Gamma(D)$, where $\Gamma(s)$ is the classical gamma function and D is the operation of differentiation. It is the purpose of the present note to show that the inversion of Phragmén and Fraser (4) is also symbolically $1/\Gamma(D)$, so that from this point of view formulas (4) and (5) are equivalent.

The exponential change of variable referred to above transforms equation (3) into

$$f(e^{-s}) = F(s) = \int_{-\infty}^{\infty} G(s-t)\Phi(t)dt,$$

$$G(t) = e^{-e^{-t}}, \qquad \Phi(t) = \phi(e^t)e^t.$$

Equation (4) becomes

$$(6) \qquad \lim_{s \to +\infty} se^t \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!} e^{kse^t} F\left(\log \frac{1}{ks}\right) = \Phi(t).$$

We should like to show that equation (6) is equivalent, in some sense, to

$$\frac{1}{\Gamma(D)} F(t) = \Phi(t).$$

If we set $x = se^t$, equation (6) becomes

$$(7) \qquad \lim_{x \to +\infty} x \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!} e^{kx} F(t - \log kx) = \Phi(t).$$

As is customary in operational theory, we define $e^{aD}F(t)$ to be $F(t+a)$, so that equation (7) may be written as

$$\lim_{x \to +\infty} \left[x \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!} \frac{e^{kx}}{(kx)^D} \right] F(t) = \Phi(t).$$

Hence our result will be established if we can show that

$$\lim_{x \to +\infty} x \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!} \frac{e^{kx}}{(kx)^y} = \frac{1}{\Gamma(y)}, \qquad (0 < y < \infty).$$

But this follows from Theorem B itself if we observe that

$$\frac{1}{s^y} = \int_0^{\infty} e^{-st} \frac{t^{y-1}}{\Gamma(y)} dt, \qquad (0 < y, s < \infty),$$

and take $\phi(t) = t^{y-1}/\Gamma(y)$, $f(s) = 1/s^y$, $t = 1$. Then Theorem B yields

$$\phi(1) = \frac{1}{\Gamma(y)} = \lim_{x\to+\infty} x \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!} \frac{e^{kx}}{(kx)^y},$$

and our assertion is proved.

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THE nTH DERIVATIVE OF A FRACTIONAL FUNCTION

V. F. IVANOFF, San Francisco

Let f be a fractional function,

$$f = \frac{g}{h}.$$

Then

$$\begin{aligned}
&hf && = g\\
&h'f + hf' && = g'\\
&h''f + 2h'f' + hf'' && = g''\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&h^{(n)}f + \binom{n}{1} h^{(n-1)}f' + \binom{n}{2} h^{(n-2)}f'' + \cdots + hf^{(n)} && = g^{(n)}.
\end{aligned}$$

Hence

$$f^{(n)} = \frac{1}{h^{n+1}} \begin{vmatrix} h & 0 & 0 & \cdots & 0 & g \\ h' & h & 0 & \cdots & 0 & g' \\ h'' & 2h' & h & \cdots & 0 & g'' \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ h^{(n)} & \binom{n}{1}h^{(n-1)} & \binom{n}{2}h^{(n-2)} & \cdots & \binom{n}{1}h' & g^{(n)} \end{vmatrix}.$$

Incidentally, the solution of Problem E 789, this MONTHLY, vol. 54, 1947, p. 471; vol. 55, 1948, pp. 366–367, can be derived immediately from the above formula.

NOTE ON A THEOREM IN n-DIMENSIONAL GEOMETRY

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L. A. WIGGLESWORTH, University College, London

The following result due to Schäfli* turns up in various connections.

THEOREM 1. *If Δ_s is the determinant of the system of linear homogeneous equations*

$$(x_{i1}U_1 + \cdots + x_{in}U_n)(x_{j1}U_1 + \cdots + x_{jn}U_n) = 0, \qquad (i, j = 1, \cdots, n),$$

with the $\frac{1}{2}n(n+1)$ quantities U_kU_l as unknowns, $(k, l=1, \cdots, n)$, then $\Delta_s \neq 0$ if $\Delta = |x_{ik}| \neq 0$.

The above result can be interpreted by means of projective geometry. For it expresses the fact that there cannot exist a quadric which has a given non-degenerate simplex both inscribed in it and self polar with respect to it; the tangent space at each vertex would have to contain all the remaining vertices.

We now give a proof of Theorem 1 by means of metrical geometry. The proof depends on the generalization of a very well known theorem in elementary geometry: *Two triangles whose sides are of equal length are congruent.* The n-dimensional analogue can be stated in the following way:

THEOREM 2. *Let $P_i = (x_{i1}, \cdots, x_{in})$, $(i=0, \cdots, n)$, be $n+1$ (not necessarily real) points where $x_{01} = x_{02} = \cdots = x_{0n} = 0$ and*

$$\Delta = \begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ x_{n1} & \cdots & x_{nn} \end{vmatrix} \neq 0.$$

Then a transformation $x_k' = \sum_j a_{kj}x_j$ which carries these points into points (say P_i') with the same squared distances must necessarily be orthogonal:

$$\sum_k a_{kj}a_{kl} = \delta_{jl}.$$

Proof of Theorem 2. Since

$$x_{ik}' = \sum_j a_{kj}x_{ij}, \qquad (i, k, j = 1, \cdots, n),$$

it follows that

$$a_{kj} = \frac{\sum_i x_{ik}'X_{ij}}{\Delta}, \tag{1}$$

where X_{ij} is the cofactor of x_{ij} in Δ. Since

$$(P_iP_j)^2 = (P_i'P_j')^2, \qquad (i, j = 0, \cdots, n),$$

it follows that

* Denkschriften d. Akad. d. Wiss. Wien (2), 1857, pp. 1–24; actually, Δ_s is the $n(n-1)/2$th power of Δ.

$$\sum_k x_{ik}x_{jk} = \sum_k x'_{ik}x'_{jk}, \qquad (i, j = 1, \cdots, n). \tag{2}$$

From (1) and (2) we obtain

$$\sum_k a_{kj}a_{kl} = \frac{\sum_{i,k,h} x'_{ik}X_{ij}x'_{hk}X_{hl}}{\Delta^2} = \frac{\sum_{i,k,h} x_{ik}x_{hk}X_{ij}X_{hl}}{\Delta^2} = \sum_k \delta_{kj}\delta_{kl} = \delta_{jl},$$

which is the required result.

Proof of Theorem 1. Assume that $\Delta_s = 0$. Let ω_{lm}, $(l, m = 1, \cdots, n)$, be a non-trivial solution of the system of linear homogeneous equations with the quantities $\sum_k a_{kl}a_{km} - \delta_{lm}$ as unknowns:

$$\sum_l \left(\sum_k a_{kl}^2 - 1\right)(x_{il} - x_{jl})^2 + \sum_{l<m}\sum_k 2a_{kl}a_{km}(x_{il} - x_{jl})(x_{im} - x_{jm}) = 0.$$

A non-trivial solution exists since the determinant Δ^* of this system coincides with Δ_s apart from a factor $(-2)^{n(n-1)/2}$. The rows

$$R_{ii} = (x_{i1}^2, x_{i2}^2, \cdots, x_{in}^2, 2x_{i1}x_{i2}, \cdots)$$

are common to both determinants. The row

$$R_{ij} = (x_{i1}x_{j1}, x_{i2}x_{j2}, \cdots, x_{in}x_{jn}, x_{i1}x_{j2} + x_{i2}x_{j1}, \cdots)$$

of Δ_s can be obtained from the row

$$((x_{i1} - x_{j1})^2, \cdots, (x_{in} - x_{jn})^2, 2(x_{i1} - x_{j1})(x_{i2} - x_{j2}), \cdots)$$

of Δ^* by adding $-R_{ii} - R_{jj}$ and multiplying by $-\frac{1}{2}$.

We can then find numbers a_{ik} such that

$$\sum_k a_{kl}a_{km} - \delta_{lm} = \omega_{lm}. \tag{3}$$

This is possible since it means to find n (not necessarily real) vectors in n dimensions $(a_{i1}, \cdots, a_{in})$ whose scalar products are given. This is equivalent to the problem of finding n (not necessarily real) points in n dimensions if their squared distances from $(0, \cdots, 0)$ and from each other are given. The latter is always possible.†

Consider then the linear transformation $x'_k = \sum_j a_{kj}x_j$ and its effect on the points P_i. Suppose that the transform of P_i is $P'_i = (x'_{i1} \cdots, x'_{in})$. Then

$$\begin{aligned}(P'_iP'_j)^2 - (P_iP_j)^2 = &\sum_l \left(\sum_k a_{kl}^2 - 1\right)(x_{il} - x_{jl})^2 \\ &+ \sum_{l<m}\sum_k 2a_{kl}a_{km}(x_{il} - x_{jl})(x'_{im} - x'_{jm}) = 0\end{aligned}$$

† See A. Wald, Erg. Math. Koll (Vienna) 5, 1933, pp. 32–42, and O. Taussky, Erg. Math. Koll. (Vienna) 6, 1935, pp. 20–23. See also L. M. Blumenthal, Distance Geometries, University of Missouri Studies, (13), 1938, pp. 70–71.

because of (3). This is a contradiction because from Theorem 2 the transformation $x \rightarrow x'$ is orthogonal so that $\omega_{lm} = 0$.

Conversely, Theorem 2 is a consequence of Theorem 1. This can be shown by a mere rearrangement of the proof given for Theorem 1.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

All material for this department should be sent to C. B. Allendoerfer, Institute for Advanced Study, Princeton, New Jersey.

TEACHING NOTE ON REPEATED ROOTS IN LINEAR DIFFERNTIAL EQUATIONS

J. B. REYNOLDS, Lehigh University

The following simple and direct method of proving the rule for double roots in the solution of linear differential equations with constant coefficients is seldom seen in elementary texts on the subject.

If

$$Ay^{(n)} + By^{(n-1)} + \cdots + Ly^{(2)} + My^{(1)} + Ny = 0 \tag{1}$$

is a differential equation in which $y^{(n)}$ stands for the nth derivative of y with respect to x and

$$f(m) = Am^n + Bm^{n-1} + \cdots + Lm^2 + Mn + N = 0, \tag{2}$$

and we substitute

$$y = c_1 e^{mx} + c_2 x e^{mx} \tag{3}$$

into (1) we find

$$[c_1 f(m) + c_2 f^1(m) + c_2 x f(m)]e^{mx} = 0 \tag{4}$$

in which $f^1(m)$ is the derivative of $f(m)$ with respect to m.

Equation (4) is satisfied, making (3) a solution of (1), if $f(m) = 0$ and $f^1(m) = 0$, simultaneously.

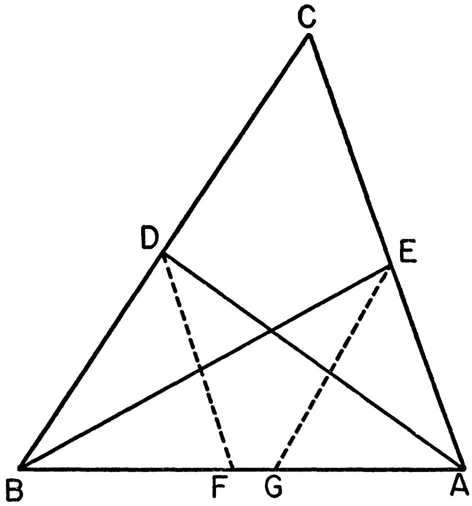
Since these are the conditions for (2) to have two equal roots, (3) is a solution of (1) in case (2) has two equal roots.

The method can readily be extended for the case of any number of multiple roots.

ANGLE BISECTORS OF AN ISOSCELES TRIANGLE

W. E. BLEICK, U. S. Naval Postgraduate School

The following is an indirect proof that a triangle ABC is isosceles if the angle bisectors AD and BE are equal. If the triangle is not isosceles, one of the bisected angles must be greater than the other. Assume that angle EAB is greater than angle ABD. Construct a line through E parallel to BC and intersecting AB at G. Construct a line through D parallel to AC and intersecting AB at F. Then $EG>EA$ since $BC>AC$ by hypothesis. Also $DF>EG$ from triangles DFA and EGB. Now $AG=AB-EG$ and $BF=AB-DF$. Hence $BF<AG$ since $DF>EG$. The triangles ABC, AGE and FBD are similar. Hence



$BF/AG=DF/EA$, from which it follows that $DF<EA$ since $BF<AG$. But $EA<EG$ from above. Hence $DF<EG$ which contradicts the inequality $DF>EG$ above. Hence the bisected angles must be equal. This proof was obtained by Maria Goeppert Mayer in 1932.

EQUATION OF VALUE IN INVESTMENT THEORY

G. F. ROSE, University of Wisconsin

Let us define an *investment system* to be any set of parties who have agreed upon a method of calculating the obligation of each one of them to each of the others.

An investment system is said to be in *equilibrium* if none of its parties has an obligation to any of the others. The *liquidation value* at a given time of any party A of an investment system is defined to be the net amount (with proper algebraic sign) that would be received by A in the process of instantaneously bringing the system into equilibrium at that time.

Any problem in the realm commonly denoted by "Mathematics of Finance"

can be reduced to the analysis of one or more investment systems. The almost universally used method of calculating obligations makes the mathematics of this analysis especially simple. Thus, it turns out that the liquidation value of any party A varies according to the following rules:

1. Upon completion of a payment to or from A, A's liquidation value is decreased or increased respectively by the amount of the payment.

2. Between payments, A's liquidation value varies at a fixed compound interest rate which will be referred to as A's *yield rate.*

The direct application of rules 1 and 2 constitutes the form of computation known as the *amortization table*, which consists of a payment-to-payment study of liquidation value. A more convenient procedure, however, is formulated by the following theorem.

THE EQUATION OF VALUE THEOREM: *The following process is valid for any party A whose liquidation value follows the above rules* 1 *and* 2.

1. *Consider the investment system over any time interval.*
2. *Choose any reference date.*
3. *Using A's yield rate, bring to the reference date A's liquidation value at the beginning of the interval and also all items leaving A's hands within the given interval. Sum the resulting values.*
4. *Using A's yield rate, bring to the reference date A's liquidation value at the end of the interval and also all items entering A's hands within the given interval. Sum the resulting values.*
5. *Equate the sum in* 3 *to that in* 4.

I wish to call attention to the fact that the equation of value process has not the status of an axiom; on the contrary, it stands on the prior concept of the amortization table procedure. To prove the equation of value theorem, let us consider a party A, with yield rate (j, m), over a time interval t_0 to t_{n+1}. Suppose that, within this interval, A makes and receives a total of n payments P_i; call their respective dates t_i $(t_0 < t_1 \leqq \cdots \leqq t_n < t_{n+1})$. Let $V(t)$ denote A's liquidation value at date t. Let r be any reference date. With rules 1 and 2 as our sole assumptions, we must establish the validity of

$$(1) \qquad V(t_0)k^{r-t_0} + \sum_{(1\leqq i\leqq n, P_i \text{ from } A)} P_i k^{r-t_i} = V(t_{n+1})k^{r-t_{n+1}} + \sum_{(1\leqq i\leqq n, P_i \text{ to } A)} P_i k^{r-t_i},$$

where $k=(1+j/m)^m$.

Let I_i be P_i or $-P_i$ according as P_i is from A or to A respectively. We will prove by induction on i that A's liquidation value after payment at date $t_i (1 \leqq i \leqq n)$ is

$$(2) \qquad V(t_i) = V(t_0)k^{t_i-t_0} + \sum_{j=1}^{i} I_j k^{t_i-t_j}.$$

In the first place $V(t_1) = V(t_0)k^{t_1-t_0} + I_1$. Furthermore,

$$
\begin{aligned}
V(t_{i+1}) &= V(t_i)k^{t_{i+1}-t_i} + I_{i+1} \\
&= (\text{by induction hypothesis})\left[V(t_0)k^{t_i-t_0} + \sum_{j=1}^{i} I_j k^{t_i-t_j}\right]k^{t_{i+1}-t_i} + I_{i+1} \\
&= V(t_0)k^{t_{i+1}-t_0} + \sum_{j=1}^{i+1} I_j k^{t_{i+1}-t_j}.
\end{aligned}
$$

Hence (2) is established. As a consequence,

$$
\begin{aligned}
V(t_{n+1}) &= V(t_n)k^{t_{n+1}-t_n} \\
&= V(t_0)k^{t_{n+1}-t_0} + \sum_{j=1}^{n} I_j k^{t_{n+1}-t_j} \\
&= V(t_0)k^{t_{n+1}-t_0} + \sum_{(1\leqq j\leqq n, P_j \text{ from } A)} P_j k^{t_{n+1}-t_j} - \sum_{(1\leqq j\leqq n, P_j \text{ to } A)} P_j k^{t_{n+1}-t_j},
\end{aligned}
$$

from which (1) follows after multiplying through by $k^{r-t_{n+1}}$.

TANGENT TO A CIRCLE FROM AN EXTERIOR POINT

F. H. YOUNG, Oregon State College

The following method of finding the equation of a line tangent to a circle and passing through a point outside the circle has proved popular with my class in elementary analytic geometry.

Consider the circle

$$(x - h)^2 + (y - k)^2 = r^2$$

and the exterior point (x_1, y_1). Denote by S the length of tangent to the circle from (x_1, y_1). Denote by m_c the slope of the line joining (h, k) and (x_1, y_1). Denote by t the tangent of the angle between the line from (x_1, y_1) to (h, k) and the tangent line. Obviously, $t=r/S$. Then, by using the formula for tan $(a\pm b)$, we obtain

$$m = \frac{m_c \pm t}{1 \pm m_c t},$$

the slopes of the tangent lines. Knowing the point and the slopes, we can easily obtain the desired equations.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 831. *Proposed by K. W. Crain, Purdue University*

If squares be constructed on the legs of a right triangle, the lines (which do not lie along the sides of the triangle) drawn from each end of the hypotenuse to a vertex of the opposite square intersect on the altitude which passes through the vertex of the right angle.

E 832. *Proposed by V. E. Dietrich, Purdue University*

If a circle has a center with at least one irrational coördinate, then there are at most two points on the circle with rational coördinates.

E 833. *Proposed by P. D. Thomas, Washington, D. C.*

A surveyor in laying out a square park area in a city found it was necessary because of obstructions to shorten two opposite sides by one foot, but both the length and width were in integral feet. To check his work he ran a diagonal of the resulting rectangle. Imagine his surprise to find that the semi-perimeter (diagonal+length+width) was in integral rods! What were the dimensions of the field?

E 834. *Proposed by Don Walter, Pomona College*

Show that

$$F_n = \begin{vmatrix} 1 & -1 & 1 & -1 & 1 & -1\cdots \\ 1 & 1 & 0 & 1 & 0 & 1\cdots \\ 0 & 1 & 1 & 0 & 1 & 0\cdots \\ 0 & 0 & 1 & 1 & 0 & 1\cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots\cdot \end{vmatrix},$$

where F_n is the nth term of the Fibonacci sequence $1, 1, 2, 3, 5, \cdots, x, y, x+y, \cdots$ and the determinant is of order $n-1$.

E 835. *Proposed by Kenneth May, Carleton College*

If x_1, y_1, and a are real numbers, and for all integral $n \geqq 1$ we have $x_{n+1} = a(x_n^2 - y_n^2)$ and $y_{n+1} = 2ax_ny_n$, for what values of x_1 and y_1 will $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$?

SOLUTIONS

Binomial Coefficients

E 794 [1947, 545]. *Proposed by Huan-Ting Kuo, National Wuhan University, China*

Show that

$$\begin{vmatrix} \binom{r+1}{r} & \binom{r+1}{r+1} & 0 & 0 & \cdots & 0 \\ \binom{r+2}{r} & \binom{r+2}{r+1} & \binom{r+2}{r+2} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \binom{n}{r} & \binom{n}{r+1} & \binom{n}{r+2} & \binom{n}{r+3} & \cdots & \binom{n}{n-1} \end{vmatrix} = \binom{n}{r}.$$

Solution by R. A. Bradley, University of North Carolina. Let us give a proof by induction. We first note that the identity holds for $n=r+1$ and $r+2$. On the assumption that the identity is true for $n=r+1, r+2, \cdots, k-1$ we now show that it is true for $n=k$. Denote the determinant for $n=k$ by D_k. Expanding D_k repeatedly with respect to the elements of the last column we obtain

$$D_k = k\binom{k-1}{r} - \frac{k(k-1)}{2!}\binom{k-2}{r} + \frac{k(k-1)(k-2)}{3!}\binom{k-3}{r} - \cdots$$
$$\pm \frac{k(k-1)\cdots(r+2)}{(k-r-1)!}\binom{r+1}{r} \mp \binom{k}{r}.$$

Factoring out $\binom{k}{r}(k-r)$ from all terms, $(k-r-1)/2$ from all but the first, $(k-r-2)/3$ from all but the first two, etc., we obtain

$$D_k = \binom{k}{r}(k-r)\left\{1 - \frac{k-r-1}{2}\left\{1 - \frac{k-r-2}{3}\left\{1 - \frac{k-r-3}{4}\left\{\cdots\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left\{1 - \frac{2}{k-r-1}\left\{1 - \frac{1}{k-r}\right\}\right\}\cdots\right\}\right\}\right\}\right\}.$$

Successively simplifying the last bracket, the jth bracket is seen to be $(k-r-j)/(k-r)$, and continuing

$$D_k = \binom{k}{r}(k-r)\left\{1 - \frac{k-r-1}{2}\,\frac{2}{k-r}\right\} = \binom{k}{r}.$$

The identity is now established for all integral $n \geqq r+1$.

Also solved by J. C. Eaves, Shai-shi Ting, and John Williamson. Irving Kaplansky pointed out that the identity is a special case of a formula on p. 257 of Netto's *Lehrbuch der Combinatorik*. Reference is made there to Zeipel, Lunds Universitets Års-Skrift, 1865.

Tetrahedron and Perspective Triangles

E 795 [1947, 546]. *Proposed by N. A. Court, University of Oklahoma*

The pairs of points U', U''; V', V''; W', W'' are marked, respectively, on the edges DA, DB, DC of a tetrahedron $ABCD$. Determine three points U, V, W on the edges BC, CA, AB, respectively, so that the three lines joining the vertices of each of the triangles $UV'W''$, $VW'U''$, $WU'V''$, UVW to the corresponding vertices of the triangles DCB, DAC, DBA, ABC, respectively, shall have a point in common.

Solution by the Proposer. If triangle $UV'W''$ is perspective to DCB, the line $V'W''$ meets BC in the harmonic conjugate P of U with respect to the points B, C. Similarly, the lines $W'U''$, $U'V''$ meet the edges CA, AB in the harmonic conjugates Q, R of V, W with respect to the pairs of points C, A; A, B. Now if triangle UVW is perspective to ABC, the three points P, Q, R are collinear, for they belong to the axis of perspectivity of ABC and UVW.

Thus a necessary condition for the problem to have a solution is that the three points $P=(V'W'', BC)$, $Q=(W'U'', CA)$, $R=(U'V'', AB)$ shall be collinear. It is readily seen that this condition is also sufficient. When this condition is fulfilled the problem has only one solution.

Coin-Tossing Experiment

E 796 [1946, 596]. *Proposed by Henry Scheffé, University of California at Los Angeles*

Describe a coin-tossing experiment in which the probability of success is one-third.

I. *Solution by R. V. Andree, University of Wisconsin.* Following are some solutions:

1. Five uniform coins are tossed until they do *not* all show the same face. Then (a) the probability of exactly two heads is $1/3$, (b) the probability of exactly three heads is $1/3$, (c) the probability of a four to one split is $1/3$.

2. Given one normal coin and one two headed coin, one coin is selected at random and tossed. It comes down heads. Then the probability that it is the normal coin is $1/3$.

3. A game of chance is played as follows: A has two coins, B has one coin. All three coins are tossed. If all three coins agree, each player takes his own coins back and no exchange is made. Otherwise A wins if both of A's coins disagree with that of B and B wins if only one of A's coins disagrees with that of B. Then the probability that A will win is $1/3$.

4. An interesting possibility is brought up by discarding the assumption that the coin is "thin" and thus that the probability of its standing on edge is negligible. Let us consider a coin the ratio of whose diameter to height is $\sqrt{3}/1$ Then the probability that this coin will land heads is $1/3$.

II. *Solution by Leo Moser, University of Manitoba.* Following are some solutions:

1. Two coins are tossed repeatedly until not both of them fall tails. If both of them are then heads the experiment is counted a success. The probability of success is 1/3.

2. A single coin is tossed repeatedly until the number of heads is one greater than, or two less than, the number of tails. The probability of the latter case is 1/3. This method can be used to obtain an experiment having any preassigned *rational* number as probability of success. For proof of this see Whitworth's *Choice and Chance*, p. 231.

3. We describe a coin-tossing experiment E_p having probability of success p, where p is any preassigned *real* number, $0 \leqq p \leqq 1$. Express p as a binary decimal, $p = \sum_{i=1}^{\infty} a_i/2^i$, $a_i = 0$ or 1. Where two expressions are possible, as for $\frac{1}{2} = .1\dot{0} = .0\dot{1}$, take either one. Now toss a coin repeatedly and let $b_i = 1$ if the ith toss results in heads and 0 if the ith toss results in tails. Let the experiment terminate when first $a_i \neq b_i$ and be a success if then $a_i > b_i$. The probability that the experiment will never terminate is clearly 0. The probability that it will terminate in success at the ith toss is $a_i/2^i$, so that the total probability of success is p.

III. *Solution by A. C. Cohen, Jr., University of Georgia.* The following simple case of the "king bee" matching scheme furnishes a solution. Three individuals agree to match, each tossing one coin. They agree to toss until one member of the trio is "odd," and the "odd" man is declared winner. The probability that any designated member of the trio be declared winner is 1/3.

IV. *Solution by J. P. Kelly, Carbide and Carbon Chemicals Corp., Oak Ridge, Tennessee.* A and B toss a coin alternately until one gets a head. If A tosses first, then the probability of B getting a head first is 1/3.

Also solved by Colin Blyth, G. F. Cramer, George Grossman, P. R. Hill, Sr., H. G. Landau, C. F. Pinzka, Hanan Rubin, E. D. Schell, R. J. Walker, and the proposer.

Ellipses in a Circle

E 797 [1947, 596]. *Proposed by C. O. Hines, University of Toronto*

If ellipses are described on diameters of a given circle as major axis, and such that they all pass through a given point (within, or on the boundary of, the circle), then they will also all pass through a second point, symmetrical about the center to the first, and the locus of their foci will be an ellipse having the two fixed points as foci and the common diameter as major axis.

Solution by Colin Blyth, University of North Carolina. Let A and B, the symmetrical fixed points, lie on the diameter MON of the circle, whose center is O. Let C and D be the foci of a general ellipse of the given family. Then $CA + AD = MN$. But, by symmetry, $CB = AD$. Therefore $CA + CB = MN$, and C lies on

the ellipse with foci A and B and major axis MN.

Also solved by R. V. Andree, W. G. Brady, Ragnar Dybvik, Bradford Hadnot, Roger Lessard, Miriam Millman, Margaret Olmsted, P. D. Thomas, Maud Willey, and the proposer.

Distribution of Suits in a Bridge Hand

E 798 [1948, 30]. *Proposed by J. M. Elkin, Chicago, Ill.*

Prove that $\Pi(n_i!)$, where Σn_i is constant, is a minimum when $\Sigma|n_i-n_j|$ is a minimum, and that, consequently, the most likely distribution of the four suits in a bridge hand is four cards of one suit and three cards each of the other three suits.

Solution by Leo Moser, University of Manitoba. If $n_i>n_j+1$ then

$$n_i!n_j! = (n_i - 1)!(n_j + 1)!n_i/(n_j + 1) > (n_i - 1)!(n_j + 1)!.$$

Hence a necessary and sufficient condition that $\Pi(n_i!)$ be minimum with Σn_i constant is that $|n_i-n_j| \leqq 1$. But, since we are dealing with integers, a necessary and sufficient condition that $|n_i-n_j| \leqq 1$ with Σn_i constant is that $\Sigma|n_i-n_j|$ be a minimum. This completes the first part of the problem.

The distribution 4, 3, 3, 3, however, is not the most probable distribution of suits in a bridge hand. The actual state of affairs and also the 'common mistake' leading to the statement made in the problem are discussed in *Mathematical Puzzles for Beginners and Enthusiasts*, Geoffrey Mott-Smith, pp. 91, 101, 214.

Also solved, partially, by Richard Courter, E. D. Schell, and the proposer.

Editorial Note. It would seem that the theorem of the problem cannot be directly applied to the problem of finding the most likely distribution of the four suits in a bridge hand. The number of bridge hands with n_1 cards of one suit, n_2 of another, n_3 of another, and n_4 of another, is

$$4C(13, n_1)C(13, n_2)C(13, n_3)C(13, n_4) = 4(13!)^4/\Pi[n_i!(13 - n_i)!],$$

where $n_1+n_2+n_3+n_4=13$. By actual computation this is found to be a maximum for the distribution 4, 4, 3, 2. This distribution, then, is the most likely. It occurs better than once in five hands, and is over twice as frequent as the 4, 3, 3, 3 distribution incorrectly supposed by the proposer to be the most frequent. For the numbers of bridge hands corresponding to the various suit distributions see *Bridge Hands*, this MONTHLY [1941, 329].

A Combinatorial Identity

E 799 [1948, 30]. *Proposed by Leo Moser, University of Manitoba*

Prove that for all n

$$\sum_{r=0}^{n}\left[2^{n-r}\binom{n}{r}^2 - \binom{2n-r}{n}\binom{n}{r}\right] = 0.$$

I. *Solution by J. G. Herriot, Stanford University.* We have

$$(x+1)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r}$$

and

$$(2+x)^n = \sum_{r=0}^{n} \binom{n}{r} 2^{n-r} x^r.$$

It is obvious that

$$\sum_{r=0}^{n} 2^{n-r} \binom{n}{r}^2$$

is the coefficient of x^n in the expansion of $(x+1)^n(2+x)^n$. But

$$(x+1)^n(2+x)^n = (1+x)^n[(1+x)+1]^n = \sum_{r=0}^{n} \binom{n}{r} (1+x)^{2n-r}.$$

The coefficient of x^n in the last expression is

$$\sum_{r=0}^{n} \binom{2n-r}{n} \binom{n}{r}.$$

Hence

$$\sum_{r=0}^{n} 2^{n-r} \binom{n}{r}^2 = \sum_{r=0}^{n} \binom{2n-r}{n} \binom{n}{r},$$

which is equivalent to the required expression.

II. *Solution by E. D. Rainville, University of Michigan.* Let $P_n(x)$ be the standard Legendre polynomial. We shall show that the left member of the equation in the problem is $[P_n(3) - P_n(3)]$, and therefore vanishes for all n.

Two standard forms for the Legendre polynomial are

$$P_n(x) = \sum_{r=0}^{n} \binom{n}{r}^2 \left(\frac{x-1}{2}\right)^r \left(\frac{x+1}{2}\right)^{n-r} \tag{1}$$

and

$$\begin{aligned} P_n(x) &= F\left(-n,\ n+1,\ 1,\ \frac{1-x}{2}\right) \\ &= \sum_{r=0}^{n} \frac{(n+r)!}{(r!)^2(n-r)!} \left(\frac{x-1}{2}\right)^r. \end{aligned} \tag{2}$$

Replacing r by $(n-r)$ and reversing the order of summation in (2) yields

$$P_n(x) = \sum_{r=0}^{n} \frac{(2n-r)!}{r![(n-r)!]^2}\left(\frac{x-1}{2}\right)^r$$

or

$$(3) \qquad P_n(x) = \sum_{r=0}^{n} \binom{2n-r}{n}\binom{n}{r}\left(\frac{x-1}{2}\right)^r.$$

Equations (1) and (3) lead us to the identity, for integral n and any x,

$$(4) \qquad \sum_{r=0}^{n} \binom{n}{r}\left[\left(\frac{x+1}{2}\right)^{n-r}\binom{n}{r} - \binom{2n-r}{n}\right]\left(\frac{x-1}{2}\right)^r = 0.$$

Put $x=3$ in (4) to get the proposer's identity.

Also solved by E. D. Schell, F. C. Smith, and F. B. Thompson. The proposer stated that he obtained the identity by solving in two entirely different ways the following problem: What is the number of "paths" from (0, 0) to (n, n), the possible "moves" being of three kinds—(1) increasing the x coördinate by 1, (2) increasing the y coördinate by 1, and (3) increasing both the x and y coördinates by 1?

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4310. *Proposed by Paul Erdös, Syracuse University*

Let $f(z)=z^n+\cdots$ be a polynomial of degree n. Denote by A_f the closed region (not necessarily connected) where $|f(z)|\leqq 1$. Prove that there always exists a z_0 in A_f with $|f'(z_0)|\geqq n$. Equality occurs only for z^n.

4311. *Proposed by V. L. Klee, Jr., University of Virginia*

If k and x are positive integers, let $f_k^1(x)=k\phi(x)$, where $\phi(x)$ is Euler's totient. For $j=2, 3, \cdots$, let

$$f_k^j(x) = f_k^{j-1}[f_k^1(x)].$$

Show that for $k \leqq 3$, the sequence, $f_k^1(x), f_k^2(x), \cdots$, is eventually constant, while for $k \geqq 4$, the sequence is eventually monotonically increasing.

4312. *Proposed by R. C. Lyness, Staffordshire, England*

In order to help our school savings campaign, Morris organized a lottery. Certain members of the savings group were each issued a book containing a gross of tickets. The tickets were sold for a penny each and the prize for the winning ticket was a number of sixpenny savings stamps. When I asked Morris how many stamps the winner got, he said that I could work it out for myself. All he need tell me was that before the winning ticket was drawn all the ticket sellers had assembled around a circular table and each had handed in his takings and his unsold tickets; that (a) no two had sold the same number; that (b) the number of unsold tickets returned by each seller was in every case equal to the product of the values in shillings of the tickets sold by his two neighbors at the table; and that (c) when the ticket sellers were rewarded with a penny each for their trouble the sum left over exactly bought the winner's stamps.

How many sellers were there and how many tickets did each sell?

4313. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Two perpendicular chords AB, CD of a circle O intersect at a point P inside the circle. There are eight circles $O_1, O_2, O_3, O_4, O_1', O_2', O_3', O_4'$ tangent to both chords and also to the circle O, the first four exteriorly, the last four interiorly and such that O_i and O_i' lie in opposite quadrants formed by the given chords. (1) The sum of the squares of the distances $O_1O_1', O_2O_2', O_3O_3', O_4O_4'$ is independent of the position of P, and the products $(O_1O_1')(O_3O_3')$ and $(O_2O_2')(O_4O_4')$ are equal. (2) The radical axes of the circles O_1, O_1', O_3, O_3' taken in pairs and those of the circles O_2, O_2' O_4, O_4' taken in pairs, intersect one another in thirty-six points of which twelve are on the circle O, four of these coinciding with the vertices of the square whose diagonals are parallel to AB and CD.

4314. *Proposed by N. J. Fine, Washington, D. C.*

Prove the identity:

$$1 + \frac{x}{(1-x)^2} + \frac{x^2}{(1-x)^2(1-x^2)^2} + \frac{x^3}{(1-x)^2(1-x^2)^2(1-x^3)^2} + \cdots$$
$$= \frac{1 - x + x^3 - x^6 + x^{10} - \cdots}{[(1-x)(1-x^2)(1-x^3)\cdots]^2}.$$

SOLUTIONS

Circles and a Locus Problem

4165 [1945, 346] Corrected. *Proposed by Victor Thébault, Tennie, Sarthe, France.*

Let A and B be two fixed points of a given circle while C and D are two variable points of the same circle such that the arc length CD remains constant.

The orthogonal projection of D on AC is P and on BC is Q. Show that (1) The Simson line of C and D for the triangles ABD and ABC have fixed directions. (2) The centers of the circles tritangent to DPQ (inscribed and escribed) describe two Pascal limaçons.

*Solution by the Proposer.** Suppose arcs AB and CD are each less than a semi-circumference and have the same sense.

1. Let B', A' be the points where the perpendiculars DP, DQ to the sides AC and BC of triangle ABC again cut the given circle (O); let O', O_1, O_2 be the midpoints of the chords CD, AB', $A'B$. The circle (O') on chord CD as diameter passes through P and Q; the arcs AB', $A'B$ are equal, each being equal to the supplement of arc CD; their chords are twice the length of OO'; the lines AA', BB', O_1O_2 are parallel; the segments O_1P, O_2Q are equal and parallel to OO'; O_1PQO_2 is a parallelogram and PQ is equal and parallel to O_1O_2. When chord CD moves so as to retain the same length, the points A', B', O_1, O_2 remain fixed; hence the direction of the Simson line PQ of point D for triangle ABC does not change.

2. The bisectors of the interior and exterior angles at D of triangle DPQ cut the circles (O') and (O) again at diametrically opposite points E, F of (O') and G, H of (O); EF is perpendicular to PQ and GH is perpendicular to $A'B'$. The lines CE, CF cut (O) again at K and L respectively. Therefore the chords GK and HL are each equal to twice the length OO'. When chord CD moves, the points G, H, K, L remain fixed and points E, F describe the circles (O_4) and (O_5) on diameters GK and HL respectively. Now the centers of the circles tritangent to triangle DPQ are the points where the two circles through P and Q with centers E and F cut the lines DG and DH respectively. The locus of these centers then are Pascal limaçons whose poles are G and H and whose director circles are (O_4) and (O_5) with parameters EP and FP for which

$$EP^2 + FP^2 = CD^2, \quad \text{and} \quad 2EP\cdot FP = CD\cdot O_1O_2.$$

A particular case arises when AB and CD are diameters of (O). The locus of centers of circles tritangent to triangle DPQ is then formed by circles through A and B with centers at E and F respectively.

Editorial Note. The translator adds the following remarks. Using isotropic coördinates, we may choose the circumcircle to be the given circle, and represent points A, B, C by unit vectors α, β and γ respectively. Let D be represented by the unit vector t^2; then, since the arc CD is constant in magnitude, we may replace γ by t_0t^2 where t_0 is constant with unit modulus. From these data we obtain $\alpha\beta t_0$ is the clinant of the Simson line of D for triangle ABC and $\alpha\beta/t_0$ as that of the Simson line of C for triangle ABD; since each of these is independent of t, the Simson lines have fixed directions. The map equation of the locus of the

* Translated from the French by O. J. Ramler, Catholic University of America.

incenter and excenter of triangle DPQ within the angle D is

$$2z = (1 + t_0)t^2 + (\alpha^{1/2} + \beta^{1/2})(1 - t_0)t + (t_0 - 1)\alpha^{1/2}\beta^{1/2}$$

which is a limaçon unless $1+t_0=0$. $t=\sqrt{\alpha}$ gives $z=\alpha$, and $t=\sqrt{\beta}$ gives $z=\beta$. Thus the locus goes through A and B. If $t_0=-1$, CD is a diameter, and the map equation reduces to that of a circle. Thus it appears that the locus degenerates to a circle when CD is a diameter, whether AB is a diameter or not.

Howard Eves notes that part (1) follows immediately from the theorem on p. 206 of Johnson's *Modern Geometry*. For, by this theorem, the Simson line of C for ABD must be perpendicular to BC', the isogonal of BC for angle ABD. But as C and D move along the circle, arc length CD remaining constant, BC' remains fixed. Similarly for the Simson line of D for ABC.

A Square with Curious Digit Pattern

4227 [1946, 594]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In what number systems is the square of a number of three digits of the form bcb of the form $aabbcc$?

Partial solution by C. R. Phelps, Rutgers University. If r is the base of the number system, we have by hypothesis

$$(1)\qquad \begin{aligned}(br^2 + cr + b)^2 &= ar^5 + ar^4 + br^3 + br^2 + cr + c\\ &= (r + 1)(ar^4 + br^2 + c).\end{aligned}$$

If $r+1$ has no square factor, then $br^2+cr+b\equiv 0 \mod r+1$, whence $2b-c=k(r+1)$. Since $0<b<r$, $0\leqq c<r$, k is either 0 or 1. Suppose $k=0$, then $c=2b$ and (1) becomes

$$(br^2 + 2br + b)^2 = (r + 1)(ar^4 + br^2 + 2b),$$

hence

$$(2)\qquad b^2(r + 1)^3 = ar^4 + br^2 + 2b.$$

Expanding, we get

$$(3)\qquad ar^4 - b^2r^3 + (b - 3b^2)r^2 - 3b^2r - (b^2 - 2b) = 0.$$

The leading coefficient b of the number bcb cannot vanish. If b is not a divisor of a, since all coefficients in (3) after the first are divisible by b, Eisenstein's criterion implies that there is no rational root r unless b^2 divides the constant term b^2-2b. This gives $b=1, 2$. With $b=1$ in (3) we have $r=\pm 1$, which is impossible. With $b=2$, $c=4$ and (3) becomes $ar^3-4r^2-10r-12=0$. 6 and 12 are the only rational roots (>4) possible; $r=12$ gives nonintegral a; but $r=6$ gives $a=1$. Thus one solution is

$$242^2 = 112244 \quad \text{to the base } 6.$$

If in (3) b is a divisor of a, say $a=nb$, then (2) becomes

$$br^3 + 3br^2 + 3br + b = nr^4 + r^2 + 2.$$

Hence, since the base r is a divisor of $b-2$, we must have $b-2=0$ and $r=6$ or 3; but this gives $n=1/2$ or $n=13/9$, both contrary to the assumption that b divides a.

The cases in which $k=1$ or in which $r+1$ has a square factor present complications. There are no instances, however, with $r<100$.

Editorial Note. The Proposer has carried considerably further the investigation of cases not included in Phelps' treatment above without, however, finding any further solutions or completing the proof that no further solutions exist.

Numbers Having Same Digits in Two Systems of Numeration

4240 [1947, 168]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Determine the relations which must connect N, B, B', in order that the number N may be written with the same three digits in the system of numeration of base B as in the system oi base B'. Having given B, find B' and N. Apply the results when $B=10$.

Solution by G. W. Walker, Buffalo, New York. Let x', y', z' be the digits x, y, z in any order and suppose

$$xB^2 + yB + z = x'B'^2 + y'B' + z'. \tag{1}$$

For fixed values of B and B', this equation can be put in the form

$$px + qy + rz = 0, \tag{2}$$

where p, q, r are integral functions of B and B'. Infinitely many sets of integral values of x, y, z exist which satisfy (2), viz.

$$x = k_3q/(p, q) - k_2r/(p, r), \qquad y = k_1r/(q, r) - k_3p/(p, q),$$
$$z = k_2p/(p, r) - k_1q/(q, r),$$

where (a, b) is the (positive) greatest common divisor of a and b, and k_1, k_2, k_3 are arbitrary integers. For our problem, we require in addition

$$0 \leqq x, \quad y, \quad z < B, \quad B'. \tag{3}$$

Values satisfying both (2) and (3) need not exist, but if they exist they are easily found by trial.

If $B=10$, we take successive values for B' and seek solutions for x, y, z subject to (3). The method is essentially one of trial, although numerous ways of saving work appear in practice. All solutions thus far discovered for $B=10$ are listed below. Numbers for which the initial digit is zero to either base have been excluded.

$265_{10} = 526_7$	$774_{10} = 477_{13}$	$825_{10} = 258_{19}$
$316_{10} = 631_7$	$834_{10} = 438_{14}$	$551_{10} = 155_{21}$
$158_{10} = 185_9$	$261_{10} = 126_{15}$	$912_{10} = 219_{21}$
$227_{10} = 272_9$	$371_{10} = 173_{16}$	$511_{10} = 115_{22}$
$445_{10} = 544_9$	$913_{10} = 391_{16}$	$910_{10} = 190_{26}$
$196_{10} = 169_{11}$	$782_{10} = 278_{18}$	$911_{10} = 191_{26}$
$283_{10} = 238_{11}$	$441_{10} = 144_{19}$	$\cdots = \cdots$
$370_{10} = 307_{11}$	$518_{10} = 185_{19}$	$919_{10} = 199_{26}$
$191_{10} = 119_{13}$	$882_{10} = 288_{19}$	$961_{10} = 169_{28}$

Note in particular: $912_{10}=219_{21}=192_{26}$; $913_{10}=391_{16}=193_{26}$.
Also partially solved by Murray Barbour.

Editorial Note. The interest in such problems as this is not so much as studies in scales of notations as for the other questions which they suggest in elementary number theory. For example, suppose in the above problem that x, y, z, are given arbitrarily with x', y', z' an arbitrary permutation of x, y, z, and suppose that xx' is not a perfect square. There are then infinitely many sets of values of B and B' which satisfy (1) and (3). In fact, if p, q are positive integers for which

$$p^2 - xx'q^2 = 1,$$

it is easy to verify the B_{n+1}, B'_{n+1} as given below, will satisfy (1) whenever B_n, B_n' do:

$$\begin{aligned} B_{n+1} &= (p^2 + xx'q^2)B_n + 2x'pqB_n' + (pqy' + q^2x'y) \\ B_{n+1}' &= 2xpqB_n + (p^2 + xx'q^2)B_n' + (pqy + q^2xy'). \end{aligned} \tag{4}$$

Now $B_0 = B_0' = 1$ satisfies (1). Therefore (4) gives an infinite sequence of sets of values of B, B' all of which, from a certain point on, will satisfy also (3). This particular sequence does not in general include all solutions.

Again, we may find in many ways parametric solutions of (1) after assigning a relation between B' and B. Thus if $B'=2B+1$, $x'=y$, $y'=z$, $z'=x$, we have

$$B^2x + By + z = (2B + 1)^2y + (2B + 1)z + x,$$

whence at once $x+y\equiv 0 \bmod B$, so that $x+y=B$. Upon eliminating x and dividing by B we find

$$B^2 - 1 = (5B + 3)y + 2z.$$

Here $2z-2y\equiv 0 \bmod B+1$ or $2z-2y=m(B+1)$ with $m=0,\ 1,\ -1$. Thus we have $B-1=5y+m$. Finally

$$(m=0),\quad B=5y+1,\quad z=y,\quad x=4y+1,\quad \text{with } y \text{ arbitrary};$$
$$(m=1),\ B=5y+2,\quad z=\tfrac{1}{2}(7y+3),\quad x=4y+2,\quad \text{with } y \text{ odd}.$$

$m=-1$ is excluded since it implies $z<0$.

A few of the many other sets which may be obtained similarly are:

$x=3y+3,\ z=3y+1,\ B=4y+3,\ B'=B+1;\ xyz_B=zxy_{B'};\ y$ arbitrary.

$x=4y-1,\ \ B=5y-2z-1,\ \ B'=2B-1;\ xyz_B=yzx_{B'};z,\ y(>2z)$ arbitrary.

$y=2x+z+1,\ B=3x+z+1,\ B'=B+1;\ xyz_B=yzx_{B'};\ x,\ z$ arbitrary.

$x=4y+1,\ B=7y+2,\ B'=2B;\ xyz_B=yxz_{B'};\ y,\ z(<B)$ arbitrary.

$x=z+2,\ y=0,\ B=2z+1,\ B'=B+2;\ xyz_B=zyx_{B'};\ z$ arbitrary.

$x=u^2,\ y=2uv,\ z=v^2,\ B=kv+1,\ B'=ku+1;\ xyz_B=zyx_{B'};$ with

$u,\ v,\ k$ arbitrary except that $u>v,\ kv>\max\ (u^2-1,\ 2uv-1)$.

A Tetrahedron of Maximum Volume

4243 [1947, 168]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The point M, situated in the interior of a tetrahedron $ABCD$, such that the volume of the tetrahedron having for vertices the points of intersection of the lines $AM,\ BM,\ CM,\ DM$, with the opposite faces of $ABCD$ be a maximum, coincides with the centroid of $ABCD$.

Solution by L. M. Kelly, University of Missouri. Let the barycentric coördinates of the point M be $x_1,\ x_2,\ x_3,\ x_4$, subject to the condition $x_1+x_2+x_3+x_4=1$. Then the coördinates of the points of intersection of $AM,\ BM,\ CM,\ DM$ with the respective opposite faces will be $(0,\ x_2,\ x_3,\ x_4)$, $(x_1,\ 0,\ x_3,\ x_4)$, $(x_1,\ x_2,\ 0,\ x_4)$, $(x_1,\ x_2,\ x_3,\ 0)$. The volume of the tetrahedron in question will be

$$V=C\begin{vmatrix} 0 & x_2 & x_3 & x_4\\ x_1 & 0 & x_3 & x_4\\ x_1 & x_2 & 0 & x_4\\ x_1 & x_2 & x_3 & 0\end{vmatrix}=Cx_1x_2x_3x_4\begin{vmatrix} 0 & 1 & 1 & 1\\ 1 & 0 & 1 & 1\\ 1 & 1 & 0 & 1\\ 1 & 1 & 1 & 0\end{vmatrix},$$

where C is a constant. It is thus a question of maximizing the product $x_1x_2x_3x_4$ subject to the condition that $x_1+x_2+x_3+x_4=1$. It is clear that this occurs when $x_1=x_2=x_3=x_4$ and hence when M is the centroid. It is equally clear that this theorem is true for a simplex of n dimensions.

Also solved by R. Bouvaist and the Proposer, who notes that the analogous problem for $n=3$ is solved in *Mathesis*, 1885, p. 87.

Perfect Squares of Special Form

4244 [1947, 232]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Consider squares of $2n$ digits which can be formed by bringing together two consecutive n-digit numbers. Show that in every system of numeration there is at least one unending sequence of such squares. Show also that, if n and B are given and if x^2 is such a square, then y^2 is another where $x+y=B^n+1$ (provided squares with initial zeros are admitted). As a particular example let $B=12$, $n=3$.

Solution by Roger Lessard, École Polytechnique, Montreal. Let A be an n-digit number in the scale of B and let

$$x^2 = AB^n + (A+1) = A(B^n+1) + 1, \qquad A + 1 < B^n, \tag{1}$$

whence

$$(x+1)(x-1) = A(B^n+1). \tag{2}$$

Consider

$$x + 1 = ab, \qquad x - 1 = cd, \qquad A = ac, \qquad B^n + 1 = bd,$$

and suppose that b and d are chosen relatively prime and having the product B^n+1. We require also b, $d>2$; otherwise we have $x\pm 1 \geqq B^n+1$, whence $A+1 \geqq B^n$. Then $ab-dc=2$ has an infinite number of positive solutions:

$$a = a_1 + kd, \qquad c = c_1 + kb,$$

where a_1 and c_1 are the unique solutions for which $0<a_1<d$, $0<c_1<b$. Since $A=ac$ is greater than B^n if $k\neq 0$, a_1 and c_1 provides the only valid solution.

We will therefore get as many solutions for given B and n as there are ways in which B^n+1 is separable into two relatively prime factors each of which exceeds 2. If B^n+1 is divisible by p distinct primes, the number of solutions will be 2^p-2, except however that if B^n+1 is even, 2 is not counted as one of the primes unless B^n+1 is divisible by 4. But B^n+1 is composite with at least two prime factors larger than 2 whenever n is not a power of 2. Therefore there are solutions for any base B and for most of the values of n.

If we substitute $x=B^n+1-y$ in (1) we have

$$(B^n + 1 - y)^2 - 1 = A(B^n + 1), \tag{3}$$

from which we get

$$y^2 = A'B^n + A' + 1, \qquad A' + x = A + y. \tag{4}$$

Therefore y is another solution whenever x is one.

If $B=12$, $n=3$, we have $B^n+1=7\cdot 13\cdot 19$. Corresponding to $d=7$, 13, 19, $7\cdot 13$, $7\cdot 19$, $13\cdot 19$ we use the above method to find that x^2 equals respectively $02E030$, 193194, 283284, 340341, 468469, $8T08T1$. These values of x^2 are written in the scale of 12 with T and E as the digits for ten and eleven.

Note that if we interchange the values of b and d we get the corresponding y instead of x.

Also solved by Murray Barbour and the Proposer.

Editorial Note. The proposed problem admits also the interpretation

$$x^2 = AB^n + (A-1). \tag{1'}$$

For such values of x the argument in (3) and (4) above still holds, for we have $x+y=B^n+1$, $x-y=A-A'$, which imply

$$y^2 = A'B^n + A' - 1. \tag{4'}$$

There are no numbers of this form for $B=12$, $n=3$ because B^n+1 is a multiple of 7 and therefore cannot be a divisor of x^2+1.

The Proposer points out that for $x=B^m$, the number x^2 consists of the $2m$-digit number $A=00\cdots 01$ followed by $2m$ zeros. Therefore for $y=B^{2m}+1-B^m$ we shall have $A'=y-x+A=B^{2m}-2B^m+2$. Thus y^2 is the $4m$-digit number

$$\overline{B-1}\;\overline{B-1}\cdots\overline{B-1}\;\overline{B-2}\;0\;0\cdots 2\;\overline{B-1}\;\overline{B-1}\cdots\overline{B-1}\;\overline{B-2}\;0\;0\cdots 0\;1$$

in which there are $m-1$ digits in each set of $(B-1)$'s and in each set of zeros. We thus have an explicit form for the members of an infinite sequence of squares as required.

The Proposer suggests also the study of squares formed by the juxtaposition of A and $A+2$, or in general A and $A+a$. For example, with $B=10$, $9011^2=81198121$, $990^2=00980100$, $7312^2=53465344$, $2689^2=07230721$.

For squares of $2n$ digits formed by bringing together two equal n-digit numbers, see the National Mathematics Magazine, Problem no. 431, May 1942, p. 409.

*n*th Derivatives

4246 [1947, 232]. *Proposed by J. A. Greenwood*

Evidently $y'=x'Dy$, $y''=(x''D+x'^2D^2)y$, $\cdots$, where $D\equiv d/dx$ and primes indicate differentiation with respect to t. Find an explicit expression for $y^{(n)}$.

Solution by Robert Breusch, Amherst College. Consider the partitions of n into positive integers,

$$n = p_1 + 2p_2 + 3p_3 + \cdots + rp_r.$$

An expression for $y^{(n)}$ may be given as

$$y^{(n)} = \sum K(p_1, p_2, \cdots, p_r)x'^{p_1}x''^{p_2}\cdots x^{(r)p_r}D^{p_1+\cdots+p_r}y, \tag{1}$$

$$K(p_1, p_2, \cdots, p_r) = \frac{n!}{(2!)^{p_2}(3!)^{p_3}\cdots(r!)^{p_r}p_1!p_2!\cdots p_r!},$$

where the summation is taken over all possible partitions of n. The formula is correct for $n=1$ and $n=2$. Assume that it holds for any particular n and dif-

ferentiate. From any term of (1) we will obtain:

a) one term $x'^{p_1+1}x''^{p_2}\cdots x^{(r)p_r}D^{p_1+1+p_2+\cdots+p_r}y$, and

b) further terms in which one exponent decreases by unity and the following exponent increases by unity, while the remaining exponents and the index of D remain unchanged; as, for example,

$$x'^{p_1}x''^{p_2-1}x'''^{p_3+1}x''''^{p_4}\cdots x^{(r)p_r}D^{p_1+\cdots+p_r}y.$$

In each case the sum of the exponents of $x', x'', \cdots, x^{(r)}$ is equal to the index of D, while the sum of the weighted exponents is $n+1$. Thus $y^{(n+1)}$ is a sum of terms:

$$y^{(n+1)} = \sum C(q_1, q_2, \cdots, q_s)x'^{q_1}x''^{q_2}\cdots x^{(s)q_s}D^{q_1+\cdots+q_s}y, \tag{2}$$
$$q_1 + 2q_2 + 3q_3 + \cdots + sq_s = n + 1.$$

Now, considering the different ways in which

$$x'^{q_1}\cdots x^{(s)q_s}D^{q_1+\cdots+q_s}y$$

can be obtained through differentiation of terms in (1), we have

$$\begin{aligned}C(q_1, \cdots, q_s) = {} & \frac{n!}{(2!)^{q_2}\cdots(s!)^{q_s}(q_1-1)!q_2!\cdots q_s!}\\ & + \frac{n!(q_1+1)}{(2!)^{q_2-1}(3!)^{q_3}\cdots(s!)^{q_s}(q_1+1)!(q_2-1)!q_3!\cdots q_s!}\\ & + \frac{n!(q_2+1)}{(2!)^{q_2+1}(3!)^{q_3-1}\cdots(s!)^{q_s}q_1!(q_2+1)!(q_3-1)!\cdots q_s!}\\ & + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ & + \frac{n!(q_{s-1}+1)}{(2!)^{q_2}\cdots\{(s-1)!\}^{q_{s-1}+1}(s!)^{q_s-1}q_1!\cdots(q_{s-1}+1)!(q_s-1)!}.\end{aligned}$$

Here, whenever $t=0$, $1/(t-1)!$ is to be understood as having the value 0, so that $1/(t-1)!=t/t!$ is generally correct. Then finally

$$\begin{aligned}C(q_1, \cdots, q_s) &= \frac{n!(q_1+2q_2+3q_3+\cdots+sq_s)}{(2!)^{q_2}(3!)^{q_3}\cdots(s!)^{q_s}q_1!q_2!\cdots q_s!}\\ &= \frac{(n+1)!}{(2!)^{q_2}\cdots(s!)^{q_s}q_1!\cdots q_s!},\end{aligned}$$

and the proof is complete by induction. Also solved by Chih-yi Wang.

Isosceles n-Points

4254 [1947, 345]. *Proposed by Paul Erdős, Syracuse University*

We have seven points in the plane. Prove that we can always select three

which do not form an isosceles triangle. For six points this does not necessarily hold. (If A, B, C are on a line we can define that they do not form an isosceles triangle if $AB \neq BC$.)

Solution by Michael Golomb, Purdue University. If every triangle formed by three of the n points, $1, 2, \cdots, n$, is isosceles we call the figure an isosceles n-point.

Isosceles 4-points. 1. Let there be three equal sides issuing from one vertex, say $12=13=14$. Then 234 is an isosceles triangle and 1 is its circumcenter. Conversely, the vertices and circumcenter of any isosceles triangle form an isosceles 4-point (type I.)

2. Suppose there is no vertex from which issue three equal sides, and suppose $12=13$. We distinguish two cases: (a) $42=43$. Then since $14\neq 12$ and $41\neq 42$, we have $21=24$. Similarly, $31=34$. Hence the 4-point is a rhombus (type II). (b) $24=23$. Then $42=41$ and $34=31$. Hence $21=13=34=a$ and $14=42=23=b$. We have a trapezoid with three equal sides a and with base $b=\frac{1}{2}a(1+\sqrt{5})$. Hence 1, 2, 3, 4 are four vertices of a regular pentagon (type III).

Isosceles 5-points. Since four vertices of an isosceles 5-point must form an isosceles 4-point, we may show by similar elementary argument that every isosceles 5-point is of one of the three types: (I) a square with its center, or (II) the vertices of a regular pentagon, or (III) the center and four of the vertices of a regular pentagon.

Isosceles 6-points. Among the six points there must be either a square or four vertices of a regular pentagon. In the first case there is only one position left for the remaining two points, hence no isosceles 6-point is obtained. In the second case, there are two positions left for the remaining two points and we obtain the only isosceles 6-point: the vertices and center of a regular pentagon.

Isoceles 7-points. Among the seven points there must be the vertices of a regular pentagon. Since there is only one position left for the remaining two points, the isosceles 7-point is impossible.

Editorial Note. See also the solution and notes for No. E 735 (1947, 227–229).

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Elementary Concepts of Mathematics. By B. W. Jones. New York, The Macmillan Company, 1947. 13+294 pages. $4.00.

The author of this book faced a problem much more difficult than that of writing a text for a standardized course in introductory college mathematics. The book is one more attempt to present a terminal course for students of minimum training who want firmer grounding in such mathematics as may be useful or interesting to them in later life. In designing such a course not only must there be kept in mind the limited ability and even more limited training of the students concerned, but it must be realized that the present day applications of mathematics are so numerous and involve so many techniques that even a minimum list of topics is larger than can be effectively dealt with.

In his book Mr. Jones has treated at about equal length topics useful to his readers and those primarily of recreational interest to them. His choice of topics, as would that of any other writer, invites argument, for if mathematics is to be a part of a liberal education we must discover what domains of mathematics throw most light upon the nature of the discipline and what ones are of most use to persons of scant training. Mr. Jones has touched upon many areas of mathematics, including several of relatively recent discovery, such as groups, non-euclidean geometry and topology, and, as indicated below, has presented the material understandably, coherently and interestingly. But there are two central ideas in mathematics of such far-reaching importance that one wonders if their omission can be justified. The idea of an infinite sequence appears more or less incidentally in the paragraphs on the integers, on the decimal representation and on prime numbers. But infinity, as a concept, and the exciting and stimulating field of the infinite are passed by. Limits enter only with regard to the convergence of a geometric series; the vast domain of the calculus, with the limit at its heart and itself the heart of the most widely applied mathematics, is wholly neglected. The reviewer seriously questions whether anyone can claim even a casual acquaintance with mathematics unless he has at least met the basic ideas of the calculus.

The first chapter of the book is a brief discussion of logic, in which Boolean diagrams are introduced, the syllogism is treated (without naming it), and types of bad argument are exposed. Much more is covered than the few pages would indicate. The exercises, of which there are many, are varied and interesting, and should stimulate the thinking of any college freshman. In Chapter 2 the author first takes up a discussion of counting and one-to-one correspondences, defining the natural numbers as those we obtain by counting things, in particular, arrays of dots. The sum of two natural numbers is defined as the natural number of the

obvious compound array of dots. Using this definition the commutative and associative laws are established. Multiplication is then similarly treated. Subtraction and division are clearly defined in terms of addition and multiplication. It is emphasized that the difference and quotient are not always defined in the system of natural numbers. The remainder of the chapter is devoted to such topics as number system bases, the game of Nim, congruences, groups, prime numbers and tests for divisibility. The pace quickens in Chapter 3 and in rapid succession the author introduces the negative integers, the rationals, the irrationals and complex numbers. At each step the point is well made that the numbers introduced are new creations, invented to serve needs not served by those of previous categories. Although no attempt is made to establish in any completeness the fact that these new numbers satisfy the usual laws, it is emphasized that this fact requires proof. There is a good discussion of the decimal notation.

Chapter 4, entitled *Algebra*, covers such topics as square numbers, Pascal's triangle, compound interest, annuities, the binomial theorem, and linear and diophantine equations. The distinction between identities and conditional equations is emphasized, and the importance of the equivalence of equations well explained. The chapter abounds in exercises of both practical and recreational interest, including several of the puzzle type. Chapter 5 is concerned with graphical matters. Measurement is introduced as an ordering process, refinement being brought about through suitable choices of units of measurement. Data graphs are defined, the straight line and its equation are discussed and a few non-linear graphs are illustrated. Curve fitting by straight lines and parabolas is treated in connection with first and second differences. Frequency tables, the histogram, the mode, the median, and the three common means are defined. The treatment here is almost too brief to be of much value. In particular, although four pages are devoted to the five averages, there is little indication of the varying importance of each of these for different types of data. Permutations, combinations and simple combinatorial probability are treated in Chapter 6. Mathematical induction makes its sole entry in the derivation of the formula for ${}_nC_m$. There is reference to mathematical expectation and its application in gaming and insurance.

The remainder of the book is devoted to geometry and topology. In Chapter 7, entitled *Mirror Geometry*, reflection, as a particular type of geometric transformation, is the central idea. Rotation and translation are defined, and the three transformations are tied together by a statement and discussion of the theorem that the product of an even number of reflections is equal to that of a translation and a rotation. Definitions of euclidean congruence and the group of euclidean displacements close the chapter. Lorentz geometry, in which the admissible transformations are $x'=kx+c$, $y'=y/k+d$, forms the subject of Chapter 8. The invariance of incidence relations and the non-invariance of (euclidean) distance are shown. Congruence is defined and the congruence of non-equal angles is illustrated. Although very brief, the material is so presented

that any intelligent student should catch from it some glimpse of the richness of non-euclidean geometries. The final chapter, *Topology*, is one of the best from the standpoint of exposition. The linear graph is introduced via the Königsberg problem and the important simple theorems established. The four-color problem is defined and the six-color and five-color theorems proved. The chapter closes with brief paragraphs on planar networks, surfaces, the Möbius strip and maps on the torus.

There is a bibliography of nearly forty items to which the student is repeatedly referred for material for additional study. The bibliography is notable in being well annotated, a feature of too infrequent occurrence.

It is scarcely a criticism of the book to say that its classroom use requires superior teaching. But it must be emphasized that it is not a book which can be handed to the pedestrian teacher with the expectation that he will impart even the substance of the material let alone the spirit. The book is not adequate as a text in the hands of such a person; too much must remain unwritten which only an enthusiastic teacher and well-trained mathematician can supply in classroom discussion.

The reviewer's major criticism of the book is that it is flippant. One feels much too strongly that the author is merely trying to show that mathematics is fun. Mathematicians know their subject is fun, but they know also that it is a serious discipline; in fact, as any first-rate mathematician will witness, the real fun comes only when one recognizes the importance of mathematics and does serious thinking about it. Motivation is not wholly lacking in this book, but that for the really important ideas is scant and subordinated. One gathers that the author feels apologetic about the importance of his subject or that he is impatient to get on with the fun.

This flippancy is aided and abetted by one feature which distinctly mars the book. The author indulges in "asides" on almost every other page. These are intended to be humorous, and in the proper atmosphere they might be, but that atmosphere is the classroom and not the text. It is annoying when reading an otherwise cogent argument suddenly to have one's train of thought upset by a facetious reference to elements which form no proper part of the argument. The use of good English is not always observed; definitely awkward constructions are too frequent.

Although it is mentioned in the preface, the title page makes no reference to the fact that the book is a rewriting of an earlier set of notes which were the joint product of several members of the staff at Cornell University.

G. B. VAN SCHAACK

Intermediate Algebra. By R. S. Underwood, T. R. Nelson, and S. Selby. New York, The Macmillan Company, 1947. 7+283 pages. \$2.60.

This is another in the growing list of textbooks written especially for college students in algebra who, for any reason, are not ready for the kind of course ordinarily given under the name of "College Algebra." It is unquestionably one

of the better books of the type and deserves consideration by anyone who is selecting a book for a course in intermediate algebra.

The discussions in the book have been written with a clarity and definiteness that should be very helpful to the students. They show a regard for mathematical accuracy that is not always found in books of this level. The number of problems should be ample for all but the very slowest classes, and the sets of problems are well graded in difficulty.

The book starts at the beginning of algebra and includes all the topics one would expect to find up through the simpler types of simultaneous quadratic equations, the factor theorem, the binomial theorem, variation, progressions and logarithms.

The publisher deserves credit for producing a book in which the typography is pleasing and the binding is quite attractive and apparently substantial.

W. C. McDaniel

A Handbook on Curves and Their Properties. By R. C. Yates. Ann Arbor, Edwards Brothers, Inc., 1947. 10+245 pages. $3.25.

This is a second edition of *Curves* by the same author which was reviewed in this Monthly, Vol. 54 (1946) pages 175–176. The chief changes from the first edition include a more substantial binding, an expanded index, additional and more complete references, and minor revisions of the text in the interest of greater accuracy and clarity.

C. B. Allendoerfer

Analytic Mechanics. (Prepared for the Department of Mathematics of the United States Naval Academy.) By A. E. Currier. Annapolis, The United States Naval Institute, 1948. 10+306 pages. $4.75.

This text was prepared to meet the requirements of the curriculum at the United States Naval Academy. The purpose, as stated in the preface, is to provide the student with an adequate understanding of mechanics to facilitate further study of the engineering and professional subjects prescribed at the Academy. In accomplishing these objectives the author has presented a well organized and concise treatment of analytic mechanics in two dimensions. A knowledge of elementary calculus and an introductory course in differential equations is assumed as a prerequisite for the reader. Indeed, the author should be commended for persistently emphasizing the role of mathematics in the derivation of the laws of Newtonian mechanics from the fundamental postulates (Newton's Laws). The definitions and the laws (theorems) are, in general, clearly and precisely stated. It is evident that a careful attempt is made to emphasize the assumptions upon which these laws as formulated are based and to enumerate the consequent limitations of the theory.

The order of presentation of the material is as follows: Chap. I. Introduction; Chap. II. Components, Sums, Moments and Equilibrium of Forces in the Plane; Chap. III. Physical Significance of a System of Forces in Equilibrium, and Ap-

plications to Statics Problems; Chap. IV. Trusses; Chap. V. Equivalent Forces, Equivalent Systems of Forces and Resultants; Chap. VI. Resultant of Gravity Forces, Center of Gravity, Center of Mass, Centroids; Chap. VII. Moment of Inertia; Chap. VIII. Normal Force and Force of Friction; Chap. IX. Strength of Materials, and Beams; Chap. X. Plane Dynamics of a Particle; Chap. XI. Work, Energy, and Potential Functions; Chap. XII. Motion of a System of Particles; Chap. XIII. Impulse and Momentum, Impulsive Forces.

The first nine chapters are devoted primarily to the theory of statics for two dimensional force systems, while the remaining four chapters treat the dynamics of particle systems and rigid bodies with plane motion. The chapter headings are in most cases sufficient to describe the contents of the chapter without the addition of further details. An excellent choice of topics is displayed throughout the text. In this respect, Chapter X on the dynamics of a particle is representative. It includes the following list of topics: 1. Assumptions Implied by Newton's Law of Motion; 2. Position, Velocity, and Acceleration; 3. The Hodograph of the Motion of a Point; 4. Motion of a Particle Acted on by the Force of Gravity, Assumed Constant; 5. Harmonic Motion with Damping; 6. Forced Harmonic Motion; 7. Tangential and Normal Components of Acceleration; 8. The Simple Pendulum; 9. Motion of a Particle Acted on by a Central Force Obeying the Inverse Square Law; 10. Effect of the Earth's Rotation, Coriolis and Centripetal Acceleration; 11. Ballistics: The Path of a Projectile.

One of the noteworthy features of the book as a text is the profusion of examples illustrating the theory. The clear explanation of the solutions of these examples and the detailed interpretation of the results will be very helpful to the student in understanding the theory and its application to the solution of problems. The student is given ample opportunity to become proficient in the solution of problems by the inclusion of an abundant set of well selected exercises. The answers are given to alternate problems. The excellent diagrams and figures illustrating the theory and the exercises are also worthy of special mention.

The format of the book is excellent. The proof has been carefully read, for there are remarkably few typographical errors.

R. C. F. BARTELS

The Early Work of Willard Gibbs in Applied Mechanics. By L. P. Wheeler, E. O. Waters, and S. W. Dudley. New York, Henry Schuman, Inc., 1947. 7+78 pages. $3.00.

The year 1947 was the first centennial anniversary of the founding of the Sheffield Scientific School of Yale University, and this little volume was published as a part of the commemoration of that occasion.

Josiah Willard Gibbs was one of the most illustrious graduates of Yale and was internationally known for his work as a mathematical physicist. The present volume, however, presents some early practical work that probably many people were unaware of. He graduated in 1863 with the degree of Doctor of Philosophy, his being the fifth such degree to be granted in the United States. The

subject of his dissertation was "On the Form of the Teeth of Wheels in Spur Gearing," but it was not printed during his lifetime.

That this was indeed his doctoral thesis is, however, a matter of inference and not of record. It was the contents of a manuscript, lacking title page and binding, which was found among the papers in the study at his home following his death. In this connection Professor Waters points out "the records of that time are confused and meager; there was no requirement that theses be published, and no complete file was kept in the Yale Library or elsewhere of theses that had been accepted." But the manuscript was examined by two of Gibbs' colleagues and they gave it as their considered opinion that it was his doctoral thesis.

The original manuscript, in Gibbs' own handwriting, was bound in 1925 and placed in the Rare Book Room of Yale University Library. It appears in printed form for the first time in this volume.

There is an Introduction contributed by Dr. Lynde Phelps Wheeler and a Commentary by Professor Everett Oyler Waters, both associated with Yale University.

Included in the book are also accounts of two other contributions of Gibbs in his early days in the practical realm. One is a discussion of An Improved Railway Car Brake by Dean Samuel Wilbur Dudley of Yale School of Engineering, and the other an account of The Gibbs Governor for Steam Engines by Dr. Wheeler.

The book throws an interesting sidelight on the early life and times of one of America's greatest scientists.

J. W. CAMPBELL

Les Principes Mathématiques de la Mécanique Classique. By M. Brelot. Grenoble, B. Arthaud, 1945. 62 pages. 120 fr.

This monograph is essentially a sketch of a modernized treatment of classical mechanics. The author considers that the usual treatments are wanting in clarity and precision, and he wishes to develop the subject in a new and concise way, on the basis of a clearly stated system of axioms. He indicates that the exposition that he has in mind would be somewhat comparable with the modern treatments of the theory of probability.

His first criticism of the traditional expositions relates to the procedure that is employed in advancing from the theory of systems consisting of finite sets of particles to the theory of systems consisting of continuous distributions of matter. He describes the practice of replacing the finite summations in the first theory by the integrations in the second theory as being "sans justification sérieuse." In order to avoid this difficulty he introduces the concept of additive set functions at an early stage, and states the dynamical formulae and theorems in terms of this concept. In brief, he postulates a point set, depending upon a parameter t, in Euclidean space, with various scalar and vector additive set functions defined on the set. These functions represent distributions of mass,

velocity, acceleration, force, and so forth. Then it turns out that it is possible to write the fundamental dynamical relations in general forms, which are immediately applicable to all particular cases.

Another of the author's criticisms relates to the way in which the notion of internal forces is usually treated. He emphasizes the fact that this notion is largely conventional, and he subjects the significance of the notion, and its applications, to a careful analysis.

At a point about half way through the booklet we find statements of the classical theorems concerning the existence and continuity of solutions of systems of differential equations. Most of the latter part of the work is devoted to a study of certain dynamical consequences of these theorems. Specifically, much of this part relates to the delicate problems presented by the motion of a system in the neighborhood of an equilibrium configuration.

The last few pages are devoted to a rapid discussion of percussions.

The author's general objective is one that will appeal strongly to all mathematicians and to many physicists. Furthermore, there will probably be general agreement that the specific proposals made are entirely sound. However, this sketch seems to be too summary to permit the implications and merits of the new ideas to become fully apparent. It is to be hoped that before long the author will expound his point of view in a full-length treatise.

L. A. MACCOLL

Curso de Matematica, en Forma de Problemas. By J. Gallego-Diaz. Madrid, Editorial Dossat, S. A., 1944. 12+344 pages. 60 pesetas.

This is a rather good collection of problems with solutions, except in a very few cases, in the subjects which are usually covered by undergraduate courses in American colleges. Some of them are in the nature of exercises, but most of them call for more than mere application of a few theorems; many of them are taken from examination papers set in some of the Spanish engineering schools in recent years. They are on the general level of the problems in the sections of this MONTHLY devoted to problems, elementary and advanced. In several instances important standard topics enter into the solutions; references to the literature would have been in order in such cases. The different topics are given treatments which vary a great deal in length. The longest section is that on limits, sequences, and infinite series and products, containing 44 problems. There are 25 problems in the section on conics, 18 on kinematics and differential geometry, 15 on probability, 10 miscellaneous problems, and groups of less than 10 each in sections on maxima and minima, transformations, quadric surfaces, and so on, several topics being represented by but a single problem.

The three problems which follow are quoted so as to give some idea of the general character of the contents of the book:

1. An urn contains one white ball and one black one. One ball is drawn and returned to the urn, with another ball of the same color as the one drawn. Con-

tinuing in this manner, after each drawing the number of balls will be increased by one. What is the probability that, when the urn contains 20 balls, there will be 10 white ones and 10 black balls? The problem is solved and generalized in obvious manner, with a reference to Polya's paper in vol. 1 of the *Annales de l'Institut Henri Poincaré*.

2. On a half line points, marked 0, 1, 2, $\cdots$, are laid off, equal distances apart. These points are projected on another line giving rise to the points P_0, P_1, P_2, $\cdots$; a uniform scale is established on this new line and in this scale the distances P_2P_5 and $P_{15}P_{25}$ are given. It is required to determine the limit of the length of the segment P_0P_n as n tends to infinity, and also the lengths in this scale, of the segments P_0P_k when P_k falls on a division point of the scale.

3. It is required to determine the conditions on 4 lines in space, R_1, R_2, R_3, and R_4, in order that, starting with an arbitrary point P_0 in space and constructing P_i as the mirror image of P_{i-1} in R_i, $i=1, 2, 3, 4$, the point P_4 will coincide with P_0.

This book could be very useful for college teachers of mathematics.

ARNOLD DRESDEN

Nomography. By A. S. Levens. New York, John Wiley and Sons, Inc., 1948. 8+176 pages. \$3.00.

A nomograph, or nomogram, or alignment chart, is a graphical device for solving an equation in three unknowns for one of the unknowns when the values of the other two are given. One draws a straight line across the three scales which constitute the nomograph, and the three scale readings at the points of intersection yield a set of values for the unknowns which satisfies the equation. Not all equations lend themselves to this graphical device. An equation in four or more unknowns may sometimes be solved by the successive use of two or more nomographs.

The present book is intensely practical. The author chooses nine type equations of general interest and gives detailed instructions for making a convenient nomograph for each type. Although the emphasis is always on the practical construction, the author does give a geometric proof of the correctness of the nomograph for each case. The explanations are clear, and are accompanied by numerous examples and figures. The use of three-rowed determinants, which plays such a large part in the theory of nomographs, is briefly presented in ten pages. There is an appendix containing about thirty nomographs intended for practical use. There is also a bibliography of eighteen references. The book is well indexed.

This book may be recommended to one who wishes to make nomographs or teach the art of making them, provided his interest is altogether on the practical side of the subject. The author's nine types do cover most of the usual cases that arise in practice, and very little mathematical background is presupposed.

D. F. BARROW

The Strange Story of the Quantum; An Account for the General Reader of the Growth of the Ideas Underlying our Present Atomic Knowledge. By Banesh Hoffmann. New York, Harper and Brothers, 1947. 11+239 pages. $3.00.

Professor Hoffmann's subtitle gives a concise description of his aim, and of his achievement; he gives an excellent account of the growths of the concepts underlying quantum mechanics and atomic and nuclear theory. Although the author is a mathematician, at present Professor of Mathematics at Queens College, New York, he tells his story, and many of its details, without the use of mathematical formalism. The contents of the theories, their relationship, "parentage," and cross-fertilization are clearly outlined, so far as it is possible without the use of the mathematical tools. The relations of theory and experiment are stressed at many points.

The book, or, following the author, the drama, starts with Hertz's discovery of electromagnetic waves, that triumph of the predictions of classical theories. Then the "Strange Story" begins to unfold; in Chapter 2 "The Quantum is Conceived" an account of Planck's work is given. The photoelectric effect is discussed in Chapter 3; this chapter and the next "Tweedledum and Tweedledee" dealing with interference, bring home, right at the start, the crucial wave-particle dualism. Chapters 4 and 5 give an exposition of the Bohr theory, the Rydberg principle, and the limitations of the Bohr model.

In "Act II" the author proceeds to follow "The Exploits of the Revolutionary Prince." De Broglie's work and the strong impetus of relativity to the development of his thoughts is mentioned in Chapter 8. In Chapter 9, headed "Laundry Lists are Discarded," the author speaks about the Heisenberg-Born Matrix Mechanics, and tries to bring out the mathematical as well as the physical problems involved. This is one of the chapters where this reader had misgivings about the reactions of the "general reader." The following two chapters "The Ascetism of Paul" and "Electrons are Smeared," on the Dirac symbolical method and Schroedinger's wave equation, are high points of the book. In Chapter 12 "Unification" the equivalence of the wave and matrix theories is discussed, and in Chapter 13 "The Strange Denouement" leads to the resolution of the wave-particle paradox, and the role therein of the Heisenberg indeterminacy principle. The last chapter of "Act II," the "New Landscape of Science," describes the general consequences of the new concepts.

In the "Epilogue" the developments of the thirties are traced. The discovery of the new particles, the positron, the neutrino, the neutron, and the meson, and the success of the quantum mechanical theory of coping with them are described.

The book is very good and absorbing reading; it is a serious attempt to show the relationship of mathematics and physics, and to trace the growths and replacement of concepts, without using the mathematical tools of quantum mechanics. It is unfortunate that in so aptly constructed a book the story is a bit overdramatized in places. Sometimes this will not help the reader much, and

may lead to misunderstandings, for instance the use of the author's "principle of perversity."

Another point is the lack of units, say on page 22, when speaking of h the author says "Its value is a mere .000 000 000 000 000 000 000 000 006 6." Also one would have liked a stronger emphasis on the permanence of classical theory, as far as macroscopic quantities are concerned. Notwithstanding these few objections, it is a fascinating book, almost in a class with Jeans and Eddington and the other great classics of Popular Science. It will be good collateral reading in courses on the history of science.

F. T. ADLER

NEW BOOKS RECEIVED

Intermediate Algebra for Colleges. By W. L. Hart. Boston, D. C. Heath and Co., 1948. 7+316 pages. $2.50.

College Algebra. By T. S. Peterson. New York and London, Harper and Brothers, 1947. 8+334 pages. $2.50.

Algebra for College Students. By J. R. Britton and L. C. Snively. New York, Rinehart and Co., 1947. 11+529 pages. $3.00.

Higher Algebra. (Sequel to Higher Algebra for Schools). By W. L. Ferrar. New York, Oxford University Press, 1948. 6+320 pages. $5.00.

Essentials of Analytic Geometry. By D. R. Curtiss and E. J. Moulton. Boston, D. C. Heath and Co., 1947. 4+259 pages. $2.80.

Analytic Geometry. By P. K. Rees and E. D. Mouzon. New York, Dryden Press, 1948. 28+305 pages. $2.75.

Analytic Geometry. By P. R. Rider. New York, The Macmillan Co., 1947. 10+383 pages. $3.25.

Solid Analytic Geometry. By J. M. H. Olmsted. New York, Appleton-Century Co., 1947. 13+257 pages. $4.00.

Plane Geometry. By D. T. Sigley and W. T. Stratton. New York, Dryden Press, 1948. 12+242 pages. $2.25.

Unified Calculus. By E. S. Smith, M. Salkover, and H. K. Justice. New York, John Wiley and Sons; London, Chapman and Hall, 1947. 10+507 pages. $3.50.

Modern Mathematics. By S. A. Walling and J. C. Hill. New York, The Macmillan Co., 1948. 6+153 pages. $1.00.

A Text-book of Mathematical Analysis. The Uniform Calculus and its Applications. By R. L. Goodstein. New York, Oxford University Press, 1948. 12+475 pages. $9.00.

Modern Operational Calculus. By N. W. McLachlan. New York, The Macmillan Co., 1948. 14+218 pages. $5.00.

Applied Differential Equations. By F. E. Relton. London, Blackie and Sons, Ltd., 1948. 6+264 pages. 20s net.

Introduction to the Differential Equations of Physics. By L. Hopf. Translated by Walter Nef. New York, Dover Publications, 1948. 5+154 pages. $1.95.

Cours de Mecanique Rationnelle. By J. Chazy. Paris, Gauthier-Villars. Vol.

I, *Dynamique du Point Materiel.* 482 pages, 900 fr.; Vol. 2, *Dynamique des Systemes Materiels.* 6+511 pages, 1100 fr.

Set Functions. By Hans Hahn and Arthur Rosenthal. Albuquerque, University of New Mexico Press, 1948. 9+324 pages. $12.50.

A Treatise on Set Topology. Part I. By R. Vaidyanathaswamy. Madras, S. Mahadevan, 1947. 6+304 pages. Rs 16-4.

Mathematics for Use in Business. By C. E. Hilbron. Boston, Houghton-Mifflin C., 1948. 7+472 pages. $3.50.

Mathematics of Finance. By J. B. Linker and M. A. Hill. New York, Henry Holt and Co., 1948. 8+175+83 pages. $2.90.

The Mathematical Theory of Finance. Revised Edition. By K. P. Williams. New York, The Macmillan Co., 1947. 10+274 pages. $4.50.

Theory of Experimental Inference. By C. W. Churchman. New York, The Macmillan Co., 1948. 11+292 pages. $4.25.

Sampling Inspection. By The Statistical Research Group, Columbia University. New York, McGraw-Hill Book Co., 1948. 20+395 pages. $5.25.

Studies and Essays. Anniversary Volume Presented to R. Courant. New York, Interscience Publishers, 1948. 470 pages. $5.50.

A Collection of Papers in Memory of Sir William Rowan Hamilton. The Scripta Mathematica Studies, No. 2. New York, Scripta Mathematica, 1945. 82 pages. $1.00.

Newton Tercentenary Celebrations. By E. N. da C. Andrade, H. J. Keynes, and Others. New York, The Macmillan Co., 1948. 15+91 pages. $3.00.

Tables of Spherical Bessel Functions. Vol. 2. Prepared by the Mathematical Tables Project, National Bureau of Standards. New York, Columbia University Press, 1947. 20+232 pages. $7.50.

The Differential Analyser. By J. Crank. New York, Longmans, Green, and Co., 1948. 8+137 pages. $2.50.

The Theory of Mathematical Machines. Revised Edition. By F. J. Murray. New York, King's Crown Press, 1948. 9+139 pages. $3.00.

Proceedings of a Symposium on Large-Scale Digital Calculating Machinery. Vol. XVI. Cambridge, Harvard University Press, 1948. 29+302 pages. $10.00.

Heat Conduction. By L. R. Ingersoll, O. J. Zobel, and A. C. Ingersoll. New York, McGraw-Hill Book Co., 1948. 12+278 pages. $4.00.

Nuclear Forces. Part I. By L. Rosenfeld. New York, Interscience Publishers, 1948. 20+181 pages. $5.00.

Mathematics for the Consumer. By R. Schorling, J. R. Clark, and F. G. Lankford, Jr. Yonkers, New York, World Book Co., 1947. 10+438 pages.

Mathematics for Radio Engineers. By L. Mautner. New York, Pitman Publishing Co., 1947. 8+327 pages. $5.00.

Two-Dimensional Fields in Electrical Engineering. By L. V. Bewley. New York, The Macmillan Co., 1948. 14+204 pages. $5.50.

Introductory Chemical Calculations. By S. J. Smith. New York, The Macmillan Co., 1947. 7+144 pages. $1.00.

Traffic Performance at Urban Street Intersections. Technical Report No. 1. New Haven, Yale Bureau of Highway Traffic, 1947. 15+152 pages.

Sobre las Orbitas Aparentes de las Estrellas Dobles Visuales. By Enrique Vidal. University of Santiago, 1947. 8+58 pages.

The Scientists Speak. By Warren Weaver. New York, Boni and Gaer, 1947. 13+369 pages. \$3.75.

Bibliographie des Multigrades avec quelques Notices Biographiques. By A. Gloden and G. Palama. Luxembourg, 1948. 4+64 pages.

Secret. By W. W. Stout. Detroit, Chrysler Corp., 1947. 67 pages.

Number Readiness in Research. By A. Riess. New York, Scott, Foresman, and Co., 1947. 70 pages. \$1.00.

Math is Fun. By J. Degrazia. New York, Gesham Press, 1948. 159 pages. \$2.75.

A Centenary of Marxism. Edited by Samuel Bernstein and the Editors of Science and Society. New York, Science and Society, 1948. 196 pages. \$2.50.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

Mathematics Club, Hunter College

The *Mathematics Club* of Hunter College held two parties, a buffet supper, and a boat ride to Bear Mountain in addition to the regular meetings at which the following papers were presented:

Philosophical problems relating to statistics and probability, by Jerzy Neyman of the University of California

Self-inductive theorems, by Professor James Singer of Brooklyn College

Transcendental numbers, by Professor J. H. Bushey

Mathematics and art, by Marian Boykan

Number scales, by Florence Burg

The number π, by Cecile Cohen

Mobius strips, by Marcia Geiger

Pythagorean numbers, by Dorothy Beck

An introduction to map projections, by Marianne Weisz.

The club was invited to join the *Physics Club* of Hunter College to hear I. F. Ritter of New York University speak on *Soap films and minimal surfaces*.

The officers for 1946–47 were: President, Wilhelmina Fluhr; Vice-President,

Florence Myres; Secretary, Ceceil Cohen; Treasurer, Ruth Friedman; Interclub Committee Representative, Marcia Geiger; Publicity Chairman, Florence Burg; Faculty Adviser, Carolyn Eisele.

Mathematics Club, University of Dayton

Biweekly meetings were held during the year at which the following papers were presented by members:

The construction of Pythagorean triangles, by Joseph R. Berry

Fundamental concepts of analysis, by J. R. Flynn

Fundamental number theory, by E. J. Freeh

Nomographic or alignment charts, by R. J. Schweller

Exterior ballistics, by R. R. Luthman

Curve fitting, by J. R. Westerheide

The irrational number system, by J. D. Griffin

Paradoxes of the infinite, by N. A. Engler

The application of Fourier's series to electrical circuits, by P. F. Swift.

Faculty talks presented during the year were:

The transcendence of e, by Dr. K. C. Schraut

A civil engineer looks at mathematics, by Prof. J. J. Chamberlain, acting head of the University Civil Engineering department

Questions after the lecture, by Prof. Merriman of the University of Cincinnati.

During the first semester the club took a special field trip to the University of Cincinnati Observatory where Dr. Paul Herget, Director of the observatory, spoke on *An application of finite differences to the computation of orbits*. The annual banquet, attended by the president of the University, the Dean of Science, several Mathematics Club Alumni including three past presidents and other distinguished guests was held at the Wishing Well Inn.

The Dean of Science Award, consisting of two volumes of *Differential and Integral Calculus*, by R. Courant, given to the student who delivers the most interesting paper during each semester, was presented to R. J. Schweller for the first semester and to J. D. Griffin for the second semester. The mathematics club alumni awards for excellence in advanced mathematics were presented to J. R. Berry, a senior, and to Charles Keller, a junior.

Several members attended and assisted in the annual symposium and dinner of the National Mathematics Honor Society of Secondary Schools, which was founded under the auspices of the Club, and aided in the installation of an additional chapter at Elder High school of Cincinnati.

The first edition of a proposed annual Mathematics Club *Bulletin*, containing reprints of all the student papers delivered before the Club and over ninety pages in length, was edited by the secretary. A new constitution, drafted by the officers, was accepted by the members.

Officers of the Club during the year were: President, John R. Westerheide; Vice-President, Edward J. Freeh; Secretary, J. Dennis Griffin; Publicity Secre-

tary, Thomas P. Hanlon; Treasurer, Peggy Ens; Faculty Advisor, Dr. K. C. Schraut.

Pi Mu Epsilon, Duke University

Two meetings of the *North Carolina Alpha* Chapter of *Pi Mu Epsilon* were held this year. At the fall meeting Dr. J. H. Roberts presented the topic *Paradoxes and mathematics*. At the spring meeting Dr. H. Hotelling presented the topic *Applications of mathematics*.

Initiation was held at both the fall and the spring meetings. In the fall nine new members were taken into the fraternity, and in the spring there were forty-nine new members taken in. The officers elected for 1947–48 are: Director, Jo Anne Walker; Vice-Director, Nancy Bloom; Treasurer, Wayne Bainbridge; Secretary, Jean Bellingrath.

Mathematics Club, Swarthmore College

The *Mathematics Club* of Swarthmore College, having lapsed into inactivity during the war, was revived last year and is now in its second year of existence. Meetings are held on alternate Thursdays, at which papers on topics of interest are given by students and faculty members. The object of these talks is to acquaint the students with areas of mathematics not taken up in the regular curriculum and to stimulate students to investigate these subjects further.

Papers given during the semester were:

Partition of numbers, by Dr. Rademacher of the University of Pennsylvania
Cryptanalysis, by Paul Mangelsdorf
Laplace transformations, by Dr. Elmore
Group theory, by Carl Levinson
Calculus of variations, by Irving Dayton
Finite fields, by Dr. Carruth.

The club also sponsors a problem contest every semester in the belief that active participation is essential to the study of mathematics. The winners of this term's contest are: Ned Freeman, Carl Levinson, Douwe Yntema, and Bob Kuller.

Officers for the semester were: President, George Yntema; Vice-President, Carl Levinson; Secretary, Patricia Plank; Treasurer, Nancy Burnholz.

For the spring semester the officers are: President, Carol Levinson; Vice-President, Edward Rawson; Secretary, Charlotte Garceau; Treasurer, Nancy Burnholz.

Mathematical Society, The University of Adelaide

The *Mathematical Society* of the University of Adelaide was conducted as a Colloquium or Seminar on the *Theory of the Gamma function* and some of its applications. The lectures of the students were prepared from typed notes on the subject by Prof. H. Schwerdtfeger. In 13 meetings the theory of the real Gamma function was covered as far as it is treated in *Cours d'analyse* by de la Vallée

Poussin, but using the method of E. Artin's *Einfuehrung* in the *Theorie der Gammafunktion* (Leipzig 1931). As an application we have dealt with Riemann-Liouville's Fractional Integration, using various sources; e.g., papers by J. D. Tamarkin (Annals of Mathematics 31, 1930, 219–229), W. L. Ferrar, (Proc. Royal Society Edinburgh 48, 1927–28, 92–105) and H. T. Davis (American Journal of Mathematics 46, 1924, 95–109).

The principle collaborators with Prof. Schwerdtfeger were: A. T. James, B.Sc., N. Munroe, B.Sc., R. B. Potts, B.Sc., Mrs. M. Sved, G. E. Wall, B. W. Worthley, B.Sc.

A colloquium on the *Theory of Matrix Functions* is contemplated along similar lines for the next school session.

Prof. Schwerdtfeger states that "the concentration upon a single subject defined in advance has, in fact, reduced the popularity of the mathematical society; but in return it gave more satisfaction to the more serious kind of students. It was therefore agreed to continue the society in this manner, changing the subject from year to year. This seems to be the more desirable under the circumstances."

Mathematics-Physics Club, College of Saint Teresa

The theme for the activities of the Mathematics-Physics Club of College of Saint Teresa for 1946–47 was *measurement*. Applications were made of the cross-staff, drum-head, clinometer, visor, range-finder, scout staff, square and cross staff, mirror, and yardstick.

Demonstrations of an oscillograph and a student-constructed radio as well as films on the parabola and crystallization were shown. Progressive mathematical games, a Christmas cryptograph, and quotations of mathematicians furnished digression at other meetings.

The year's activities were closed with a mathematical scout picnic at which the participants were tested in mapping, surveying, estimating measurements, the compass, the scout code, and laws. Appropriate badges were awarded.

The officers for 1946–47 were: President, Jean Stephany; Vice-President Gladys Schmitz; Secretary, Mary Vroman; Treasurer, Clarice Abts.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

RESEARCH FELLOWSHIPS IN PSYCHOMETRICS

The Educational Testing Service, Princeton, New Jersey is offering for 1949–50 two research fellowships in psychometrics leading to the Ph.D. degree at Princeton University. These fellowships carry a stipend of $2,200 a year.

They are available to men who are acceptable to the Princeton University Graduate School and who show exceptional aptitude for advanced training in psychology, psychological measurement and statistics.

Applications must be received in Princeton by January 15, 1949. Application blanks may be obtained from the Director of Psychometric Fellowship Program, Educational Testing Service, Box 592, Princeton, New Jersey.

SYMPOSIUM ON MODERN CALCULATING MACHINERY

A Symposium on Modern Calculating Machinery and Numerical Methods was held at the University of California at Los Angeles on July 29–31, 1948. It was sponsored by the Departments of Astronomy, Engineering, and Mathematics of the University and by the Institute for Numerical Analysis in cooperation with the Association and other scientific societies. The program included reports on computing machine developments in various research centers and on numerical methods in applied mathematics.

PERSONAL ITEMS

Professor W. D. Reeve, retiring head of the Department of Mathematics of Teachers College, Columbia University, has been honored by the establishment of the William David Reeve Scholarship in the Teaching of Mathematics.

Carnegie Institute of Technology anounces the following appointments for the year 1948–49: Professor F. D. Murnaghan as visiting professor for the first semester and Professor E. R. Lorch as visiting professor for the second semester; Dr. Abraham Charnes of the University of Illinois to an assistant professorship; Assistant Professors H. J. Greenberg and George Handelman of Brown University to assistant professorships.

Cornell University announces: Associate Professor C. J. Thorne of the University of Utah has been appointed Visiting Associate Professor; Associate Professor R. J. Walker has been promoted to a professorship; Assistant Professor Harry Pollard has been promoted to an associate professorship; Professor W. B. Carver has retired.

At Kent State University Assistant Professor R. Y. Iwanchuk of St. Basil's College and Mr. B. B. Dressler have been appointed to assistant professorships.

Rutgers University makes the following announcements: Mr. C. W. Saalfrank of Franklin and Marshall College, Mr. A. G. Makarov of the University of Pennsylvania, and Mr. D. A. Darling of University College have been appointed to assistant professorships; Mr. J. N. Livingood and Mr. E. D. Nering have resigned.

The University of Michigan announces the following promotions: Associate Professor S. B. Myers to a professorship, Assistant Professor R. M. Thrall to an associate professorship, Mr. P. S. Jones to an assistant professorship.

The University of Wyoming announces: Dr. W. N. Smith has been appointed Assistant Professor of Mathematics; Assistant Professor S. R. Smith has been promoted to an associate professorship.

Dr. H. G. Apostle, formerly of Amherst College, has been appointed to an associate professorship at Grinnell College.

Professor A. S. Besicovitch is serving as visiting professor at the University of Pennsylvania during the academic year 1948–49.

Dr. P. T. Bateman of Yale University is now at the Institute for Advanced Study.

Assistant Professor Truman Botts of the University of Delaware has been appointed to an acting assistant professorship at the University of Virginia.

Professor Louis Brand of the University of Cincinnati has received an appointment as visiting professor at the University of Hawaii for the year 1948–49.

Dr. C. T. Bumer of the Massachusetts Institute of Technology has been appointed Professor of Mathematics and Chairman of the Department of Mathematics at Clark University.

Associate Professor W. J. Combellack of Northeastern University has been appointed Professor of Mathematics and Head of the Department of Mathematics at Colby College.

Assistant Professor W. W. Dolan of the University of Oklahoma has accepted an appointment as professor and head of the Department of Mathematics at Linfield College.

Miss Geneva E. Durham of Atlantic Union College has been appointed Assistant Professor of Mathematics and Astronomy at Pacific Union College.

Assistant Professor Howard Eves of Oregon State College has been promoted to an associate professorship.

Professor H. P. Fawcett of Ohio State University has been appointed Chairman of the Department of Education.

Dr. R. D. Gordon of Indiana University has been appointed to an assistant professorship at the University of Buffalo.

Assistant Professor F. G. Graff of Amherst College has been appointed to an assistant professorship at Oberlin College.

Professor E. H. Hadlock of Hastings College has been appointed to an associate professorship at the University of Florida.

Dr. G. P. Hochschild of Harvard University has been named Assistant Professor of Mathematics at the University of Illinois.

Assistant Professor Carl Holtom of Purdue University has accepted an appointment as associate professor of mathematics at the U. S. Air Force Institute of Technology, Wright-Patterson Air Force Base, Dayton, Ohio.

Dr. L. H. Kanter of the University of Wisconsin has been appointed to an assistant professorship at the University of Arkansas.

Dr. Samuel Kaplan has been appointed Assistant Professor of Mathematics at Wayne University.

Professor C. E. Melville has retired from his position as professor of mathematics and chairman of the Department of Mathematics after forty-six years of teaching at Clark University.

Professor G. W. Mullins of Columbia University has been given the title of Emeritus Professor of Mathematics.

Dr. Edward Paulson of the University of Washington has been promoted to an assistant professorship.

Miss Lucille K. Pinette of Colby College has been promoted to an assistant professorship.

Mr. I. H. Rose of Pennsylvania State College has been appointed to an assistant professorship at the University of Massachusetts.

Associate Professor Wladmir Seidel has obtained a year's leave of absence from the University of Rochester in order to work at the Institute for Numerical Analysis, University of California at Los Angeles.

Dr. Shien-Sin Shu has received an appointment as associate professor of mathematics and research associate in mechanics at Illinois Institute of Technology.

Professor R. R. Shumway of the University of Minnesota has retired.

Professor E. R. Sleight of Albion College has retired.

Assistant Professor W. D. Temple, Agricultural and Mechanical College of Texas, has been appointed to an associate professorship at Louisiana Polytechnic Institute.

Brother L. Thomas is now a member of the faculty of Christian Brothers College, Memphis, Tennessee.

Assistant Professor E. P. Vance has been appointed Chairman of the Department of Mathematics at Oberlin College for the year 1948–49.

Dr. W. E. Wilson has been appointed President of the South Dakota School of Mines and Technology.

The following appointments to instructorships are announced:

Dartmouth College: Mr. D. G. Dickson

Franklin and Marshall College: Mr. J. R. Holzinger

Illinois Institute of Technology: Mr. R. B. Brady

Kent State University: Mr. A. P. Boblett, Mr. W. C. Lowry, Mr. F. R. Olson, Mr. E. T. Stapleford, Miss Jane A. Uhrhan.

Rutgers University: Mr. W. W. Boone

University of Arkansas: Mr. W. C. Guenther

University of Buffalo: Miss Janet E. Abbey

University of Oregon: Mr. W. L. Shepherd

Lehigh University announces the appointment of Mr. R. R. Townsend to a graduate assistantship.

Professor Mae R. Andersen of Concordia College died on Marsh 28, 1948.

Sister M. Borgia Clarke of Webster College died on January 21, 1948. She had been a member of the Association for twenty-five years.

Professor Emeritus R. M. Ginnings of Western Illinois State Teachers College died on June 19, 1948. He was a charter member of the Association.

Assistant Professor T. R. Long of the University of Rochester died on July 11, 1948.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following seventy persons have been elected to membership by the Board of Governors on applications duly certified:

REV. H. B. ALBISER, S.S.E., A.B. (St. Michael's) Instructor, St. Michael's College, Winooski, Vt.

P. M. ANSELONE, Student, College of Puget Sound, Tacoma, Wash.

L. G. ARNOLD, B.A. (Oberlin) Graduate Student, University of Michigan, Ann Arbor, Mich.

H. G. H. BARTRAM, B.A. (Colorado) Part-time Instructor, University of Colorado, Boulder, Colo.

V. N. BEHRNS, B.A. (Buffalo) Instructor, University of Buffalo, N. Y.

E. J. Berger, M.A. (Colorado State) Instructor, Monroe High School and College of St. Catherine, St. Paul, Minn.

LIPMAN BERS, Ph.D. (Syracuse) Asso. Professor, Syracuse University, N. Y.

JACOB BORSUK, B.A. (Brooklyn) Mathematician, U. S. Coast & Geodetic Survey, N. Y.

LEONARD BRISTOW, Ph.D. (Illinois) Asst. Professor, University of Wyoming, Laramie, Wyo.

C. G. BURGER, JR., B.S. (Rensselaer Polytechnic Institute) Instructor, Rensselaer Polytechnic Institute, Troy, N. Y.

NIKOLINE A. BYE, A.M. (Michigan) Asst. Professor, Central Michigan College of Education, Mount Pleasant, Mich.

HELEN SELFRIDGE (Mrs.) CLUCAS, M.A. (Oregon) Public Assistance Worker, Kern Country Welfare Dept., Bakersfield, Calif.

W. B. CONROY, JR., M.A. (St. Lawrence) Asst. Professor, Clarkson College, Potsdam, N. Y.

VIOLET B. DAVIS, M.A. (Toledo) Asst. Professor, University of Toledo, Ohio

REV. H. F. DEBAGGIS, C.I.C., Ph.D. (Notre Dame) Instructor, University of Notre Dame, Ind.

M. W. DEJONGE, M.A. (Illinois) Asst. Professor, Purdue University, Lafayette, Ind.

B. K. DICKERSON, M.A. (Northwestern) Instructor, Syracuse University, N. Y.

W. M. DUKE, Sc.M. (New York) Chairman of Research Division, Cornell Aeronautical Laboratory, Buffalo, N. Y.

M. B. EVANS, B.S.E. (Michigan) Instructor, Missouri School of Mines and Metallurgy, Rolla, Mo.

KY FAN, D.Sc. (Paris) Asst. Professor, University of Notre Dame, Ind.

W. E. FELLING, B.S.E.E. (St. Louis) Graduate Student, St. Louis University, Mo.

H. H. FOX, M.A. (Wisconsin) Assistant, University of Illinois, Urbana, Ill.

C. V. FRONABARGER, M.A. (Geo. Peabody) Asst. Professor, Southwest Missouri State College, Springfield, Mo.

J. B. GARRETT, Student, Siena College, Loudonville, N. Y.

SIDNEY GLUSMAN, M.A. (Columbia) Instructor, Seton Hall College, South Orange, N. J.

N. H. GOTTESMAN, B.S. (Notre Dame) Teaching Fellow, University of Notre Dame, Ind.

G. F. GUILFORD, JR., M.S. (Syracuse) Asst. Professor, Rensselaer Polytechnic Institute, Troy, N. Y.

SMITH HIGGINS, JR., M.S. (Notre Dame) Instructor, University of Notre Dame, Ind.

H. K. HILTON, M.A. (Northwestern) Instructor, University of Wyoming, Laramie, Wyo.

K. E. HOGGATT, Student, College of Puget Sound, Tacoma, Wash.

V. E. HOGGATT, JR. Student, College of Puget Sound, Tacoma, Wash.

R. J. HOWERTON, M.S. (Northwestern) Instructor, Regis College, Denver, Colo.

N. R. HUGHES, A.B. (Wabash) Instructor, Wabash College, Crawfordsville, Ind.

T. R. HUMPHREYS, M.A. (Oregon) Professor, Bergen Junior College, Teaneck, N. J.

R. E. HUSTON, Ph.D. (Chicago) Professor, Rensselaer Polytechnic Institute, Troy, N. Y.

T. C. HUTCHISON, Student, Pennsylvania State College, State College, Pa.

RUTH E. KENNEDY, B.S. (Louisiana Polytechnic Institute) Instructor, Louisiana Polytechnic Institute, Ruston, La.

P. J. KIERNAN, A.M. (Columbia) Instructor, Lawrenceville School, N. J.

F. P. KOWALEWSKI, JR., M.A. (Buffalo) Instructor, University of Buffalo, N. Y.

SIDNEY KRAVITZ, M.Ae.E. (New York) Mathematician, Ballistic Research Laboratory, Aberdeen Proving Ground, Md.

W. G. LEAVITT, Ph.D. (Wisconsin) Instructor, University of Nebraska, Lincoln, Neb.

J. V. LIMPERT, A.M. (Syracuse) Instructor, Syracuse University, N. Y.

R. L. LOKENSGARD, Ed.D. (Columbia) Head of Dept., Winona State Teachers College, Minn.

R. B. MCHUGH, B.A. (Minnesota) Teaching Asst., University of Minnesota, Minneapolis, Minn.

R. A. MILLER, M.A. (Mississippi) Asst. Professor, University of Mississippi, University, Miss.

TEXAS MORRIS (Miss) 1516 Chestnut St., Wilmington, N. C.

HELEN DOROTHY MORTELL, M.A. (Catholic University) Instructor, Catholic University of America, Washington, D. C.

F. R. OLSON, B.A. (Alfred) Instructor, Hamilton College, Clinton, N. Y.

S. R. ORR, B.A. (Hiram) 129 North Elm St., Columbiana, Ohio

FLORENCE E. POOL, M.A. (Nebraska) Instructor, University of Nebraska, Lincoln, Neb.

W. G. PREBLE, B.S. (Tulane) Computer, Corps. of Engineers, U. S. Army, New Orleans, La.

HENRY RAINBOW, B.A. (Cambridge) Shell Research Laboratory, Bellaire Road, Houston, Tex.

R. R. REED, A.B. (Union) Instructor, Clarkson College, Malone, N. Y.

G. X. SALTARELLI, M.S. (Notre Dame) Teaching Fellow, Notre Dame University, Ind.

A. C. SCHAEFFER, Ph.D. (Massachusetts Institute of Technology) Professor, Purdue University, Lafayette, Ind.

SISTER MARY VIRGILIA DRAGOWSKI, O.S.F., Ph.B. (Detroit) Student, University of Notre Dame, Ind.

SISTER MARY FERRER MCFARLAND, M.S. (De Paul) Student, University of Notre Dame, Ind.

RUBIN SMULIN, B.S. (Miami) Electrical Engineer, Unity Electric Co., Inc., Miami Beach, Fla.

F. M. STEIN, M.S. (State University of Iowa) Instructor, Iowa Wesleyan College, Mount Pleasant, Iowa

EUGENE STEPHENS, M.S. (Washington) Asst. Professor Emeritus, Washington University, St. Louis, Mo.

R. A. STRUBLE, B.S. (Notre Dame) Graduate Assistant, University of Notre Dame, Ind.

ANNA K. SUTER, A.M. (Indiana) Instructor, Purdue University Extension Division, Indianapolis, Ind.

A. K. TERZUOLI, M.S. (New York) Instructor, Polytechnic Institute of Brooklyn, N. Y.

W. J. THRON, Ph.D. (Rice) Asst. Professor, Washington University, St. Louis, Mo.

G. L. TILLER, Ph.D. (Kentucky) Asst. Professor, Utica College of Syracuse University, Utica, N. Y.

G. R. TROTT, Ph.D. (Johns Hopkins) Professor, University of Mississippi, University, Miss.

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MARCH MEETING OF THE SOUTHEASTERN SECTION

The annual meeting of the Southeastern Section of the Mathematical Association of America was held at The Citadel, Charleston, South Carolina, on Friday and Saturday, March 19–20, 1948. Professor J. W. Cell, Chairman of the Section, presided at the Friday afternoon and Saturday morning meetings, except for the meetings of the subsections, which were presided over by Major

L. A. Dye, Vice-Chairman, and Professor H. K. Fulmer. Colonel C. F. Myers, Jr., of The Citadel presided on Friday evening at the informal dinner given in honor of the visiting speaker, Lt. Colonel R. C. Yates.

After the dinner, an informal discussion was led by Professor Tomlinson Fort on *Common Problems Due to Overcrowding, Poor Preparation, and Inexperienced Instructors.*

There were about one hundred and fifty present, including the following seventy-eight members of the Association: R. H. Ackerson, Louise Adams, G. E. Albert, J. C. Barnes, D. F. Barrow, Helen Barton, W. S. Beckwith, R. C. Blackwell, R. G. Blake, J. P. Brewster, N. R. Bryan, Berdie J. Buffkin, R. C. Bullock, E. A. Cameron, J. W. Cell, B. G. Clark, A. C. Cohen, J. B. Coleman, W. J. Conner, R. W. Cowan, Nelle C. Douglas, Jeanette R. Durst, L. A. Dye, R. B. Folsom, Tomlinson Fort, Jack Frierson, H. K. Fulmer, L. L. Garner, W. W. Graham, C. L. Hair, E. A. Hedberg, R. A. Hefner, C. W. Hook, H. B. Hoyle, Jr., V. A. Hoyle, G. B. Huff, L. P. Hutchinson, J. A. Hyden, Rosa L. Jackson, F. W. Kokomoor, G. B. Lang, J. W. Lasley, Jr., R. J. Levit, Anne L. Lewis, F. A. Lewis, Nathaniel Macon, C. F. Martin, W. J. Mays, S. W. McInnis, W. G. Miller, J. D. Novak, W. V. Parker, William Pennington, Jr., R. I. Pepper, Lillian Perkins, Mary Pettus, C. G. Phipps, Alice B. Rabon, Ellen F. Rasor, B. P. Reinsch, G. E. Reves, H. A. Robinson, L. V. Robinson, C. L. Seebeck, Jr., E. B. Shanks, D. C. Sheldon, C. Eucebia Shuler, T. M. Simpson, W. B. Stovall, Jr., Cora Strong, C. S. Sutton, J. M. Thomas, R. Z. Vause, J. A. Ward, W. W. Weber, W. L. Williams, R. C. Yates, G. C. Zader.

At the business session the following officers were elected for the coming year: Chairman, L. A. Dye, The Citadel; Vice-Chairman, F. A. Lewis, University of Alabama; Secretary-Treasurer, H. A. Robinson, Agnes Scott College. The Section voted to hold its March 1949 meeting at the University of Alabama.

The program consisted of the following papers:

1. *Dissection of the euclidean plane by the conic sections*, by Lt. W. J. Conner, The Citadel.

The plane is divided into not more than $2n^2-n+1$ regions by n parabolas. Similar formulas were obtained for the other conic sections.

2. *On the characteristic equations of certain matrices*, by Professor W. V. Parker, University of Georgia.

This speaker generalized a theorem proved by Brauer, and published under the same title in the *Bulletin of the American Mathematical Society*, Vol. 52, pp. 605–607. The generalized theorem was stated as follows: Let A, C_1, and C_2 be n-rowed square matrices such that $C_1A=AC_2=0$. If $C=C_1+C_2$, and if B is an arbitrary n-rowed square matrix, then AB and $A(B+C)$ have the same characteristic equation.

3. *The effect of hyperbolic grooves on the stress distribution in a wood plate*, by Professor C. B. Smith, University of Florida, introduced by H. A. Robinson.

This paper consisted of an analysis of the stress distribution existing in a wood plate of large length with hyperbolic grooves. Assuming a wood plate had two perpendicular axes of elastic symmetry in the plane of the plate, the analysis furnished a formula for the factor of stress concentra-

tion, that is, the ratio of the maximum stress occurring at the bottom of the grooves to the average tensile stress over the cross section.

4. *The exponential function and the Laplace transform*, by Professor J. W. Cell, North Carolina State College.

With this expository paper, some fifty slides were shown which illustrated the following ideas: (1) the meaning of the transformation $x'=bx$ and $y'=y/a$ to reduce $y=ae^{bx}$ to the simpler form $y=e^x$ (both graphical meaning and the interpretation in terms of dimensionless variables); (2) applications of this law of growth or decay to many fields of science and technology; (3) functions related to the exponential function by compounding this function with other basic functions; (4) the meaning of the Laplace transformation, with graphical illustrations showing the transforms of several elementary functions; (5) types of problems to which the Laplace transform method applies.

5. *Groups of birational transformations transitive on the rational points of a plane curve*, by Professor G. B. Huff, University of Georgia.

It was shown that for any elliptic cubic curve, there exists a group $G(c)$ of Cremona transformations which is transitive on the rational points of the cubic, and which has a finite number of generators. These results were capable of being generalized in several directions.

6. *The expansion of x^n in terms of $\binom{x}{j}$*, by Professor G. B. Lang, University of Florida.

Explicit formulas were obtained for the expansion. The results included a solution to Problem 4276 in this MONTHLY, vol. 54, 1947, p. 601.

7. *A generalization of Taylor's theorem*, by Professors C. L. Seebeck, Jr., and P. M. Hummel, University of Alabama.

A set of series expansions for functions of one variable which reduced to Taylor's series with remainder as a special case was developed. For a symmetric case, the error for n terms of the series was found to be less than that for a Taylor's series of $2n$ terms.

8. *Quasi-monotonic series*, by Professor Tomlinson Fort, University of Georgia.

Series of the type $a_1+a_2+a_3+\cdots$, where $a_n \geqq a_{n+1} \geqq 0$ are known as monotonic series. Such series have been extensively studied and many properties developed. The present paper considered certain generalizations and analogues of the relationship $a_n = a_{n+1} > 0$ which assures for the series many of the well known properties of monotonic series.

9. *Some properties of plane curves*, by Lt. Colonel R. C. Yates, United States Military Academy.

Colonel Yates discussed the following topics: (1) folding and creasing the conics; (2) the projection upon a focal radius of the normal length N of a conic (constant and equal to the semi-latus rectum); (3) instantaneous centers and the construction of tangents to such curves as the ellipse, limaçon, conchoid, strophoid, and cycloid; (4) caustics, particularly the evolute of a central conic formed by light rays refracted through a plane surface; (5) the rose curves and their generation as special hypotrochoids; (6) the carpenter's square and its use in describing the conics, cardioid, strophoid, cissoid, and conchoid.

10. *Determinants and undergraduate mathematics*, by Major L. A. Dye, The Citadel.

The content and methodology of most undergraduate courses in mathematics are long overdue for a critical evaluation as to their justification and merit. It was suggested that the study and use of determinants in courses up to and including the calculus was an essential waste of time.

11. *An expression for a general law of mortality*, by Mr. W. J. Mays, Assistant Actuary, Liberty Life Insurance Company.

It was assumed that the probability of joint survival of m lives over any period of time may be expressed as a function of n variables ($n < m$). This assumption led to a set of linear differential equations involving the probabilities of survival of single lives. The solution was remarkable in that it comprehended almost every function that had been used successfully on empirical grounds to represent the mortality function l_x, that is, the number of persons who attain exact age x according to the mortality table.

12. *Imaginary graphs of elementary real functions of "excluded" portions of the x-axis*, by Professor J. W. Ward, University of Georgia.

Let the X, Y_r and Y_i axes be three mutually perpendicular axes in space. Real values of y are plotted along the Y_r-axis, and imaginary values of y along the Y_i-axis. The coördinate X is restricted to be real. The real graph of the real function $f(x, y) = 0$ will lie in the XY_r-plane. Real values of x that give pure imaginary values of y will give points in the XY_i-plane. For example, $x^2 - y^2 = 1$ is a hyperbola in the XY_r-plane, and a circle in the XY_i-plane. Professor Ward showed the imaginary parts of several elementary functions that are generated by "excluded" values of x.

13. *On the Lagrange multiplier*, by Professor C. G. Phipps, University of Florida.

This speaker considered the Lagrange multiplier as a constant, as a function of the variables involved, and as a function of certain parameters. The usual manipulation was justified; the multiplier was interpreted geometrically; and finally the problem of working backwards from the first results was discussed.

14. *On the sign of certain minors arising in the expansion of a Vandermonde determinant*, by Dr. R. J. Levit, University of Georgia.

In the theory of extremal properties of functions on a finite set it is important to determine the algebraic sign of certain determinants closely related to Vandermonde's. In this paper it was shown by an indirect method that the determinants are all non-negative.

15. *Iterations of quadratic polynomials*, by Lt. C. S. Sutton, The Citadel.

A set D, associated with any quadratic polynomial $P(x)$ is defined as the set of all points in the complex plane whose iterates under $P(x)$ are uniformly bounded. The author proved that D is connected if and only if it contains the point $P(\rho)$, where ρ is the zero of the equation $P'(x) = 0$, and that otherwise D consists of a non-denumerable number of closed disjoint sets.

16. *Two dimensional Riemannian spaces that admit continuous groups of homothetic transformations*, by Mr. E. B. Shanks, Vanderbilt University.

It was shown that there exists only a one parameter family of Riemannian spaces admitting a two parameter group, while euclidean space alone has a group associated with it with a greater number of parameters than two. The latter admitted a four parameter group.

17. *Some examples in which the Pearson criteria for classifying frequency curves break down*, by Professor A. C. Cohen, Jr., University of Georgia.

The Pearson criteria for typing frequency curves indicate that the distributions of sample

means from various Pearson populations are of the same type as the populations from which the samples are selected. In this paper, certain problems connected with the classification of frequency distributions were discussed. By employing the Pearson moment recursion formulas, it was demonstrated that the typing criteria break down in classifying distributions of sample means from Type II and Type VII populations.

18. *On properties of polynomial solutions of certain linear differential equation*, by Dr. R. W. Cowan, University of Florida.

The general solution of the differential equation was obtained by Frobenius' method. The infinite series thus obtained reduced to polynomials for certain integral values of the parameter. Dr. Cowan established an orthogenality relation, and the integrated square of the polynomials was evaluated.

19. *Some peculiar integral operators*, by Professor L. V. Robinson, University of South Carolina.

Two of the operators discussed were $x \pm D^{-1}$, and the third was $(x-D^{-1})(x+D^{-1})$. Interesting properties were found, especially for the third.

20. *The integration of* $\csc^n v\, dv$, by Professor W. G. Miller, Clemson College.

The author developed methods of integration involving odd powers of the secant and cosecant by the use of trigonometric manipulation only, the primary value of which was to establish a continuity in an elementary calculus course.

21. *Formulas for expressing a function of three variables in nomograph form*, by Professor D. F. Barrow, University of Georgia.

Professor Barrow gave two formulas for expressing a function of x, y, z in the form of a 3-rowed determinant, the elements of the first, second, and third rows being functions of x alone, of y alone, and of z alone, respectively (provided such an expression was possible). This was the first step in constructing a nomograph for the solution of the equation obtained by equating the function to zero.

H. A. ROBINSON, *Secretary*

SPRING MEETING OF THE MICHIGAN SECTION

The Spring meeting of the Michigan Section of the Mathematical Association of America was held in conjunction with the meeting of the Michigan Academy of Science, Arts, and Letters at the University of Michigan at Ann Arbor on Saturday, April 3, 1948. This meeting also constituted the meeting of the Mathematics Section of the Michigan Academy of Science, Arts, and Letters. Morning and afternoon sessions and a luncheon-business meeting were held, at all of which the Chairman, Professor Harold Blair, presided.

About ninety persons attended the meeting including the following fifty-four members of the Association: N. H. Anning, J. W. Baldwin, J. H. Bell, H. L. Black, Harold Blair, W. M. Borgman, J. W. Bradshaw, D. M. Brown, W. H. Cain, R. E. Carr, R. V. Churchill, C. J. Coe, A. H. Copeland, P. C. Cox, M. L. DeMoss, P. S. Dwyer, C. M. Erickson, F. D. Faulkner, C. H. Fischer, J. W. Foust, J. S. Frame, G. W. Grotts, G. E. Hay, Fritz Herzog, T. H. Hildebrandt, E. E. Ingalls, L. A. Jehn, L. S. Johnston, Wilfred Kaplan, A. E. Lampen, H. D. Larsen, G. E. Markle, E. D. McCarthy, L. E. Mehlenbacher, D. C. Mor-

row, A. L. Nelson, E. A. Nordhaus, Sister M. Paula, George Piranian, H. H. Pixley, P. H. Raker, M. O. Reade, C. C. Richtmeyer, Hans Samelson, E. R. Sleight, Wallace Smith, T. H. Southard, H. E. Stelson, B. M. Stewart, L. O. Thompson, R. M. Thrall, Leonard Tornheim, C. P. Wells, R. L. Wilder.

The following officers were elected for the coming year: Chairman, Professor B. M. Stewart, Michigan State College; Secretary-Treasurer, L. J. Rouse, University of Michigan.

At the morning and afternoon sessions the following program of seven papers was presented:

1. *Polygenic functions and monogeneity*, by Professor Emeritus V. C. Poor, University of Michigan, introduced by the Secretary.

This paper includes a fundamental theorem on polygenic functions and a discussion of a certain type of monogeniety of such functions. A rather explicit characterization of such monogenic functions is exhibited in a particular theorem, and the dual relation between this and a second class of monogenic functions is considered.

2. *Some elementary proofs concerning binomial coefficients*, by Professor Fritz Herzog, Michigan State College.

Let p be a given prime and $c_{n,k}$ be a given binomial coefficient. The familiar formula for the highest power of p dividing $c_{n,k}$ can be interpreted in an elementary way in terms of the addition of k and $n-k$, written in the scale of p. This interpretation furnishes a simple method of proving certain divisibility theorems concerning binomial coefficients. As examples, explicit expressions for the greatest common divisor and for the least common multiple of $c_{n,1}, c_{n,2}, \cdots, c_{n,n-1}$ are derived.

3. *The other half of euclidean geometry*, by Dr. Kenneth Leisenring, University of Michigan, introduced by the Secretary.

With respect to the undefined terms "point" and "line," euclidean plane geometry is not dual; parallelism is defined for lines but not for points. Since, however, the euclidean plane can be described in purely projective terms, it follows from the duality of the projective plane that there must exist a geometry dual to the euclidean plane, obtainable by dualizing the projective description. In this dual we have a point at infinity rather than a line, and parallelism is defined for points. If we choose as a model for the projective plane the conventional euclidean plane augmented by the line at infinity, we can choose an arbitrary point as the point at infinity in the dual metric, and the two metrics can be simply related to each other. This dual metric can be intuitively grasped and systematically exploited. Results indicated are the obtaining of new theorems by dualization, (which can be repeated at will) and the unification and simplification of metric theorems on conics.

4. *An approximation to the quotient of Gamma functions*, by Professor J. S. Frame, Michigan State College.

A close approximation to the ratio of two nearby values of the Gamma function is obtained by neglecting the remainder $R_n(w)$ in the formula

$$\left(n+\frac{1+w}{2}\right)+\left(n+\frac{1-w}{2}\right)=\left(n^2+\frac{1-w^2}{12}\right)^{w/2} e^{-R_n(w)}.$$

This remainder vanishes for all n when $w=0$, ± 1, or ± 2. It is less than $1/2(4n)^{-4}$ for $w<1$, so that it may be neglected for $n>2.5$ and $w<1$ if only four significant figures are required. Furthermore, the Gamma function arguments are so chosen that $R_n(w)$ is an even function of n and an odd func-

tion of w. Approximations to the values of certain binomial coefficients and Beta functions follow immediately.

5. *What Ohio colleges are doing about mathematics*, by Professor Wayne Dancer, University of Toledo.

This paper presents the results of a recent questionnaire regarding policies and practices in mathematics in the various Ohio colleges. The study deals with college requirements in mathematics, organization of courses, the proper curriculum for a student majoring in mathematics, poorly prepared students, supervision of instruction, and other problems encountered by the departments of mathematics.

6. *Report of committee on high school mathematics*, by Professor C. C. Richtmeyer, Central Michigan College.

7. *The evaluation of determinants with pivotal methods*, by Professor P. S. Dwyer, University of Michigan.

After a brief discussion of the more general elimination, synthetic, and condensation methods, the treatment was directed to pivotal methods. Special emphasis was given to the exact pivotal method of multiplication and subtraction with exact division. Applications were made to the compact solution of simultaneous linear equations, to determinants whose elements are complex numbers, to determinants whose elements are approximate numbers, to determining all the principal minors of a determinant, and so on. A plea was made for a greater use of pivotal methods in elementary and advanced instruction dealing with the evaluation of determinants.

L. J. ROUSE, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-Second Annual Meeting, Columbus, Ohio, December 31, 1948.

Thirty-First Summer Meeting, Boulder, Colorado, September, 1949.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, University of Pittsburgh, November 6, 1948.
ILLINOIS, Peoria, May 13–14, 1949
INDIANA
IOWA, Drake University, Des Moines, April 15–16, 1949
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, University of Mississippi, Oxford, Spring, 1949
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May, 1949
NORTHERN CALIFORNIA, San Francisco, January 29, 1949
OHIO, Ohio State University, Columbus, April 2, 1949
OKLAHOMA
PACIFIC NORTHWEST, Oregon State College, Corvallis, Spring, 1949
PHILADELPHIA, Philadelphia, November 27, 1948
ROCKY MOUNTAIN, Colorado School of Mines, Golden, April, 1949
SOUTHEASTERN, University of Alabama, University, March 18–19, 1949
SOUTHERN CALIFORNIA, John Muir Jr. College, Pasadena, March 12, 1949
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, University of Buffalo, May, 1949
WISCONSIN, Lawrence College, Appleton, May 14, 1949

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ON THE APPLICATION OF VECTOR ALGEBRA TO PROJECTIVE GEOMETRY

SHOU CHEN, National Nankai University, Tientsin, China

1. Introduction. The study of projective geometry by vector methods was originated in principle by Grassman in his *Ausdehnungslehre.* It was carried out in detail by E. Study in his *Einleitung in die Theorie der Invarianten linearer Transformationen auf Grund der Vektorenrechnung* (1923), and appears also in the book of H. G. Forder, *The Calculus of Extension* (1941). Since this work does not appear to be generally known, it is the purpose of this paper to publicize it by giving a connected treatment of some of the chief theorems of two-dimensional projective geometry based upon this vector approach.

We define $\mathbf{a}=(a_1, a_2, a_3)$ to be a vector representing a point in the projective plane. The usual definitions and identities of vector algebra hold, and we give the most important ones below:

$$\mathbf{a}+\mathbf{b} = (a_1+b_1,\ a_2+b_2,\ a_3+b_3) \tag{1}$$

$$\lambda\mathbf{a} = (\lambda a_1,\ \lambda a_2,\ \lambda a_3). \tag{2}$$

For $\lambda\neq 0$, this represents the same point as the vector $\mathbf{a}$.

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1+a_2b_2+a_3b_3 \tag{3}$$

$$\mathbf{a}\times\mathbf{b} = (a_2b_3-a_3b_2,\ a_3b_1-a_1b_3,\ a_1b_2-a_2b_1). \tag{4}$$

We have then, besides the well known associative, commutative, and distributive laws:

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = [\mathbf{abc}] \tag{5}$$

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a} \tag{6}$$

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}'\times\mathbf{b}') &= [\mathbf{aa'b'}]\mathbf{b}-[\mathbf{ba'b'}]\mathbf{a}\\ &= [\mathbf{abb'}]\mathbf{a}'-[\mathbf{aba'}]\mathbf{b}'\end{aligned} \tag{7}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}'\times\mathbf{b}') = (\mathbf{a}\cdot\mathbf{a}')(\mathbf{b}\cdot\mathbf{b}')-(\mathbf{a}\cdot\mathbf{b}')(\mathbf{a}'\cdot\mathbf{b}). \tag{8}$$

This relation is known as Lagrange's identity. From (8), we derive:

$$\begin{aligned}[\mathbf{a}\times\mathbf{a}',\ \mathbf{b}\times\mathbf{b}',\ \mathbf{c}\times\mathbf{c}'] &= [\mathbf{abb'}][\mathbf{a'cc'}]-[\mathbf{acc'}][\mathbf{a'bb'}]\\ &= [\mathbf{bcc'}][\mathbf{b'aa'}]-[\mathbf{baa'}][\mathbf{b'cc'}]\\ &= [\mathbf{caa'}][\mathbf{c'bb'}]-[\mathbf{cbb'}][\mathbf{c'aa'}]\end{aligned} \tag{9}$$

which will be used frequently later on.

We shall also introduce the vector $\boldsymbol{\alpha}=(\alpha_1, \alpha_2, \alpha_3)$ to represent the line coordinates of a line in the plane. As above $\lambda\boldsymbol{\alpha}=(\lambda\alpha_1, \lambda\alpha_2, \lambda\alpha_3)$ represents the same line. We shall henceforth use Latin letters for points and Greek letters for lines.

2. Determination of points and lines in the plane. We see at once that:

(i) $\mathbf{a}\cdot\boldsymbol{\alpha}=0$ is the condition for $\mathbf{a}, \boldsymbol{\alpha}$ to be in incidence.

(ii) $\mathbf{a}\times\mathbf{b}=0$ is the condition for a and b to coincide. By 0 we mean the vector $(0, 0, 0)$.

(iii) $\mathbf{a}\times\mathbf{b}\neq 0$ is the line joining a and b.

(iv) $[\mathbf{abc}]=0$ is the condition for $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ to be collinear. We also have the duals of (ii), (iii), and (iv). From these it follows that:

(v) $(\mathbf{a}\times\mathbf{b})\times\boldsymbol{\alpha}=(\mathbf{a}\cdot\boldsymbol{\alpha})\mathbf{b}-(\mathbf{b}\cdot\boldsymbol{\alpha})\mathbf{a}$ is the point of intersection of the line joining $\mathbf{a}$, $\mathbf{b}$ with the line $\boldsymbol{\alpha}$;

(vi) $(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}'\times\mathbf{b}')=[\mathbf{aa'b'}]\mathbf{b}-[\mathbf{ba'b'}]\mathbf{a}$ is the point of intersection of the lines joining $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a}'$, $\mathbf{b}'$;

(vii) $(\mathbf{a}\times\mathbf{b})\cdot(\boldsymbol{\alpha}\times\boldsymbol{\beta})=0$ or $(\boldsymbol{\alpha}\cdot\mathbf{a})(\boldsymbol{\beta}\cdot\mathbf{b})=(\boldsymbol{\alpha}\cdot\mathbf{b})(\boldsymbol{\beta}\cdot\mathbf{a})=0$ is the condition for the line joining $\mathbf{a}$, $\mathbf{b}$ to pass through the point of intersection of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$;

(viii) $[\mathbf{a}\times\mathbf{a}', \mathbf{b}\times\mathbf{b}', \mathbf{c}\times\mathbf{c}']=0$ is the condition for the three lines joining $\mathbf{a}, \mathbf{a}'$; $\mathbf{b}, \mathbf{b}'$; $\mathbf{c}, \mathbf{c}'$ in pairs to be concurrent. From (9), we see that this condition may be written in three different ways.

In this manner, our list may be extended indefinitely. All may be dualized.

The advantage of the present representation in numerical problems is obvious; it saves considerable calculation. It can also be used to prove in a simple and concise manner many theorems of fundamental importance in plane projective geometry. We shall give a few examples of such theorems.

3. Diagonal triangle of a complete quadrangle. As a first illustration, let us prove the

THEOREM. *The three diagonal points of a complete quadrangle form a triangle.*

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be the vertices of a complete quadrangle so that no three of them are collinear. The diagonal points of the quadrangle are the intersections of opposite sides; namely,

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}),\qquad (\mathbf{a}\times\mathbf{c})\times(\mathbf{d}\times\mathbf{b}),\qquad (\mathbf{a}\times\mathbf{d})\times(\mathbf{b}\times\mathbf{c}).$$

These three points are not collinear; for, applying (9) twice, we find that their determinant is equal to

$$-2[\mathbf{bcd}][\mathbf{acd}][\mathbf{abd}][\mathbf{abc}]$$

which is not zero by (iv), since no three of the points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are collinear.

4. Desargues' theorem on perspective triangles. Next, let us take up

DESARGUES' THEOREM. *The corresponding vertices of two triangles lie on concurrent lines if and only if the points of intersection of corresponding sides meet in collinear points.*

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be the vertices of one triangle and $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$, the corresponding

vertices of the other. The condition for the lines joining corresponding vertices to be concurrent is

$$[\mathbf{a} \times \mathbf{a}', \mathbf{b} \times \mathbf{b}', \mathbf{c} \times \mathbf{c}'] = 0 \tag{10}$$

and the condition for the points of intersection of corresponding sides to be collinear is

$$[(\mathbf{b} \times \mathbf{c}) \times (\mathbf{b}' \times \mathbf{c}'), (\mathbf{c} \times \mathbf{a}) \times (\mathbf{c}' \times \mathbf{a}'), (\mathbf{a} \times \mathbf{b}) \times (\mathbf{a}' \times \mathbf{b}')] = 0. \tag{11}$$

Applying (9) again, we find that (10) is equivalent to

$$[\mathbf{abb}'][\mathbf{a}'\mathbf{cc}'] - [\mathbf{acc}'][\mathbf{a}'\mathbf{bb}'] = 0 \tag{12}$$

and (11) to

$$[\mathbf{abc}][\mathbf{a}'\mathbf{b}'\mathbf{c}']([\mathbf{abb}'][\mathbf{a}'\mathbf{cc}'] - [\mathbf{acc}'][\mathbf{a}'\mathbf{bb}']) = 0. \tag{13}$$

Since $\mathbf{a}, \mathbf{b}, \mathbf{c}$ as well as $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ are non-collinear, $[\mathbf{abc}][\mathbf{a}'\mathbf{b}'\mathbf{c}'] \neq 0$ and the conditions (12) and (13) are equivalent.

In particular, if $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ lie on the sides of the other triangle opposite to $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively, we may put

$$\begin{aligned} \mathbf{a}' &= \lambda_1 \mathbf{b} + \mu_1 \mathbf{c}, \\ \mathbf{b}' &= \lambda_2 \mathbf{c} + \mu_2 \mathbf{a}, \\ \mathbf{c}' &= \lambda_3 \mathbf{a} + \mu_3 \mathbf{b}. \end{aligned} \tag{14}$$

Substituting into (12), we have the condition

$$(\lambda_1\lambda_2\lambda_3 - \mu_1\mu_2\mu_3)[\mathbf{abc}]^2 = 0 \tag{15}$$

or

$$\lambda_1\lambda_2\lambda_3 - \mu_1\mu_2\mu_3 = 0.$$

This gives the Theorem of Ceva. Its dual, the Theorem of Menelaus, is proved in a dual fashion.

5. Pole and polar w.r.t. a triangle. Again, let us prove the

THEOREM. *If three points, each lying on a side of a triangle, are collinear, the lines joining their harmonic conjugates w.r.t. the vertices of the triangle to the opposite vertices are concurrent and conversely.*

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be the vertices of the triangle and $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ be points on the sides opposite to $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively. If we assume that $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ are given by (14), the condition for these points to be collinear is

$$[\lambda_1\mathbf{b} + \mu_1\mathbf{c}, \lambda_2\mathbf{c} + \mu_2\mathbf{a}, \lambda_3\mathbf{a} + \mu_3\mathbf{b}] = 0,$$

or

$$(\lambda_1\lambda_2\lambda_3 + \mu_1\mu_2\mu_3)[\mathbf{abc}] = 0. \tag{16}$$

If $\mathbf{a}''$, $\mathbf{b}''$, $\mathbf{c}''$ are the harmonic conjugates of $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ w.r.t. $\mathbf{b}, \mathbf{c}$; $\mathbf{c}, \mathbf{a}$; $\mathbf{a}, \mathbf{b}$ respectively, we may put

$$\mathbf{a}'' = \lambda_1\mathbf{b} - \mu_1\mathbf{c},$$
$$\mathbf{b}'' = \lambda_2\mathbf{c} - \mu_2\mathbf{a},$$
$$\mathbf{c}'' = \lambda_3\mathbf{a} - \mu_3\mathbf{b},$$

so that the condition for the lines joining $\mathbf{a}''$, $\mathbf{b}''$, $\mathbf{c}''$ to $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ to be concurrent is, by (15),

$$(17) \qquad [\mathbf{a} \times \mathbf{a}'', \mathbf{b} \times \mathbf{b}'', \mathbf{c} \times \mathbf{c}''] = (\lambda_1\lambda_2\lambda_3 + \mu_1\mu_2\mu_3)[\mathbf{abc}]^2 = 0.$$

Since $[\mathbf{abc}] \neq 0$, the conditions (16) and (17) are equivalent.

6. Pappus' Theorem. We now prove

PAPPUS' THEOREM. *If a, b, c are three points on a line and* $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ *three points on another, the three points of intersection of the pairs of lines joining* $\mathbf{b}$ *and* $\mathbf{c}'$, $\mathbf{b}'$ *and* $\mathbf{c}$; $\mathbf{c}$ *and* $\mathbf{a}'$, $\mathbf{c}'$ *and* $\mathbf{a}$; $\mathbf{a}$ *and* $\mathbf{b}'$, $\mathbf{a}'$ *and* $\mathbf{b}$ *are collinear.*

We have to show that

$$(18) \quad [(\mathbf{b} \times \mathbf{c}') \times (\mathbf{b}' \times \mathbf{c}), (\mathbf{c} \times \mathbf{a}') \times (\mathbf{c}' \times \mathbf{a}), (\mathbf{a} \times \mathbf{b}') \times (\mathbf{a}' \times \mathbf{b})] = 0$$

which, according to (9), is equivalent to

$$(18') \qquad [\mathbf{bb'c}][\mathbf{cc'a}][\mathbf{aa'b}][\mathbf{a'b'c'}] - [\mathbf{c'b'c}][\mathbf{a'c'a}][\mathbf{b'a'b}][\mathbf{abc}] = 0.$$

Now since $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are collinear, $[\mathbf{abc}] = 0$ and similarly $[\mathbf{a'b'c'}] = 0$, and so (18′) is satisfied.

This result extends immediately to give a proof of

PASCAL'S THEOREM: *If* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ *are six points on a conic, the three points of intersection of the pairs of lines joining* $\mathbf{b}$ *and* $\mathbf{c}'$, $\mathbf{b}'$ *and* $\mathbf{c}$; $\mathbf{c}$ *and* $\mathbf{a}'$, $\mathbf{c}'$ *and* $\mathbf{a}$; $\mathbf{a}$ *and* $\mathbf{b}'$, $\mathbf{a}'$ *and* $\mathbf{b}$ *are collinear.*

Let us think of (18′) as an equation in the variable point $\mathbf{c}$. It appears at once that this equation is of the second degree in the coordinates of $\mathbf{c}$ and hence represents a conic. Further it is easy to see that the equation is identically satisfied when $\mathbf{c}$ is equal to any one of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a}'$, $\mathbf{b}'$, or $\mathbf{c}'$; and hence this is an equation of the conic passing through these last five points. Since the point $\mathbf{c}$ given in the theorem lies on this conic by hypothesis, (18′) is satisfied and the conclusion follows. This demonstration illustrates the fact that Pappus' Theorem is a degenerate form of Pascal's Theorem.

From (18′) we also have the following joint converse of these two theorems:

CONVERSE THEOREM: *If six points* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ *are so located that the three points of intersection of the pairs of lines joining* $\mathbf{b}$ *and* $\mathbf{c}'$, $\mathbf{b}'$ *and* $\mathbf{c}$; $\mathbf{c}$ *and* $\mathbf{a}'$, $\mathbf{c}'$ *and* $\mathbf{a}$; $\mathbf{a}$ *and* $\mathbf{b}'$, $\mathbf{a}'$ *and* $\mathbf{b}$ *are collinear, then*

(1) *if no three points are collinear, the six points lie on a non-degenerate conic;*

(2) *if three points (say* **a**, **b**, *and* **c**) *are collinear either: one or more of* **a**′, **b**′, **c**′, *lies on the line* **abc** *or:* **a**′, **b**′, *and* **c**′ *are collinear.*

In interpreting these results it should be noted that the exact form of (18′) is not unique. Indeed it has 60 equivalent forms obtained from the one given by permuting any five of the given points. The equivalence of these forms can be shown directly by careful algebraic manipulation in the reduction of (18) to (18′) or from the fact that the equation of a conic through five points clearly is independent of the order in which these points are chosen.

MUSIC AND TERNARY CONTINUED FRACTIONS*

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1. Introduction. The most important interval in music is the octave, expressed by the ratio, 2:1. The number of notes within the octave has varied at different times and in different places. The pentatonic or black-key scale is very old and is widely distributed, occurring in such countries as Scotland and China. The heptatonic, diatonic, or white-key scale, in its various major and minor forms, is the foundation of our classic musical system. The duodecuple or chromatic scale includes both the white keys and the black keys of the piano, and has become increasingly important during the past two or three centuries.

After harmony was introduced into music during the late Middle Ages, major and minor triads emerged as the principal chords. The major triad, as $C\,E\,G$, was regarded with especial favor, because it occurs naturally in the harmonic series, as on bugles, and can be expressed by the simple ratios, 4:5:6. A system of tuning for the diatonic scale known today as just intonation gained support in the 16th century, because its principal triads, $C\,E\,G$, $F\,A\,C$, and $G\,B\,D$, had these just ratios. But an important minor triad, $D\,F\,A$, is harsh in just intonation, and other unsatisfactory triads result when this tuning is extended to the complete chromatic scale.

To correct the worst of these pitch errors, various systems of temperament were devised, in which the intonation of some (or all) notes was altered slightly, so that no intervals would be unusable. In practice, temperament was often

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haphazard, a rule of thumb which some musical theorists exalted into definite systems. More important are the regular systems, those in which a particular value of the perfect fifth (as $C\,G$) is used as the tuning unit. The meantone temperament is a regular system in which the fifth (3/2) is flatted by 1/4 syntonic comma, and the major third is pure (5/4). A comma is a very small interval, the difference between two larger intervals. The syntonic, or Ptolemaic, comma is the difference between two tones, $(9/8)^2$, and a pure major third, (5/4). Since the difference between intervals is expressed by the quotient of their ratios, this comma has the ratio of 81/64 to 5/4, or 81:80. A fifth tempered by 1/4 syntonic comma then has the ratio $(3/2)(80/81)^{1/4}=5^{1/4}:1$, in place of 3:2. Other regular systems in which the fifth is tempered by an aliquot part of the comma, such as 1/3 or 1/5, may be thought of as varieties of the meantone temperament.

Today the only tuning system in common use is equal temperament, in which the fifth is tempered by 1/12 ditonic comma. The ditonic, or Pythagorean, comma is the difference between six tones, $(9/8)^6$, and the pure octave, (2/1); that is, the ratio 531441:524288, which lies between 75:74 and 74:73. A fifth tempered by 1/12 ditonic comma has the ratio $(3/2)(8/9)^{1/2}2^{1/12}=2^{7/12}:1$. This same ratio is obtained directly from the consideration that the interval of the fifth contains 7 semitones; that is, it is 7/12 of an octave.

Equal temperament was used on fretted instruments (lutes and viols) in the early 16th century. It was advocated for keyboard instruments before the end of the 16th century, but was not universally accepted until the middle of the 19th century. The reason for its slow acceptance was the sharpness of its major thirds. These deviations will be clearer if the intervals are expressed in a logarithmic unit known as the cent, $2^{11/200}$. The octave contains 1200 cents; the fifth of equal temperament, 700; its third, 400. The pure fifth, with ratio 3:2, contains $1200\,(\log 3/2)\div\log 2=702$ cents, and thus has been slightly flatted in equal temperament; but the pure third, with ratio 5:4, contains 386.3 cents, and thus has been sharped by almost 14 cents. The syntonic comma contains 21.5 cents; the ditonic comma, 23.5.

The tremendous advantage of equal temperament over other regular systems of 12 notes is that it is a closed, or cyclic, system. In a cyclic tuning system the nth power of the value chosen for the fifth will be a higher octave of the initial note. A great many suggestions have been made from time to time for improving the quality of the thirds by increasing the number of notes in the octave. The best of these systems operate upon the principle that an equal division of the octave is much to be preferred to an extension of just intonation. In making the division, care must be taken that the improvement of the thirds does not result in too great an impairment of the fifths. An increase in the number of notes in the octave has the further advantage that such enharmonic pairs as $G\sharp$ and $A\flat$ can be differentiated.

2. Systems of multiple division. In the middle of the 16th century Nicola

Vicentino described his six-manual Archicembalo, in which the octave is divided into 31 equal parts. Since he wrote before the invention of logarithms, he had no way of expressing these intervals by numbers on a monochord. But he directed that the fifths are to be tempered "according to the usage and tuning common to all the keyboard instruments," that is, according to the 1/4 comma meantone temperament. It remained for Christian Huyghens to show that the two systems are almost identical. The meantone fifth contains 696.6 cents; its third 386.3. The fifth of the 31-division contains 696.8 cents; its third, 387.1.

The division of the octave into 19 equal parts was accomplished a few years after Vicentino's successful attempt at multiple division. Both Zarlino and Salinas discussed harpsichords with 19 keys to the octave, and the latter advocated a system in which the fifth is tempered by 1/3 comma. This corresponds very closely to the equal 19-division. The fifths of both systems contain 694.7 cents; their thirds, 378.9. The large distortion of both intervals would make this system inferior to the ordinary 1/4 comma meantone temperament were it not that it is a cyclic system of only a few more notes than our present equal temperament of 12 notes. There have been eloquent advocates of the 19-division even in our own day, including the Russian-American, Joseph Yasser.

The best cyclic division with fewer than 100 parts is the 53-division. It was implied by the Pythagoreans, who held that there are 4 commas in the diatonic semitone and 9 in a tone, whence the octave will contain $5\cdot 9+2\cdot 4=53$ commas. Since $1200\div 53=22.6$ cents, this is a mean comma between the syntonic and ditonic commas. It was appreciated by a medieval Chinese theorist, King Fâng, and was referred to by 17th century European musical theorists (Mersenne and Kircher); it was used by Nicholas Mercator as a "common measure" for all intervals, and applied by Bosanquet to the "generalized" keyboard of his Enharmonic Harmonium. The fifths of the 53-division are practically perfect (701.9 cents), and its thirds are slightly flat (384.9 cents). Since this is a "positive" system, with fifths larger than those of equal temperament, its third must be formed as a diminished fourth, as C $F\flat$. (The "negative" definition of the third in terms of fifth and octave is: $T=4F-2O$; the "positive," $T=5O-8F$.) This would be a confusing feature to a performer used to thirds that are really thirds.

Of the three systems [1] of multiple division discussed above, only the 19-division would have any practical value. The others would have been too expensive to construct and too cumbersome to play. And so, as we discuss various theories of the division of the octave, it must be realized that such a scheme is useless to the musician. Of course, some of the workers in this field have tried seriously to perfect instruments for multiple division, believing they would help the cause of music thereby. Events have proved them wrong. But others, who glibly talked of dividing the octave into more than 100 parts, had nothing to say about how their theories could be applied to instruments. They may have been deceiving themselves, but probably were aware that this was nothing but mathematical speculation.

3. Theories of multiple division. Joseph Sauveur, the French geometer and acoustician, was greatly interested in multiple division. In one of his articles [2] he gave a very impressive list of octave divisions, consisting of 25 terms: 12, 17, 19, 31, 43, 50, 53, 55, 67, 74, 98, 105, 112, 117, 122, 136, 141, 153, 160, 177, 184, 189, 208, 232, 256. All of these except 12, 17 (which he calls an Oriental system), and 53 are what Bosanquet called "negative," that is, they have fifths that are flatter than those of equal temperament, and thus form their major thirds properly.

Sauveur's theory is based on the division of the tone, as *C D*, into a diatonic semitone, as *C D*♭, and a chromatic semitone, as *D*♭ *D*. In just intonation, if the ratio 16:15 or 112 cents is taken for the diatonic semitone, and the ratio 135:128 or 92 cents for the chromatic semitone, the ratio of the diatonic to the chromatic semitone will be 112:92 or 28:23. Convenient approximations to this ratio are 5:4, 6:5, and 11:9. Since the octave contains 7 diatonic and 5 chromatic semitones, we may use the formula, $O=7d+5c$, where $d>c$. If $d/c=5/4$, $O=7\cdot 5+5\cdot 4=55$ parts; if $d/c=6/5$, $O=7\cdot 6+5\cdot 5=67$ parts; if $d/c=11/9$, $O=7\cdot 11+5\cdot 9=122$ parts.

In Sauveur's list there are combined several independent series, each determined by the formula $d=c+i$, where $i=1, 2, 3, 4$. If, for example, $i=4$, $O=7(c+4)+5c=12c+28$. Using only odd values of c from 7 through 19, Sauveur obtained 112, 136, 160, 184, 208, 232, 256. The ratio of the third to the fifth to the octave in this series is $(4c+8):(14c-5):(24c-8)$. The limit of these ratios is 4:7:12, as in equal temperament. Therefore, as c increases beyond a certain value, the fifths continue to improve, but the thirds become sharper and sharper, since equal temperament has good fifths and very sharp thirds. This will be true of each of Sauveur's series. It would seem to be a false theory by which series are generated in which the middle terms are the best. Romieu [3] and Bosanquet [4], both of whom did outstanding work in this field, had no better theories than Sauveur.

M. W. Drobisch [5] made the first notable contribution to the theory of multiple division of the octave through the use of continued fractions. Having expressed the ratio of the fifth to the octave (log 3/2:log 2) as a decimal, 0.5849625, or as a fraction, 46797/80000, he used ordinary continued fractions to find successive approximations to this ratio. He obtained for his denominators (the octaves) this series: 2, 5, 12, 41, 53, 306, 665, [15601], · · · . Then, apparently not trusting these results, he found all the powers of 3/2 from the 13th to the 53rd, to ascertain which of them approach a pure octave. This should have checked closely with his previous list, to which 17 and 29 would be semiconvergents. His complete list is: 17, 19, 22, 29, 31, 41, 43, 46, 51, 53. Having eliminated all positive divisions (those with raised fifths), he still had 19, 31, and 43 to add to his previous list.

Drobisch's continued fraction expansion was the first really scientific method of dividing the octave with regard to the principal consonances, the thirds and the fifths. The difficulty with it is that there are three magnitudes to be com-

pared (third, fifth, octave), but only one ratio (third to octave, or fifth to octave) can be approximated by binary continued fractions. If we must choose a single ratio, it is better to use the ratio of fifth to octave, as Drobisch did, since the third may be expressed in terms of the fifth. But the "negative" formula, $T=4F-2O$, is valid only through $O=12$. The syntonic comma (81:80 or 21.5 cents) is about 1/56 octave; hence this formula will fail to give a correct number of parts for the third in any octave division greater than 28. For example, if $O=41$ and $F=24$, the formula gives $T=14$, whereas the correct value is 13. Knowing the approximate size of the comma, we can correct the formula to read: $T=4F-2O-[O/56]$. But even this would give only by accident a value for the third with as small an error as that for the fifth in the same division.

4. Use of ternary continued fractions. The desired solution can be obtained only by ternary continued fractions, which are a means by which the ratios of three numbers may be approximated simultaneously, just as the ratios of two numbers may be approximated by binary continued fractions. C. G. J. Jacobi [6] was the first person to explore the possibilities of ternary continued fractions. The most thorough study of the subject has been made by Oskar Perron in half a dozen articles over a period of 30 years [7]. American workers in this field include D. H. Lehmer, J. B. Coleman, and P. H. Daus, most of their articles being published in the *American Journal of Mathematics*. The principal questions discussed in these papers are convergence and periodicity, especially the relation between periodic ternary continued fractions and the roots of cubic equations. None of the theory developed by these men appears to have any bearing upon the present problem.

Jacobi's algorithm. P. H. Daus has stated Jacobi's algorithm for ternary continued fractions as follows [8]:

If u_1, v_1, w_1 be any three numbers, we define a ternary continued fraction expansion for them by the equations

$$u_{n+1} = v_n - p_n u_n; \qquad v_{n+1} = w_n - q_n u_n; \qquad w_{n+1} = u_n. \tag{1}$$

. . . .

The numbers (A, B, C), defined by the recursion formulas (3) below, form the nth convergent set to the ternary continued fraction

$$\left(\frac{v_1}{u_1}, \frac{w_1}{u_1}\right) = (p_1, q_1; p_2, q_2; \cdots; p_n, q_n; \cdots).$$

$$\begin{aligned} A_n &= q_n A_{n-1} + p_n A_{n-2} + A_{n-3}, \\ B_n &= q_n B_{n-1} + p_n B_{n-2} + B_{n-3}, \\ C_n &= q_n C_{n-1} + p_n C_{n-2} + C_{n-3}, \end{aligned} \tag{3}$$

with the initial conditions

$$\begin{bmatrix} A_{-2} & A_{-1} & A_0 \\ B_{-2} & B_{-1} & B_0 \\ C_{-2} & C_{-1} & C_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In his earlier article [9], Daus had shown that

$$\begin{bmatrix} A_{n-2} & A_{n-1} & A_n \\ B_{n-2} & B_{n-1} & B_n \\ C_{n-2} & C_{n-1} & C_n \end{bmatrix} = \prod_1^n \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & p_n \\ 0 & 1 & q_n \end{bmatrix},$$

from which the formulas in (3) follow. In [9], Daus, following Jacobi's erroneous example, had interchanged p's and q's in the formulas for A_n, B_n, and C_n, given correctly in (3).

Jacobi's method in the tuning problem. For our musical problem

$$u_1 = \log 5/4; \qquad v_1 = \log 3/2; \qquad w_1 = \log 2.$$

The ratios $A_n{:}B_n{:}C_n$ represent successive approximations to the ratios of these three logarithms. The musical interpretation is that if the octave is divided into C_n parts, A_n parts are a good approximation to the major third, B_n to the perfect fifth. In Table I are shown the results of applying Jacobi's expansion to the three logarithms.

TABLE I. JACOBI'S TERNARY CONTINUED FRACTIONS, WITH APPLICATION TO THE TUNING PROBLEM

p_n	q_n	A_n	B_n	C_n
1	3	1	1	3
0	1	1	2	3
1	7	8	15	25
0	1	9	16	28
0	1	10	18	31
0	2	28	51	87
3	8	478	263	817

There are two serious faults with these results. In the first place, the expansion converges too rapidly, and we are interested chiefly in small values, those for which $C<100$. In the second place, the first few terms are foreign to every other proposed solution, such as those by Sauveur and Drobisch on previous pages. Compare this table with Table IV that follows, which is a composite of Tables I–III.

Reversed Expansion. Jacobi's method is not the only way to obtain a ternary continued fraction expansion. What might be called the reverse of Jacobi's expansion is defined as follows:

$$u_{n+1} = u_n - p_n v_n; \qquad v_{n+1} = w_n - q_n v_n; \qquad w_{n+1} = v_n.$$

Here

$$\begin{bmatrix} A_{n-2} & A_{n-1} & A_n \\ B_{n-2} & B_{n-1} & B_n \\ C_{n-2} & C_{n-1} & C_n \end{bmatrix} = \prod_1^n \begin{bmatrix} 1 & 0 & p_n \\ 0 & 0 & 1 \\ 0 & 1 & q_n \end{bmatrix}$$

from which

$$A_n = q_nA_{n-1} + A_{n-2} + p_n,$$
$$B_n = q_nB_{n-1} + B_{n-2},$$
$$C_n = q_nC_{n-1} + C_{n-2},$$

with the same initial conditions as before.

TABLE II. REVERSED TERNARY CONTINUED FRACTIONS, WITH APPLICATION TO THE TUNING PROBLEM

p_n	q_n	A_n	B_n	C_n
0	1	0	1	1
0	1	0	1	2
1	2	1	3	5
2	2	4	7	12
0	3	13	24	41
0	1	17	31	53
0	5	98	179	306
1	2	214	389	665

Since v_1 is log 3/2 and w_1 is log 2, and since B_n and C_n are independent of p_n, the series for the ratio of perfect fifth to octave (B_n: C_n) will be the same as that obtained by Drobisch by binary continued fractions. However, the correction for the major third for octave divisions greater than 28 is no longer needed, since the correction is made directly by the formula for A_n, with its extra member.

It probably is an accident that in this particular problem the solution by the reversed method yields a more familiar series than Jacobi's expansion does. (Again compare with Table IV.) Its rate of convergence is slightly lower, but it is still somewhat high. The method is presented here chiefly as an added basis for the expansion next to be discussed.

Defects of both expansions. Since neither the Jacobi nor the reversed ternary continued fraction expansion converges slowly, some method must be devised for slower convergence. In binary continued fractions, about half of the semi-convergent sets that lie between two convergent sets form better approximations to the ratio of the given numbers than the former of the two convergent sets does. But semi-convergents are a doubtful expedient in ternary continued fractions. Although their peculiarities have never been completely charted, a brief study of them by the present writer shows that a careful selection of coefficients provides semi-convergents superior to the *latter* of the two convergent sets, something impossible in binary continued fractions. For example, between

the second and third sets of Table I may be inserted the semi-convergent sets (taking q_3 as 5 and 6, respectively) 6 11 19 and 7 13 22, both of which, as can be seen from Table IV, are superior to the next convergent set of Table I, 8 15 25.

Mixed expansion. It is possible, however, to obtain fairly slow convergence without recourse to semi-convergents. At any stage of the expansion there are two possible divisors, u_n and v_n, the former being used in the Jacobi expansion and the latter in the reversed method. If either of these expansions is used, the rate of convergence is likely to be somewhat irregular, but approaches a mean value. But if fast convergence is desired, one should consistently divide by whichever of u_n and v_n is the smaller. In our tuning problem, slow convergence is the desideratum. Hence one should always divide by the larger of u_n and v_n.

This mixed expansion, as it may be called, presents certain difficulties, since it is no longer possible to express an approximation in terms of the three immediately preceding approximations. No new definitions of u_{n+1}, v_{n+1}, and w_{n+1} are needed; for at any stage either the Jacobi (J) or the reversed (R) expansion is to be used.

Then, in terms of a matrix product, for a J stage,

$$\begin{pmatrix} A_{n-2} & A_{n-1} & A_n \\ B_{n-2} & B_{n-1} & B_n \\ C_{n-2} & C_{n-1} & C_n \end{pmatrix} = \begin{pmatrix} A_{n-k-3} & A_{n-2} & A_{n-1} \\ B_{n-k-3} & B_{n-2} & B_{n-1} \\ C_{n-k-3} & C_{n-2} & C_{n-1} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & p_n \\ 0 & 1 & q_n \end{pmatrix},$$

where k is the number of R stages between this stage and the last previous J stage. For an R stage, substitute on the right the matrix for the reversed expansion,

$$\begin{pmatrix} 1 & 0 & p_n \\ 0 & 0 & 1 \\ 0 & 1 & q_n \end{pmatrix}.$$

From the above we obtain,

for an R stage: $S_n = q_nS_{n-1} + S_{n-2} + p_nS_{n-k-3}$;

for a J stage: $S_n = q_nS_{n-1} + p_nS_{n-2} + S_{n-k-3}$.

It is understood that

$$S_n = A_n, \ B_n, \quad \text{or} \quad C_n.$$

5. Mixed expansion in the tuning problem. In the application of the mixed method to our tuning problem, we shall use the Jacobi (J) method whenever $u_n > v_n$, and the reversed (R) method whenever $u_n < v_n$. Hence all the p's will be 0. In Table III this application is shown, together with a determinant in which A_n, B_n, and C_n are the coefficients of a, b, and c, respectively, in the expansion of the determinant.

TABLE III. MIXED TERNARY CONTINUED FRACTIONS, WITH APPLICATION TO THE TUNING PROBLEM

a	-1	0	0	0	0	0	0	0	0	0	0
c	0	-1	0	0	0	0	0	0	0	0	0
b	p_1	q_1	-1	0	0	0	0	0	0	0	0
0	p_2	1	q_2	-1	0	0	0	0	0	0	0
0	1	0	p_3	q_3	-1	0	0	0	0	0	0
0	0	0	1	p_4	q_4	-1	0	0	0	0	0
0	0	0	0	p_5	1	q_5	-1	0	0	0	0
0	0	0	0	1	0	p_6	q_6	-1	0	0	0
0	0	0	0	0	0	p_7	1	q_7	-1	0	0
0	0	0	0	0	0	p_8	0	1	q_8	-1	0
0	0	0	0	0	0	p_9	0	0	1	q_9	-1
0	0	0	0	0	0	1	0	0	0	p_{10}	q_{10}

	p_n	q_n	A_n	B_n	C_n
	0	1	0	1	1
R	0	1	0	1	2
J	0	1	1	1	2
J	0	1	1	2	3
R	0	1	2	3	5
J	0	1	2	4	7
R	0	1	4	7	12
R	0	1	6	11	19
R	0	1	10	18	31
J	0	1	11	20	34
J	0	1	17	31	53
R	0	1	28	51	87
J	0	1	38	69	118
R	0	4	180	327	559
J	0	1	197	358	612

Table IV contains all the terms from the other three tables, through $C=118$, together with a few other good approximations not found in any of the previous tables. The errors in cents for the major third (in some cases to 0.1 cent) have been computed by the formula: Error $A_n=386.3-1200A_n/C_n$; the errors for the fifth by: Error $B_n=702-1200B_n/C_n$. The total error is taken as $|\text{Error } A_n| + |\text{Error } B_n|$.

TABLE IV. COMPOSITE OF APPROXIMATIONS TO TUNING RATIOS (LOG 5/4: LOG 3/2: LOG 2), WITH BEST APPROXIMATIONS STARRED

A_n	B_n	C_n	Error A_n	Error B_n	Total Error
* 0	1	1	−386	498	884
* 0	1	2	−386	−102	488
* 1	1	2	214	−102	316
* 1	1	3	14	−302	316
* 1	2	3	14	98	112
* 2	3	5	94	18	112
1	3	5	−146	18	164
* 2	4	7	− 43	− 16	59
* 3	5	9	14	− 35	49
* 3	6	10	− 26	18	44
* 4	7	12	14	− 2	16
* 6	11	19	− 7	− 7	14
* 7	13	22	− 4	7	11
8	15	25	− 2	18	20
9	16	28	− 1	− 16	17
*10	18	31	.8	− 5.2	6.0
*11	20	34	1.9	3.9	5.8
13	24	31	− 6	1	7
*17	31	53	− 1.4	− .1	1.5
*28	51	87	− .1	1.4	1.5
*38	69	118	.2	− .5	.7

Of the starred best approximations in Table IV, only $C=9$, 10, and 22 are missing from Table III; but none of these occurs in Tables I or II either. The only serious omission from Table III is the Hindoo division, $C=22$, and it would occur as a semi-convergent, by assuming a J instead of an R for the term where 31 now stands. Note the two terms, 31 and 34, nearly the same size: it would have been impossible to include them both (even if one had been a semi-convergent) if binary or Jacobi ternary continued fractions had been used. Both 53 and 118 are present; but between them lies 87, a term superior to 65, the middle term in the binary series. The last term shown, 612, was said by Bosanquet [4] to have been considered excellent by Capt. J. Herschel; it would have had no place in any other series derived by continued fractions in which 559 was also present.

6. Euler's problem. This mixed method for ternary continued fraction expansions has been shown to satisfy the need for slow convergence in our special problem. It should be equally valuable wherever slow convergence is desired. In this connection, let us return to Jacobi. His article on ternary continued fractions was preceded, both literally and logically, by an article [10] in which he treated integral solutions of a linear equation in any number of variables. He made special reference, however, to Euler's problem: to solve in integers the equation $Aa+Bb+Cc=0$, where the capital letters are given integers, possibly integral approximations to irrational numbers.

Euler's first example was $49a+59b+75c=0$. To obtain a general solution, he placed u on the right, and then replaced it by 0 for the particular solution. His method was to divide two coefficients by the remaining coefficient, always dividing by the *smaller* remainder in each stage. Thus his method made for rapid convergence. It is irregular, but for a purpose, as is our method of dividing by the larger remainder. In the example just given, the coefficients are so chosen that the smaller remainder is always on the left, thus agreeing with Jacobi's method, which he first used in this connection before applying it to continued fractions. Both Euler and Jacobi obtained as the smallest solution: $a=-13$, $b=-7$, $c=14$.

Euler's second example was $1{,}000{,}000a+1{,}414{,}214b+1{,}732{,}051c=u$, where the second and third coefficients will be recognized as approximations to $\sqrt{2}$ and $\sqrt{3}$, with the decimal point omitted. Euler again divided by the smaller remainders, and thus got a different result from Jacobi, who divided by the left-hand remainders. With $u=0$, Euler's smallest solution was: $a=8104$, $b=-6889$, $c=946$; Jacobi's, $a=282$, $b=-2377$, $c=1778$.

7. Mixed expansion in Euler's problem. The results already obtained in this paper suggest that Jacobi's results may have been better than Euler's in this second problem, not because his method was regular, but because of the slower convergence arising from a casual mixture of small and large remainders as divisors. If, then, the larger remainders are used uniformly as divisors, as in the tuning problem, better results than Jacobi's might be expected. As applied

to Euler's first problem, our mixed method gives: $a=u-8i+j$, $b=3u+13i-11j$, $c=-3u-5i+8j$. If $u=0$, $i=1$, $j=1$, then $a=-7$, $b=2$, $c=3$. The Euler-Jacobi solution above is then obtained by letting $i=2$, $j=3$.

Again, in Euler's second problem, division by the larger remainders yields superior results. This method gives: $a=51u-2403\alpha+895\beta$, $b=264u+1829\alpha-1402\beta$, $c=-245u-106\alpha+628\beta$. If $u=0$, $\alpha=1$, $\beta=1$, then $a=-1508$, $b=427$, $c=522$. Euler's solution is obtained by letting $\alpha=-3$, $\beta=1$; Jacobi's, with $\alpha=1$, $\beta=3$.

8. Conclusion. Insufficient study has been given the mixed expansion for ternary continued fractions to ascertain all of its possibilities and weaknesses. Without the use of semi-convergents, the mixed method clearly provides slower convergence than Jacobi's method. It can be used to good advantage also when fast convergence is desired, by a simple reversal of the hypotheses. In this case the divisors are so selected than none of the p's will be 0. Thus the mixed expansion would seem to fulfill a definite need which Jacobi's expansion does not meet.

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CLAIRAUT AND THE ORIGIN OF THE DISTANCE FORMULA

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The familiar distance formula of plane analytic geometry

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

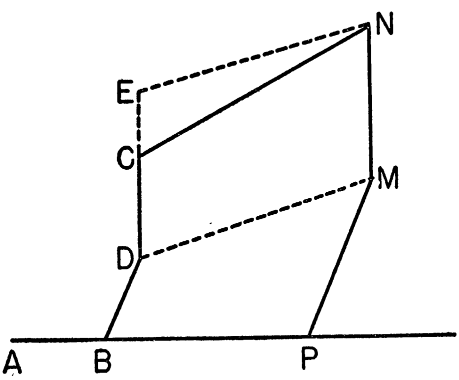
is ascribed [1] universally to Lacroix, who published it [2] in 1797 and 1798. Paradoxically, the corresponding formula for space of three dimensions, once ascribed to Monge [3], has been traced back to Clairaut's classical *Recherches sur les courbes à double courbure* of 1731. Coolidge consequently expressed the traditional view when he wrote:

> Clairaut published the distance formula in three dimensions three-quarters of a century before LaCroix gave the corresponding formula in plano.

Coolidge prudently added, however,

> My conclusion is that LaCroix's formula had been for a long time more or less common property and could be found by sufficiently diligent research [4].

It is the object of this brief note to point out that Coolidge was quite correct in his conclusion. The distance formula for two dimensions can indeed be found before LaCroix, and exactly where one would expect to find it—in Clairaut!



The *Recherches* of Clairaut, composed when he was only sixteen years old, is devoted primarily to twisted curves in three-space, and hence the formula for distance in a plane appears but incidentally, as a preliminary to the determination of the equation of the sphere. Let the center of the sphere be C with coördinates $AB = \pm a$, $BD = \pm b$, and $DC = \pm c$, with respect to the axis AB and the reference plane ABD; and let N be any point on the sphere with coördinates $AP = x$, $PM = y$, and $MN = z$. Then Clairaut wrote [5] that $EN = MD$

$=\sqrt{\overline{x\mp a}^2+\overline{y\mp b}^2}$, possibly the first time that the distance formula had appeared in print. He gave the analogue for three-space immediately afterwards as $f=\sqrt{\overline{x\mp a}^2+\overline{y\mp b}^2+\overline{z\mp c}^2}$, where $f=CN$. Consequently, to Clairaut, pending further evidence, must go the credit for the distance formula, both in two dimensions and in three, for his forms differ but slightly from the modern equivalents, mainly in the failure to regard the literal constant quantities a, b, and c as indifferently either positive or negative. More modern forms, also overlooked by historians, were given in 1773 by Lagrange for both two and three dimensions. In the preface to Clairaut's *Recherches* there is a hint of a system of spherical coördinates, but Lagrange was one of the first to develop this; and it is in his transformation from Cartesian to spherical coördinates that one finds [6] distances given by the expressions $\sqrt{(x-a)^2+(y-b)^2}$ and $\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$.

The contribution of Clairaut in this connection should not, however, be exaggerated. The distance formulas are, after all, obvious analytical expressions of an ancient theorem, named for Pythagoras, but known to the Babylonians of some four thousand years ago. There can be little doubt but that their equivalents were familiar to the earliest analytical geometers, including the inventors, Fermat and Descartes. The equations of circles and spheres, given long before Clairaut, are tantamount to distance formulas; and the rectification of curves, known since 1659, is dependent upon some such equivalent. The formulas for distance in infinitesimal analysis, $\sqrt{dx^2+dy^2}$ and $\sqrt{dx^2+dy^2+dz^2}$, also appear in Clairaut's *Recherches*; but these are not attributed to him. It would not be surprising if "sufficiently diligent research" would reveal explicit, as well as implicit, anticipations of Clairaut's distance formulas, not only for the calculus, but also for analytic geometry.

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MATHEMATICAL NOTES

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DETERMINANTS OF FOURTH ORDER MAGIC SQUARES

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1. Introduction. In the editorial note to problem E 674 [this MONTHLY, vol. 53, 1946, p. 99] the question was raised, "What is the minimum necessary condition that the determinant of a magic square be zero?" The conditions are given below for fourth order squares whose elements are in arithmetic progression.

Dudeney, in his "Amusements in Mathematics" (1917), page 120, classifies fourth order magic squares into twelve types according to the relative positions of the conjugate pairs whose sum, X, is one-half the constant. These twelve types fall into four "determinant" types. As indicated on the opposite page, Types I, II, III are included in Determinant Type A; Types IV, V, VI are in Determinant Type B; Types VII, VIII, IX, X are in Determinant Type C; and Types XI, XII are in Determinant Type D. Type I is a Nasik (pandiagonal) square; Types II, III, IV and V are semi-Nasik squares; Type VI may be semi-Nasik or simple; and Types VII, VIII, IX, X, XI and XII are simple magic squares. The type numbers and patterns (in which connected cells contain conjugate pairs) are Dudeney's.

2. Transitions. The transitions within the "determinant" types are as follows:

Type A.

Type II. Interchange 2nd and 3rd rows, and then interchange 2nd and 3rd columns to get Type I.

Type III. Interchange 3rd and 4th columns, and then interchange 3rd and 4th rows to get Type I.

Type B.

Type IV. Place first row below the 4th row to get Type VI.

Type V. Interchange 3rd and 4th rows to get Type VI.

Type C.

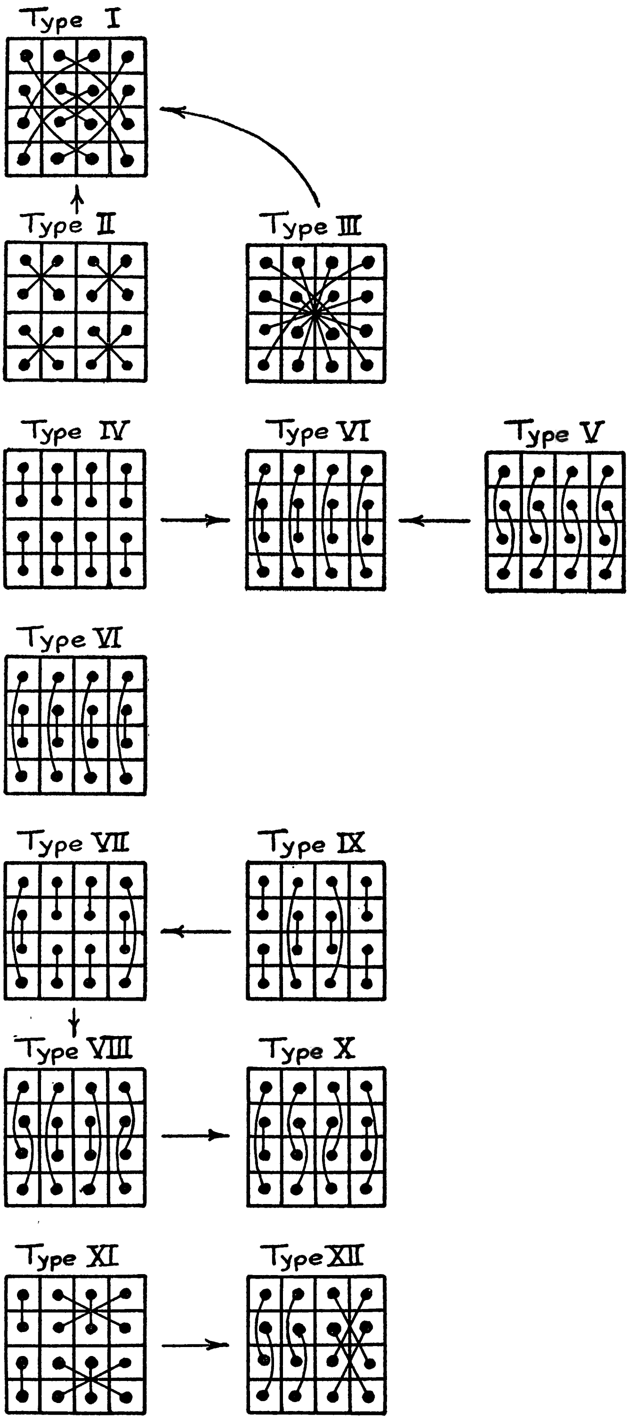
Type VII. Interchange 3rd and 4th rows, and then place first row below the 4th row to get Type VIII.

Type VIII. Interchange first and 2nd rows to get Type X.

Type IX. Place first row below 4th row to get Type VII.

Type D.

Type XI. Interchange 2nd and 3rd columns and then 2nd and 3rd rows to get Type XII.



Type I

1	14	7	12
15	4	9	6
10	5	16	3
8	11	2	13

Type II

1	7	14	12
10	16	5	3
15	9	4	6
8	2	11	13

A

Type VI

1	14	12	7
4	15	9	6
13	2	8	11
16	3	5	10

Type VI

1	12	14	7
4	15	9	6
13	2	8	11
16	5	3	10

B

Type VII

9	16	2	7
12	1	15	6
5	4	14	11
8	13	3	10

Type VIII

16	4	5	9
14	2	7	11
1	15	10	8
3	13	12	6

C

Type XI

15	6	9	4
2	13	8	11
7	12	1	14
10	3	16	5

D

3. Evaluations. Since the absolute value, Δ, of a determinant is unaltered by transposition of columns or of rows, by reflection or by rotation, it will be necessary to evaluate only one of each determinant type, provided that no property of the diagonals is used in evaluating the representative chosen. The general square may be represented conveniently by

$$\begin{matrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ M & N & P & Q \end{matrix}$$

and the conjugate sum by X so that the constant is $2X$. Evidently the sum of each of the columns, rows and principal diagonals in all of the squares is $2X$.

Type A. Schell showed in E 674 that $\Delta=0$ for Type I. All members of Type A therefore have $\Delta=0$. This may be shown also by evaluating Type II, which may be written

$$\begin{vmatrix} A & B & C & D \\ X-B & X-A & X-D & X-C \\ I & J & K & L \\ X-J & X-I & X-L & X-K \end{vmatrix}.$$

By subtracting the first column from the 2nd and the 3rd column from the 4th, *then* subtracting the 2nd row from the first and the 4th row from the 3rd, *then* adding the first and 3rd columns (noting that $A+B+C+D=2X=I+J+K+L$),

$$\begin{vmatrix} A+B-X & 0 & 0 & 0 \\ X-B & B-A & 2X-B-D & D-C \\ I+J-X & 0 & 0 & 0 \\ X-J & J-I & 2X-J-L & L-K \end{vmatrix} = 0$$

is secured.

Type B. It is evident upon inspection that if the first row of Type IV be added to the second row, and the third row be added to the fourth, a determinant is secured with two rows of identical elements, namely X. Therefore, for Type IV and for all of Type B, we have $\Delta=0$.

Types C and D. Type VII may be written

$$\begin{vmatrix} A & B & C & D \\ E & X-B & X-C & H \\ X-E & J & K & X-H \\ X-A & X-J & X-K & X-D \end{vmatrix}.$$

From the second and third rows, $2X-B-C+E+H=2X$ and $2X-E-H+J+K=2X$, so $B+C=E+H=J+K$. The determinant is evaluated by adding the first three columns to the fourth, then the first three rows to the fourth, and factoring out $4X$. Then adding the first row to and subtracting the fourth row from the second row, we obtain

$$\Delta = -4X(A+E-X)\begin{vmatrix} B & C & 1 \\ J & K & 1 \\ X & X & 2 \end{vmatrix}.$$

The evaluation is completed by adding the first column to the second, and then subtracting the first row from the second. Thus

$$\begin{aligned}\Delta &= -8X[A-(X-E)][B-J][C-(X-B)] \\ &= -8X(A-I)(B-J)(C-F).\end{aligned}$$

Therefore, if all elements are distinct, we have $\Delta \neq 0$ for Type VII and for the other members of Type C.

The other members of Type C may be evaluated by expansion, or by transposing Type VII and setting the square thus secured into a one-to-one correspondence with the standard square. Types XI and XII are separately evaluated. Thus for the members of Types C and D:

Type VII	$\Delta = -8X(A-I)(B-J)(C-F)$
Type VIII	$\Delta = -8X(A-E)(B-F)(C-N)$
Type IX	$\Delta = -8X(A-I)(B-J)(C-N)$
Type X	$\Delta = 8X(A-E)(B-F)(C-J)$
Type XI	$\Delta = 8X(A-I)(B-D)(C-A)$
Type XII	$\Delta = 8X(A-E)(B-A)(C-D).$

4. Conclusion. It is therefore apparent that in order for a fourth order magic square with elements in arithmetic progression to have a zero determinant, it is necessary and sufficient that it meet the conditions for a semi-Nasik square or that it be of Type VI.

Since the sum S of all the elements of the square is $8X$, a secondary result is that if all the elements are integers then Δ/S is an integer. This property is proven in another manner in the solution to problem E 813, where it is also shown that Δ/S is an integer for squares of order n when n is odd and also when n is even if the difference d is even.

THE NUMBER OF DISTANCES IN A CUBICAL NETWORK

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We consider the set of points $\{(a, b, c)\}$ in a three dimensional Euclidean space, where each coordinate a, b and c assumes the values $0, 1, \cdots, n$ independently of the values of the other coordinates. Let A_k be the number of straight lines of length k which connect one point of the set with another, and let A_0 equal the total number of points in the set. Then the following identity holds:

$$(1) \qquad A_0 + 2\sum_{d\geqq 1} A_d x^{d^2} = \left[(n+1) + 2\sum_{\nu=1}^{n}(n-\nu+1)x^{\nu^2}\right]^3.$$

Proof: Consider the six fold sum

$$(2) \qquad \sum_{\alpha=0}^{n}\sum_{\beta=0}^{n}\sum_{\gamma=0}^{n}\sum_{\lambda=0}^{n}\sum_{\mu=1}^{n}\sum_{\nu=0}^{n} x^{(\alpha-\lambda)^2+(\beta-\mu)^2+(\gamma-\nu)^2}.$$

Each term of this sum corresponds to either (1) a point of the set $\{(a, b, c)\}$ or (2) a pair of points of this set.

Case (1). For every point (a', b', c') there will be exactly one term in the sum (2) for which

$$\alpha = \lambda = a', \qquad \beta = \mu = b', \qquad \gamma = \nu = c'$$

and the value of this term is 1.

Case (2). For every pair of distinct points (a'', b'', c'') and (a''', b''', c''') there will be two terms with the exponent $(a''-a''')^2+(b''-b''')^2+(c''-c''')^2$. This exponent is the square of the distance between the points. Since every term of the sum (2) arises under case (1) or case (2), the coefficient of x^{k^2} when the terms are combined, will be equal to $2A_k$. ($A_k=0$ if there are no terms with exponent k^2.) The constant term is equal to A_0. Hence (2) is equal to the left hand side of (1).

The following transformations show that (2) is also equal to the right hand side of identity (1). We write (2) as follows:

$$(3) \qquad \left[\sum_{\alpha=0}^{n}\sum_{\beta=0}^{n} x^{(\alpha-\beta)^2}\right]^3$$

Let $\alpha-\lambda=\nu$, then ν takes on the values $-n, -n+1, \cdots, -1, 0, 1, \cdots, n$. For each value of ν, there are $n-|\nu|+1$ pairs α, λ for which $\alpha-\lambda=\nu$. Hence (3) can be written

$$(4) \qquad \left[\sum_{\nu=-n}^{n}(n-|\nu|+1)x^{\nu^2}\right]^3,$$

and finally as

$$(5) \qquad \left[(n+1) + 2\sum_{1}^{n}(n-\nu+1)x^{\nu^2}\right]^3,$$

which is the form of the right hand side of (1).

Identities of a similar nature can be obtained for a set of points in n dimensional space, and also for points whose array is rectangular rather than cubical.

NOTE ON THE GAMMA FUNCTION

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In this note we use the inequality of Hölder to establish the classical asymptotic relation

$$(1) \qquad \lim_{x\to\infty} \frac{\Gamma(x+a)}{x^a\Gamma(x)} = 1$$

for real a and x.

Proof: Let $0<a<1$, $p=1/a$, $q=p/(p-1)=1/(1-a)$, $f(t)=e^{-at}t^{ax}$, $g(t)=e^{-(1-a)t}t^{(1-a)x+a-1}$,

and apply the inequality

$$\int_0^\infty f(t)g(t)dt \leqq \left\{\int_0^\infty f(t)^p dt\right\}^{1/p}\left\{\int_0^\infty g(t)^q dt\right\}^{1/q}$$

to obtain

$$(2) \qquad \Gamma(x+a) = \int_0^\infty e^{-t}t^{x+a-1}dt \leqq \left\{\int_0^\infty e^{-t}t^x dt\right\}^a \cdot \left\{\int_0^\infty e^{-t}t^{x-1}dt\right\}^{1-a}$$
$$= [\Gamma(x+1)]^a[\Gamma(x)]^{1-a}.$$

Since

$$(3) \qquad \Gamma(x+1) = x\Gamma(x),$$

(2) becomes

$$(4) \qquad \Gamma(x+a) \leqq x^a\Gamma(x).$$

Replacing a by $1-a$ in (4) we get

$$(5) \qquad \Gamma(x+1-a) \leqq x^{1-a}\Gamma(x),$$

from which we obtain

$$(6) \qquad \Gamma(x+1) \leqq (x+a)^{1-a}\Gamma(x+a),$$

by substituting $x+a$ for x.

Combining (4) and (6) we get

$$\frac{x}{(x+a)^{1-a}}\Gamma(x) \leqq \Gamma(x+a) \leqq x^a\Gamma(x).$$

Therefore

$$\left(\frac{x}{x+a}\right)^{1-a} \leqq \frac{\Gamma(x+a)}{x^a\Gamma(x)} \leqq 1. \tag{7}$$

Letting x tend to infinity in (7) yields (1) for $0<a<1$. The extension to all real a is immediate on repeated application of (3).

THE NUMBER OF r-TUPLES OF PAIRS OF INTEGERS

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Our problem may be stated as follows:

Let $n>1$ be a positive integer and let $(s_1, t_1), \cdots, (s_r, t_r)$ be an r-tuple of pairs of integers satisfying

$$n \geqq s_i > t_i \geqq 1,\ s_{i+1} > s_i, t_{i+1} < t_i, \qquad (i = 1, \cdots, r). \tag{1}$$

If $S(k, m)$ is the number of such r-tuples beginning with (k, m), and K is the total number of r-tuples, show that

$$S(k, m) = \binom{n-k+m-1}{n-k},$$

and hence that $K=2^{n-1}-1$.

(For example, if $n=5$, the 15 possible r-tuples are (5, 4); (5, 3); (5, 2); (5,1); (4, 3); (4, 3), (5, 2); (4, 3), (5, 1); (4, 2); (4, 2), (5, 1); (4, 1); (3, 2); (3, 2), (4, 1); (3, 2), (5, 1); (3, 1); and (2, 1).)

We first show that

$$S(k, m) = \sum_{i=k}^{n} S(i, m-1), \quad S(k, 1) = 1, \qquad (m > 1). \tag{2}$$

Indeed (2) follows from the fact that the r-tuples $(k_1, m_1), (k_2, m_2), \cdots$, with $m_2>m_1-1$, may be put into $1-1$ correspondence with the r-tuples with first term (k_1, m_1-1) via

$$(k_1, m_1), (k_2, m_2), (k_3, m_3), \cdots \leftrightarrow (k_1, m_1-1), (k_2, m_2), (k_3, m_3), \cdots;$$

while the r-tuples $(k_1, m_1), (k_2, m_1-1), (k_3, m_3), \cdots$, may be put into $1-1$ correspondence with the r-tuples beginning with (k_2, m_1-1) via

$$(k_1, m_1), (k_2, m_1-1), (k_3, m_3), \cdots \leftrightarrow (k_2, m_1-1), (k_3, m_3), \cdots.$$

To show then that

$$S(k, m) = \binom{n - k + m - 1}{n - k} = \binom{n - k + m - 1}{m - 1}, \tag{3}$$

we first observe that (3) holds for $m = 1$, since

$$S(k, 1) = \binom{n - k}{n - k} = 1$$

for all admissible k (i.e., for $n \geqq k > 1$). Likewise, for $m = 2$, the r-tuples beginning with $(k, 2)$ are enumerated by remarking that there is one r-tuple consisting of $(k, 2)$ alone while the choices for a second term are $(k+1, 1), \cdots, (n, 1)$. Thus

$$S(k, 2) = n - k + 1 = \binom{n - k + 1}{1}.$$

Assume, then, that (3) holds for $S(k, i)$, for $i < m$ and for any admissible k. Then (2) implies

$$S(k, m) = \sum_{i=k}^{n} \binom{n - i + m - 2}{m - 2} \tag{4}$$

under our induction assumption that $m > 2$, so that the proof of (3) reduces to a proof of

$$\sum_{i=k}^{n} \binom{n - i + m - 2}{m - 2} = \binom{n - k + m - 1}{m - 1}, \qquad (m > 2). \tag{5}$$

With $s = m - 2$, $p = n - k + s$, $j = n - i + m - 2$, (5) becomes

$$\sum_{j=s}^{p} \binom{j}{s} = \binom{p + 1}{s + 1}. \tag{6}$$

Now (6) holds for $p = s$ or $s + 1$ for any s, since

$$\binom{s}{s} = \binom{s + 1}{s + 1} = 1,$$

and

$$\binom{s}{s} + \binom{s + 1}{s} = \binom{s + 2}{s + 1};$$

and (6) also holds for $p = 1$ and any admissible s (i.e., here, $s = 1$), since

$$\binom{1}{1} = \binom{2}{2} = 1.$$

We suppose (6) holds for the values 1, 2, $\cdots$, $p-1$ for any admissible s (i.e., $s<p$) and re-write (6), for $p\neq s$ or $s+1$, as

$$\frac{s+1}{p-s}\binom{p}{s+1}+\sum_{j=s}^{p-1}\binom{j}{s}=\binom{p+1}{s+1}.$$

But

$$\binom{p-1}{s}+\cdots+\binom{s}{s}=\binom{p}{s+1}$$

by our induction hypothesis for $s<p-1$; and since

$$\frac{s+1}{p-s}\binom{p}{s+1}+\binom{p}{s+1}=\frac{p+1}{p-s}\binom{p}{s+1}=\binom{p+1}{s+1},$$

it follows that (6), and hence (5) and (3), are proved.

Then

$$\sum_{k=m+1}^{n} S(k,k-m)=\sum_{k=m+1}^{n}\binom{n-m-1}{k-m-1} \tag{7}$$

by (3). We let $n-m-1=t$, $k-m-1=i$ so that (7) becomes

$$\sum_{i=0}^{t}\binom{t}{i}=\binom{t}{0}+\sum_{i=1}^{t}\binom{t}{i}=1+(2^t-1)=2^{n-m+1}. \tag{8}$$

Finally, since it is clear that

$$K=\sum_{m=1}^{n-1}\left(\sum_{k=m+1}^{n} S(k,k-m)\right),$$

we have, by (7) and (8),

$$K=\sum_{m=1}^{n-1}2^{n-m-1},$$

which is a geometric progression of $n-1$ terms with first term 2^{n-1} and common ratio $\frac{1}{2}$, so that

$$K=2^{n-1}\left[\frac{1-(\frac{1}{2})^{n-1}}{1-(\frac{1}{2})}\right]=(2^{n-2}-2^{-1})2=2^{n-1}-1.$$

By varying the conditions (1) a number of similar problems may be stated, while extensions may be made by considering r-tuples of triples etc. The problem which has $t_{i+1}<t_i$ in (1) replaced by $t_{i+1}>t_i$ is considered in a study of the ideals of a certain type of nilpotent algebra being made by Sam Perlis and the author.

CLASSROOM NOTES

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POINTING OFF IN SLIDE RULE WORK

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For placing the decimal point in answers to problems solved on the slide rule there are available the method of the characteristic and the method of estimation. The first of these is precise although it has relatively few users. The second, although it borders on mysticism, seems entirely satisfactory in practice. In stating here a rigorous method of estimation, our purpose is to illuminate rather than to modify present procedure.

Suppose that a is the exact value of a fraction whose numerator and denominator are products of decimal numbers or roots of such numbers. To simplify language, suppose further that the slide rule gives the exact answer and that all numbers have exactly three significant digits. These assumptions in no way limit the generality of the results. Without using the characteristic we find from the slide rule merely the set of values

$$\cdots, 10^{-2}a, 10^{-1}a, a, 10a, 10^2a, \cdots \tag{1}$$

which contains the answer but we do not know which number of the set to identify with a.

We interpret the method of estimation to involve rounding the given factors to one digit. The extreme case of rounding which *decreases* a replaces a factor 199 in the numerator by 100 and consequently multiplies a by more than $1/2$. Similarly, the extreme case of rounding which *increases* a replaces a factor 199 in the denominator by 100 and consequently multiplies a by less than 2. If there are m factors in the fraction to be computed, rounding by decreasing and rounding by increasing give a lower bound a_1 and an upper bound a_2 for a and these bounds satisfy

$$(1/2)^m a < a_1 \leqq a_2 < 2^m a. \tag{2}$$

Once a_1, a_2 have been computed mentally, it is easy to pick out which of (1) lie on the interval $[a_1, a_2]$. If only one number of (1) lies on $[a_1, a_2]$, the identification is complete.

From (1), (2) it is seen that the method is successful for all values of the factors if and only if

$$\frac{a}{10} \leqq (1/2)^m a < 2^m a \leqq 10a.$$

Solving for m by logarithms or by trial gives $m \leqq 3$. The foregoing *method of bounds* therefore always works if no more than three factors are involved. More-

over, in any particular case of more than three factors it is apparent whether or not the method is reliable.

The usual method of estimation, however, employs only a single rounding, which we suppose is to the nearest single digit. The result b is not known to be either a lower bound or an upper bound in general. The extreme case of rounding replaces 149 by 100. Hence if n roundings decrease and p roundings increase the fraction, the factor by which a is multiplied is bounded below by $(2/3)^n$ and above by $(3/2)^p$:

$$(2/3)^n < \frac{b}{a} < (3/2)^p. \tag{3}$$

For the product $1.42 \times 1.43 \times 1.44 = 2.92$ the set (1) is

$$\cdots, .0292, .292, 2.92, 29.2, 292., \cdots \tag{4}$$

and the rounded value is 1. We might be tempted to select the value in (4) nearest 1. Since this gives .292, which is not the correct value, we see that some other principle of selection must be used for comparing (1) with the rounded values in a rigorous statement of the rule of estimation, although even a novice would intuitively point off correctly in the present case.

Let us agree to select the number in (1) so that the relative error of the estimated value b is numerically less with respect to that number than with respect to any other number in (1). To determine when this principle gives the correct answer, we find a necessary and sufficient condition that

$$\left|\frac{b}{a} - 1\right| < \left|\frac{10^{-k}b}{a} - 1\right| \tag{5}$$

for all integers k except 0.

Put $c = b/a$. It is found $(k \neq 0)$ that $0 \leqq k(1-c)$ implies (5) and that $k(1-c) < 0 < (1-c)(1-10^{-k}c)$ contradicts (5). Hence only the cases where $k(1-c) < 0$ and $(1-c)(1-10^{-k}c) \leqq 0$ need be considered. It is thus found that in the presence of $k \neq 0$ (5) is equivalent to $0 < k[2-(1+10^{-k})c]$ and also to the two inequalities got from this by making $k=-1$, $k=1$. Hence we have as the final condition

$$\frac{2}{11} < \frac{b}{a} < \frac{20}{11}. \tag{6}$$

Segment (3) lies in segment (6) if and only if

$$\frac{2}{11} \leqq (2/3)^n, \qquad (3/2)^p \leqq \frac{20}{11}. \tag{7}$$

Making the calculations by means of logarithms or by trial we find that for nonnegative integers n, p (7) is equivalent to $n \leqq 4$, $p \leqq 1$. Hence we have

THEOREM 1. *If in rounding to the nearest single digit the fraction is decreased at most four times and increased at most once, the decimal point is correctly placed by requiring that the relative error of the rounded value be least.*

It will be noted that the principle identifies 2.92 as the answer in (4) and that the decision is not even close.

The result given is the best possible unless the method of estimation is changed. It is questionable whether this can be done without sacrificing generality or introducing complications which would offset the advantages of using the slide rule.

The limits given in the theorem can in general be exceeded with impunity because (i) the multipliers have values much nearer 1 than the extreme values which it is necessary to use in the above calculations and (ii) the increases and decreases tend to cancel. A lower bound $P(n, p)$ for the probability that the decimal point is correctly placed when $4<n$ or $1<p$ could be calculated. The results in practice, however, might well be better than this bound would indicate because the operator intuitively modifies the process so as to keep b near a. In fact, the writer has yet to have knowledge of a point misplaced by error of judgment on the part of an experienced operator.

A result which seems to cover the usual situations in the solution of triangles is given in Theorem 2, which is easily proved by the law of sines. The angle 6° can be replaced by the principal value of $\sin^{-1}.1$, but there is usually no advantage in doing so.

THEOREM 2. *If the ratio of two sides of a triangle is at least* 10, *the angle opposite the smaller side is less than* 6°.

SOME APPLICATIONS OF AITKEN'S METHOD OF INTERPOLATION

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Aitken's method of interpolation published in 1930 [1] seems relatively unknown, although it has been discussed by various authors [2], [3], [4], [5], [6], [8] in some detail. With the rapid increase of interest in computational techniques it should be one of the tools familiar to every student of mathematics. We shall point out some applications of the method to the mathematics of finance. We have used it for some years in those classes with considerable success. Since the references may not be readily available, we quote the elegant proof of Feller [6].

"Let it be required to compute $f(\xi)$ of a polynomial of the nth degree $f(x)$, given $f_k=f(x_k)$ for $k=0, 1, \cdots, n$. We note

$$f^1(x) = \begin{vmatrix} f_0 & x_0 - \xi \\ f(x) & x - \xi \end{vmatrix} \div (x - x_0) \tag{1}$$

is a polynomial of degree $n-1$, and $f^1(\xi)=f(\xi)$. Hence we are required to find

$f(\xi)$ knowing

$$f_k^1 = \begin{vmatrix} f_0 & x_0 - \xi \\ f_k & x - \xi \end{vmatrix} \div (x_k - x_0), \tag{2}$$

for $k=1, 2, \cdots, n$. Thus the problem is reduced from n to $n-1$. In like fashion the problem is further reduced to $n-2$. We note that $x_k - x_0 = (x_k - \xi) - (x_0 - \xi)$." We change our notation to that conventional in finite differences and illustrate the process for four points (a, u_a), (b, u_b), (c, u_c), (d, u_d), to determine u_x given $a<x<d$. A table is constructed

$$\begin{array}{lllll}
u_a & & & & a - x \\
u_b & u(a, b) & & & b - x \\
u_c & u(a, c) & u(a, b, c) & & c - x \\
u_d & u(a, d) & u(a, b, d) & u(a, b, c, d) & d - x
\end{array}$$

where $a-x$, $b-x$, $c-x$, $d-x$ are called "parts" and

$$u(a, b) = \begin{vmatrix} u_a & a - x \\ u_b & b - x \end{vmatrix} \div (b - a), \qquad u(a, c) = \begin{vmatrix} u_a & a - x \\ u_c & c - x \end{vmatrix} \div (c - a),$$

$$u(a, d) = \begin{vmatrix} u_a & a - x \\ u_d & d - x \end{vmatrix} \div (d - a), \quad u(a, b, c) = \begin{vmatrix} u(a, b) & b - x \\ u(a, c) & c - x \end{vmatrix} \div (c - b),$$

$$u(a, b, d) = \begin{vmatrix} u(a, b) & b - x \\ u(a, d) & d - x \end{vmatrix} \div (d - b),$$

$$u(a, b, c, d) = \begin{vmatrix} u(a, b, c) & c - x \\ u(a, b, d) & d - x \end{vmatrix} \div (d - c) = u_x.$$

The iteration is discontinued as soon as the desired number of significant figures is determined.

We stress the peculiar advantages of the method. The intervals need not be equal; x may be anywhere among the arguments, that is, the interpolation may be central or non-central; it is well adapted to machine calculations; no differencing is required; and finally it may be used for inverse interpolation as well as direct interpolation merely by interchanging the argument and the function. Of course the method is not intended for routine calculations in the construction of tables or for the purposes of subtabulation.

We give only one example in the mathematics of finance, a problem involving the yield of a bond, although the method is applicable wherever interpolation is needed. Thus the power of the usual short tables of interest functions given in the typical text is greatly increased making unnecessary seven place logarithms so frequently advocated. The method is quite effective in the solution

of any equation $g(x)=0$. A $5\frac{1}{2}\%$ bond, coupons payable semi-annually, redeemable in 25 years at 110 is purchased for 105; what is the yield? [7, pp. 190, 226]. Denote the yield for the half year by i, and the equation we must solve is

$$f(i) = 5 + (2.75 - 110i)a_{\overline{50}|i} = 0.$$

The table will be

1.75					−32.34150
2.00	2.5538343				−22.28299
2.50	2.6371542	6 612588			− 5.00000
3.00	2.7243090	6 746793	6 6001		9.15137
3.50	2.8150126	6 889149	6 6618	552	20.80118

Hence $i=2.66552\%$, the extra figures are carried since errors accumulate rapidly in successive multiplications. It should be noted that in iterating the process one multiplies only where the figures disagree. Since Todhunter insisted that the answer was 2.6656%, $f(4\%)=30.44560$ was added to the table resulting in $i=2.665651\%$. Using $f(2.665651\%)=-.001176$ and $f(2.6656\%)=.000314$, we obtain by linear interpolation $i=2.665611\%$ correct to seven significant figures. If an interval of $\frac{1}{4}\%$ is taken using 2%, $2\frac{1}{4}\%$, $2\frac{1}{2}\%$, $2\frac{3}{4}\%$, and 3%, then $i=2.665607\%$ by Aitken's method. Most texts give values only at intervals of $\frac{1}{2}\%$ after $2\frac{1}{2}\%$. For an interval of $\frac{1}{2}\%$ our experience shows that at most an error of 2 or 3 will occur in the fifth significant figure in a bond yield problem. Even the utility of a bond table is extended by Aitken's method.

Other uses are finding $(1+i)^n$, v^n, $a_{\overline{\ }|}$, $s_{\overline{n}|}$, if the value of i is not given in the tables. Generally, it is advisable to start with five points. One might think it worthwhile to make a change in origin and scale, but in our experience the extra time spent was not justified. Inverse interpolation will be rather difficult for most students and a little patience is required. It is best to start a class on direct interpolation with equal intervals. In statistics, Aitken's method makes it easy to extend the linear interpolation formula employed in determining the median; essentially one solves the equation $f(x)=.5$, assuming that the probability function is continuous.

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INTRODUCING $e=2.718+$

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Textbooks of calculus usually introduce the number e by considering the limit of $(1+1/n)^n$. Such treatment frequently omits many of the essential steps and leaves the student not much enlightened. An alternative approach is suggested in this note which leads to 2.718 by a more natural and elementary consideration.

The most familiar exponential is $y=2^x$. The slope of this curve at the point (0, 1) can be found by interpolation using a four place table as 0.70, while the slope of the exponential $y=3^x$ comes out 1.10. These two results suggest the query: is there some number between 2 and 3, for which the corresponding slope is 1? Let us assume there is and call this number e, and thus assume that when Δx is the smallest value for which we can interpolate we shall have $y=e^x$ yielding $\Delta y/\Delta x=1$.

For Δy we have $e^{\Delta x}-1$ at the point (0, 1), and so we must solve the equation $(e^{\Delta x}-1)/\Delta x=1$, or $e^{\Delta x}=1+\Delta x$, whence $\Delta x(\log e)=\log(1+\Delta x)$.

From a seven place table we have

$$\log(1-.001) = -.000\ 4345$$

$$\log(1+.001) = +.000\ 4341$$

and these items suggest that when Δx is nearer to zero than either plus or minus .001, we should use the intermediate decimal, .0004343, and the interpolation formula

$$\log(1+\Delta x) = .000\ 4343\ (\Delta x/.001).$$

Putting this into the equation for $\Delta x(\log e)$ above we obtain $\log e=.4343$, whence $e=2.718+$.

If a four place table is used, a less accurate value, 2.7+, is obtained. But the assumption that there is a number e such that for $y=e^x$, $dy/dx=e^x$ exactly, makes it possible to obtain the familiar expansion for e^x, from which the value of e can be determined as accurately as desired.

TANGENT LINES AND PLANES

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Although the problem of finding a tangent line to a conic or a tangent plane to a quadric presents no great difficulty, the problem can be solved by a method even simpler than that ordinarily seen.

Consider a given circle and a fixed point on the circle. Write the equation of a circle with its center at the point and of radius less than that of the given circle. If the two equations are normalized, the difference between the equations of the first and second circles is then the radical axis, a straight line passing through the points of intersection of the circles. Now, if the radius of the second circle were decreased, the intersection points would approach the tangent line at the fixed point. This suggests the following method: solve simultaneously the equation of the given circle and the equation of the null circle at the fixed point. The method is so simple that in many cases the answer may be obtained by inspection.

In the case of an ellipse, much the same method may be used. At a point on the given ellipse, consider a null ellipse of the same orientation whose eccentricity is the same as that of the given ellipse. Then the difference between the two equations will represent the tangent line.

In the case of a hyperbola, we can make use of the degenerate hyperbola of the same eccentricity and orientation as the given hyperbola. Geometrically, this means the equation of two straight lines, parallel to the asymptotes, intersecting at the fixed point on the given hyperbola. Again the difference is the required tangent.

For a parabola, we use the equation of a degenerate parabola with vertex at the fixed point and axis parallel to that of the given parabola. The method still works.

Geometrically, our method consists of employing a degenerate form of the conic section of the same eccentricity and orientation as that of the curve on which the fixed point lies. Algebraically, we subtract the equation satisfied by the coördinates of the point from the given equation, thus obtaining a linear equation.

The general equation of the conic, and this includes the degenerate, null, and imaginary cases, is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0. \tag{1}$$

Consider the point (u, v) satisfying the above equation. Then our null or degenerate form of the same eccentricity will be

$$a(x - u)^2 + b(x - u)(y - v) + c(y - v)^2 = 0. \tag{2}$$

Expanding and subtracting, we obtain

$$(2au + bv + d)x + (bu + 2cv + e)y - (au^2 + buv + cv^2 - f) = 0.$$

It is then easily verified that this is the equation of the line tangent to the given curve at the point (u, v). If equation (1) represents a null circle or null ellipse, it will be identical with equation (2). This is the only case in which a tangent line cannot be obtained. It is interesting to note that the method works equally well for points on imaginary conics.

An especially simple example of this method is the case of the tangent line at the origin to a conic passing through the origin. Consider, then, the conic

$$ax^2 + bxy + cy^2 + dx + ey = 0.$$

The desired tangent is merely $dx+ey=0$.

It is apparent that this method may be extended to the case of the general second degree surface in n dimensions to obtain the equation of the tangent plane.

RATIONAL SOLUTIONS OF A CERTAIN TRIGONOMETRIC EQUATION

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To students of trigonometric equations it may be of interest to find the relationship between the constants of the trigonometric equation

$$a \cos \omega + b \sin \omega = c \tag{1}$$

such that the solutions for $\cos \omega$ and $\sin \omega$ will be rational numbers.

Certain values of the constants can be found by inspection. Thus, when one function $\sin \omega$ or $\cos \omega$ is zero the other must be $+1$ or -1. Then $c = \pm a$, or $c = \pm b$, which are the relations existing for quadrantal angles, are the relations which will assure rational values for the two functions. However, more general conditions on the constants of equation (1) can be found by turning to a general solution of the given equation.

In the April 1947 issue of this MONTHLY, R. W. Wagner uses effectively the substitution $t = \tan(\omega/2)$ which leads to the following values for $\sin \omega$, $\cos \omega$ and $\tan(\omega/2)$:

$$\begin{gathered} \sin \omega = 2t/(1 + t^2), \qquad \cos \omega = (1 - t^2)/(1 + t^2), \\ \tan \frac{\omega}{2} = \sin \omega/(1 + \cos \omega). \end{gathered} \tag{2}$$

This shows that the functions $\sin \omega$ and $\cos \omega$ are rational if and only if $\tan(\omega/2) = t$ is rational.

By means of (2) equation (1) becomes the quadratic in t:

$$(a + c)t^2 - 2bt + c - a = 0. \tag{3}$$

Let $4k^2$ represent the discriminant of this equation. The values of t, which are the roots, will be rational if and only if $4k^2$ is zero or a perfect square. This leads

to the condition

$$a^2 + b^2 - c^2 = k^2, \qquad (k = 0, \text{ or an integer}). \tag{4}$$

At the same time it is noted that the values of t are

$$t = (b \pm k)/(a + c). \tag{5}$$

From the conditions in (4) it is evident that $a^2+b^2 \geqq c^2$, that the constants a, b and c occur as squares, also that a and b can be interchanged. Let the equation (1) which contains the constants a, b and c be denoted by the symbol (a, b, c). Then (4) which is the necessary and sufficient condition for rational values of $\sin\omega$ and $\cos\omega$ in the equation (a, b, c) will also assure rational values of the functions in the seven related equations which may be designated by the symbols (b, a, c), $(-a, b, c)$, $(-b, a, c)$, $(-a, -b, c)$, $(-b, -a, c)$, $(a, -b, c)$ and $(b, -a, c)$.

The equation (4) can be rewritten so as to introduce the positive integer N;

$$N = a^2 + b^2 = c^2 + k^2. \tag{6}$$

Since the problem now may be reduced to finding the integers N which may be written as the sum of two squares in more than one way, a notation for N equivalent to equation (6) may prove convenient:

$$N = (a, b; c, k). \tag{7}$$

Such numbers N, as implied in Dickson's *The History of the Theory of Numbers*, Vol. II, p. 228, can be found by the following method: all the products and doubles of these products of two or more of the prime numbers of the form $4n+1$ (n an integer) including multiple powers of these numbers are numbers that may be expressed as the sum of two squares in more than one way. Thus 5, 13, 17, 29 $\cdots$ are such prime numbers. From these a few of the products which may be the numbers N are 25, 50, 85, 100, 125, 130, $\cdots$. By the notation in (6) a few numbers for N are $N=25=(3, 4; 5, 0)$, $N=50=(1, 7; 5, 5)$, $N=65=(1, 8; 4, 7)$, $\cdots$.

In $(a, b; c, k)$ there are four basic sets of constants that may be used in (1) to give rational solutions. They are (a, b, c), (a, b, k), (c, k, a) and (c, k, b). Each set has the seven related equations described above, making 32 possible combinations of constants for each N, if a, b, c and k are distinct. As an example, from $N=65=(1, 8; 4, 7)$, the four basic sets are $(1, 8, 4)$, $(1, 8, 7)$, $(4, 7, 1)$ and $(4, 7, 8)$. Of these $(1, 8, 4)$ will represent the equation

$$\cos\omega + 8\sin\omega = 4. \tag{8}$$

Substitution in (5) leads to $t=3$ and $\sin\omega=3/5$, $\cos\omega=-4/5$ or $t=1/5$ and $\sin\omega=5/13$, $\cos\omega=12/13$.

SOLUTIONS

A Ruler Construction

E 793 [1947, 545]. *Proposed by Joseph Rosenbaum, The Milford School, Connecticut*

With straight edge alone construct a hexagon which can possess both an inscribed and a circumscribed conic.

Solution by R. T. Hood, University of Wisconsin. Let us show how, given five lines, a sixth may be determined so that the resulting hexagon shall possess both an inscribed and a circumscribed conic.

Draw any four lines a, b, c, d, forming a quadrilateral with vertices $ab=1$, $bc=2$, $cd=3$, $da=4$. Let the sides a and c intersect in a point P and let b and d intersect in a point Q. Let any fifth line be drawn intersecting c in a point 5 and d in a point 6. Call this line e. The problem is now to determine a sixth line to go with a, b, c, d, e.

Draw the diagonals 13 and 24 of the quadrilateral; call their point of intersection 7. Draw the lines 57 (intersecting a in a point 8) and 67 (intersecting c in a point 9). Draw the line 89 and call it f. Let it intersect the line e in a point R. Then the figure 489256 (or $afbced$) is the desired hexagon.

For, by construction, the three diagonals (24, 58, 69) of the hexagon are concurrent at 7. Hence, by the converse of Brianchon's theorem, there is a conic having the sides of the hexagon as tangents. Now lines 13, 85, and 96 are also concurrent at 7, and these are lines joining corresponding vertices of the two triangles 189 and 356. Hence, by Desargues' theorem, the corresponding sides of these two triangles intersect in three collinear points. That is, a and c intersect in P, b and d intersect in Q, e and f intersect in R, and P, Q, and R are collinear. But a and c, b and d, e and f are simply pairs of opposite sides of the hexagon. Hence, by the converse of Pascal's theorem, there is a conic having as six of its points the six vertices of the hexagon.

Thus this hexagon can have both an inscribed and a circumscribed conic.

Also solved by Frederick Bagemihl and the proposer.

Associated Conjugates

E 800 [1948, 30]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The polar planes, with respect to a tetrahedron, of the isotomic conjugates of a set of collinear points are coaxal.

Note. If the cevians AP, BP, CP, DP, for a tetrahedron $ABCD$, of a point P, cut the planes of the faces BCD, CDA, DAB, ABC, in A', B', C', D', then the isotomic conjugate P' of P is the point of concurrency of the lines joining A, B, C, D to the isotomic conjugates A'', B'', C'', D'' of A', B', C', D' in the faces of the tetrahedron.

Solution by L. M. Kelly, University of Missouri. Let a_1, a_2, a_3, a_4 be the tetrahedral coördinates of one of the collinear points. Then $1/a_1$, $1/a_2$, $1/a_3$, $1/a_4$ are the coördinates of its isotomic conjugate. The polar plane of this point is

$a_1x_1+a_2x_2+a_3x_3+a_4x_4=0$. Thus if (a) and (b) are two fixed points of the line, and (c) any third point of the line, the matrix of the coördinates of these points has rank less than three, and hence the matrix of the coefficients of the corresponding polar planes has rank less than three. But this is the condition that the planes be coaxal.

Also solved by the proposer.

Editorial note. This problem extends to the tetrahedron a problem proposed by this editor in the *National Mathematics Magazine* (vol. XX, no. 2, pp. 95–97).

Let (p_1, p_2, p_3, p_4) be tetrahedral coördinates of any point P and let P' be the point $(k_1/p_1, k_2/p_2, k_3/p_3, k_4/p_4)$, where k_1, k_2, k_3, k_4 are fixed non-zero constants. Then we may call P' the *associated conjugate* of P, for the given tetrahedron and the constants k_1, k_2, k_3, k_4. The polar plane, p', of P', with respect to the tetrahedron, may be called the *associated polar* of P, and P the *associated pole* of p'. If P and P' coincide, P will be called a *self-conjugate* point. The following theorems are easily established.

1. The associated polars of a set of collinear points are coaxal, and dually, the associated poles of a set of coaxal planes are collinear.

2. The associated polars of two points collinear with a self-conjugate point intersect on the polar plane of that self-conjugate point.

3. The associated polars of a set of coplanar points are copunctual, and dually.

4. The common point of a set of copunctual planes is the associated pole of the plane of complanarity of their associated poles, and dually.

5. Cross-ratio is invariant under an *associate correlation.*

6. The tetrahedron determined by four points P, Q, R, S is perspective with the tetrahedron determined by their associated polars p', q', r', s', the center and plane of perspectivity being corresponding associated pole and polar.

7. The locus of points lying on their own associated polars is the conicoid Σ: $x_1^2/k_1+x_2^2/k_2+x_3^2/k_3+x_4^2/k_4=0$.

8. A point and its associated polar are pole and polar with respect to the conicoid Σ.

9. A point and its isotomic conjugate are associated conjugates for the constants 1, 1, 1, 1. A point and its isogonal conjugate are associated conjugates for the constants $A_1^2, A_2^2, A_3^2, A_4^2$, where A_1, A_2, A_3, A_4 are the areas of the faces of the given tetrahedron.

10. The polars of the isotomic conjugates of any two points collinear with the centroid of the tetrahedron are parallel.

Bidding Sequences

E 801 [1948, 95]. *Proposed by J. P. Ballantine, University of Washington*

How many different bidding sequences are possible in one hand of Bridge?

Solution by Roger Lessard, Ecole Polytechnique, Montreal. Consider any specific set of n bids. The bidding will cease if and only if three passes occur in succession. Since a bid can be made in place of the first, second, or third pass, there are three times as many ways of continuing the bidding as of finishing it. Let us count the ways the bidding can be finished. The three passes can be preceded by any one of the following sequences: B, BD, BDR, $BPPD$, $BPPDR$, $BDPPR$, $BPPDPPR$. After each bid there are, then, seven ways of finishing the bidding, and twenty-one ways of continuing the bidding. Thus any specific set of n bids can be made in $7(21^{n-1})$ ways. But, because there are five suits and seven levels, there are 35 possible bids. Thus the number of sets of n bids is $C(35, n)$. Now

$$\sum_{n=1}^{35} C(35, n)(7)(21^{n-1}) = (7/21)\sum C(35, n)(21^n)$$
$$= [(1 + 21)^{35} - 1]/3 = (22^{35} - 1)/3.$$

But each player can make the first bid. Therefore the number of bidding sequences is $4(22^{35}-1)/3$. If we count the case in which no bid is made we have $[4(22^{35})-1]/3$.

Also solved by J. F. Darling, Karl Itkin, and the proposer. Itkin said that, in expanded form, the final result above is

128,745,650,347,030,638,120,231,926,111,609,371,363,122,697,577.

The Sum of Cubes

E 802 [1948, 95]. *Proposed by E. A. Nordhaus, Michigan State College*

(1) Find the smallest positive integer N having the property that the sum of its digits does not divide the sum of the cubes of its digits.

(2) Find the two consecutive positive integers each of which equals the sum of the cubes of its digits.

Solution by J. M. McLynn, Washington, D.C. (1) Since $a+b$ is a factor of a^3+b^3 no two digit number will satisfy the problem. However, $a+b+c$ is not a factor of $a^3+b^3+c^3$, and on second trial (since we need not consider numbers containing zeros) we find 112 to be the required number.

(2) If the required numbers are N and $N+1$, then N must end in 0, because 0 and 1 are the only cubes whose difference is 1. No two digit number will satisfy the problem because $a^3+b^3=10a+b$ (where b must be 0) has no integral solution. The equation $a^3+b^3+c^3=100a+10b+c$ (where a and b are digits and c is 0) has only one solution: $a=3$, $b=7$, $c=0$. Thus the required numbers are 370 and 371. The solution is unique since $a^3+b^3+c^3+d^3=1000a+100b+10c+d$ (where $d=0$ and a, b, c are digits) has no satisfactory solution. There can be no five digit solution, since $5(9^3)$ is only a four digit number.

Also solved by Colin Blyth, W. E. Buker, R. E. Crane, Monte Dernham, L. A. Dwight, Frank Herlihy, Frank Kocher, R. S. Lehman, Roger Lessard,

Leo Moser, C. S. Ogilvy, George Shapiro, C. W. Trigg, G. W. Walker, and the proposer.

Moser pointed out that there are just four numbers having the property of being equal to the sum of the cubes of their digits: 153, 370, 371, 407. This fact is given in Ball-Coxeter, *Mathematical Recreations and Essays*, p. 13, 1944 edition. There a further reference is given to *Sphinx*, 1937, pp. 72, 87.

Triangle with Squares of Sides in Arithmetic Progression

E 803 [1948, 95]. *Proposed by J. H. Butchart, Arizona State College*

If the squares of the sides of a triangle form an arithmetic progression, the line joining the centroid and the symmedian point is parallel to one side of the triangle.

I. *Solution by L. M. Kelly, University of Missouri.* If b is the middle side in the progression the sum of the squares of the sides is $3b^2$. The distance to the side b from the symmedian point is given by

$$2\Delta b/(a^2 + b^2 + c^2) = 2\Delta/3b = h_b/3,$$

where Δ is the area of the triangle and h_b is the length of the altitude on side b. But the distance of the centroid from side b is likewise $h_b/3$.

II. *Solution by the Proposer.* Let A, B, C, K, G be position vectors of the vertices, symmedian point, and centroid of the triangle. Then

$$K = \frac{a^2A + b^2B + c^2C}{a^2 + b^2 + c^2} = \frac{(b^2 - d)A + b^2B + (b^2 + d)C}{3b^2}$$
$$= \frac{A + B + C}{3} + \frac{d}{3b^2}(C - A) = G + \frac{d}{3b^2}(C - A).$$

Thus the join of K and G is parallel to the join of C and A.

Also solved by Yu Yuan Chin, K. W. Crain, L. A. Dwight, Frank Herlihy, B. R. Leeds, D. W. Matlack, N. C. Scholomiti, P. D. Thomas, and C. W. Trigg.

Editorial Note. For a similar problem where the tangents of the angles of the triangle are in arithmetic progression see E 259 [1937, 541]. For the case where the sines of the angles (or the sides) of the triangle are in arithmetic progression see E 411 [1940, 708].

The Euler Line of a Tetrahedron

E 805 [1948, 96]. *Proposed by N. A. Court, University of Oklahoma*

If two coplanar edges of a tetrahedron are each equal to the respectively opposite edge, the remaining two opposite edges are each coplanar with the Euler line of the tetrahedron.

Solution by P. D. Thomas, U. S. Coast and Geodetic Survey, Washington, D. C. (Numbers in parentheses refer to articles in the proposer's *Modern Pure Solid*

Geometry, Macmillan, 1935.) The bimedians of a tetrahedron meet in the centroid. Under the conditions given, one of the bimedians is also a bialtitude (201). The circumcenter is the point common to the mediators of the six edges. But two mediators must have in common the bimedian which is also a bialtitude, whence the circumcenter lies on this bialtitude. That is, the bimedian which is also a bialtitude is the Euler line of the tetrahedron.

Also solved by L. M. Kelly, C. W. Trigg, and the proposer.

The proposer established the more complete theorem: *In a tetrahedron* (T) *consider the bimedian relative to a pair of opposite edges of* (T), *the bialtitude relative to the same pair of edges, and the Euler line of* (T). *If any two of these three lines coincide, the third line coincides with them.* If two of the bimedians of a tetrahedron coincide with the respective bialtitudes, then, by this theorem, the Euler line of the tetrahedron would coincide with two distinct lines. The paradox is resolved by the fact that in this case the centroid and the circumcenter coincide, and the Euler line becomes indeterminate (297, 298).

The Two Pedestrians

E 806 [1948, 158]. *Proposed by Leo Moser, University of Manitoba*

Lewis Carroll once proposed the following problem.

"Two travellers spent from 2 o'clock till 9 in walking along a level road, up a hill, and home again; their pace on the level being x miles per hour, up hill y, and down hill $2y$. Find the distance walked."

In the original problem x and y were given integers. Deduce the solution to the original problem without *a priori* knowledge of what these integers are.

Solution by Monte Dernham, San Francisco. If it is r miles along the level road and h miles up hill, then

$$2r/x + h/y + h/2y = 7,$$

whence

$$4yr + 3xh = 14xy,$$

where r and h are the variables, and x and y unknown integers. Clearly, r and h, which need not be integers, are severally indeterminate; also, their sum is indeterminate except in the single case where $4y=3x$. Thus the unique answer for a pedestrian is given by $x=4$, $y=3$, from which $r+h=14$; and, presuming that Lewis Carroll proposed a problem having a determinate solution, we conclude that each of the travellers walked 28 miles.

Also solved by R. V. Andree, D. M. Brown, W. E. Buker, W. B. Campbell, Richard Courter, R. E. Crane, J. S. Cromelin, Jerome Drexler, R. L. Greene, Sam Kravitz, C. S. Ogilvy, S. T. Parker, E. D. Rainville, F. Underwood, G. W. Walker, and the proposer.

Several solvers pointed out that the assumption that the original problem has a determinate solution is equivalent to insisting that the average rate on the hill be equal to the rate on the level (for otherwise the total distance would be a

function of the ratio of the length of the hill to the length on the level). Hence the harmonic mean between y and $2y$ is x, or $(4/3)y=x$.

The Lewis Carroll problem appears on p. 1025 of *Complete Works of Lewis Carroll,* Modern Library.

Center of Liquid Pressure

E 807 [1948, 158]. *Proposed by R. V. Andree, University of Wisconsin*

An elliptical endgate of a reservoir is to be mounted with the minor axis parallel to the water's surface and in such a manner that it will turn about a horizontal axis in the plane of the gate. Where should this axis be placed so that the gate will not tend to rotate in either direction when the water level is at a given distance above the top of the gate? Generalize to a gate of any shape.

Solution by S. T. Parker, Kansas State College. Let the equation of the ellipse be

$$a^2x^2 + b^2y^2 = a^2b^2,$$

and the height of the liquid c units above the top of the gate. Let the desired axis be denoted by the equation $y=-z$. We have

$$\begin{aligned} \Delta A &= 2x\Delta y, \\ \Delta F &= 2wx(c + a - y)\Delta y, \\ \Delta M &= 2wx(c + a - y)(y + z)\Delta y, \end{aligned}$$

which gives us

$$\int_{-a}^{a} 2wx(c + a - y)(y + z)dy = 0.$$

Therefore

$$\int_{-a}^{a} (c + a - y)(y + z)(a^2 - y^2)^{1/2}dy = 0.$$

On evaluation, this leads to the result

$$z = a^2/4(c + a).$$

This can be generalized to a gate of any shape as follows. Let the x axis be taken horizontally through the centroid of the area in question. Let the limits of the area be $y=-b$, $y=+a$, and let H $(\geqq a)$ be the height of the liquid above the centroid of the gate. As before, we obtain

$$\int_{-b}^{a} x(H - y)(y + z)dy = 0,$$

or

$$\int_{-b}^{a} x(H-y)y\,dy = -\int_{-b}^{a} x(H-y)z\,dy.$$

Now

$$\int_{-b}^{a} xy\,dy = 0.$$

Therefore

$$-\int_{-b}^{a} xy^2 dy = -Hz\int_{-b}^{a} x\,dy,$$

or

$$z = \frac{\int_{-b}^{a} xy^2 dy}{H\int_{-b}^{a} x\,dy}.$$

This is equivalent to saying that $z=k^2/H$, where k is the radius of gyration of the area of the gate with respect to the horizontal centroidal axis.

Also solved by D. M. Brown, W. B. Campbell, Roger Lessard, Leo Moser, C. S. Ogilvy, Arthur Pancoe, and the proposer.

Editorial Note. The problem of locating the center of liquid pressure is treated in many elementary texts. Thus Moser mentioned *Calculus*, L. M. Kells, pp. 337–338, and Ogilvy mentioned *Differential and Integral Calculus*, Ross R. Middlemiss, p. 311, 2nd (1946) edition.

An Interesting Perfect Cube

E 808 [1948, 158]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that the number

$$N = 19000458461599776807277716631$$

is a perfect cube and that the twenty-eight numbers which are formed by cyclic permutations of its digits are all divisible by the cube root of N.

Solution by C. W. Trigg, Los Angeles City College. In the conventional manner we find

$$M = N^{1/3} = 2668425111.$$

We also observe that

$$10^{29} - 1 = 9(2668423111)(4163924028879808001).$$

Represent N by $a_1a_2a_3 \cdots a_{29}$, N_1 by $a_2a_3 \cdots a_{29}\, a_1$, N_2 by $a_3 \cdots a_{29}a_1a_2$, *etc.* Then

$$N_1 = 10N - a_1(10^{29}) + a_1 = 10N - a_1(10^{29} - 1),$$
$$N_2 = 10N_1 - a_2(10^{29} - 1),$$

etc. Since N and $10^{29}-1$ are each divisible by M, then N_1 is also. It follows that all the N_i are divisible by M.

In general, if N has p digits and M is a divisor of 10^p-1, then M is a divisor of the cyclic permutations of N. The eight smallest values of N having this property are tabulated below:

p	$10^p - 1$	N	M
2	$3^2 \cdot 11$	27	3
3	$3^3 \cdot 37$	729	9
4	$3^2 \cdot 11 \cdot 101$	1331	11
5	$3^2 \cdot 41 \cdot 271$	68921	41
6	$3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$	250047	63
		456533	77
		753571	91
		970299	99

Also solved by D. M. Brown, J. F. Darling, Roger Lessard, and Alvin and Martin Milgram (jointly).

A Divergent Series

E 809 [1948, 158]. *Proposed by P. L. Chessin, New York, N. Y.*

Show that the following series diverges:

$$\sum_{n=1}^{\infty} n^{-(n+1)/n}$$

I. *Solution by P. A. Clement, University of California at Los Angeles.* For a series, $\sum a_n$, of positive monotone decreasing terms, the well known theorem of Olivier (see Knopp, *Theory and Application of Infinite Series*, Blackie, 1928, p. 124) provides as a necessary condition for convergence that $\lim_{n\to\infty} na_n = 0$. By an easy calculation we find this limit to be 1 for the given series of positive monotone decreasing terms; hence the series diverges.

As a second solution, it is readily established that half the harmonic series may be used in a comparison test; *i.e.*, the given series is termwise greater than $\sum 1/2n$, and thus must diverge.

II. *Solution by D. W. Matlack, North American Aviation, Inc.* We use the well known theorem (see Hardy, *Pure Mathemtaics*, 6th edition, p. 309): If $\sum u_n$ is divergent and u_n/v_n tends to a limit other than zero as $n\to\infty$ then $\sum v_n$ is divergent. In the present case let $u_n=n^{-1}$, $v_n=n^{-(n+1)/n}$. Then $u_n/v_n=n^{1/n}\to 1$.

Also solved by J. O. Blumberg, W. G. Brady, W. H. Breen, D. M. Brown, R. L. Clayton and Eugene McLachlan (jointly), Ragnar Dybvik, Karl Itkin, M. S. Klamkin, Roger Lessard, Michael Martino, Jr., Norman Miller, Leo Moser, C. S. Ogilvy, W. Seidel, George Shapiro, E. J. Stulken, F. Underwood, W. R. Van Voorhis, and the proposer.

Editorial Note. As Miller pointed out, this problem occurs in Knopp, *Theory and Application of Infinite Series*, Blackie, 1928, p. 119, ex. h, where another proof of divergence is indicated. The problem is also found as ex. 34, p. 125 of the same work, where the application of Olivier's theorem is indicated. Lessard established the more general theorem: The series $\sum 1/n^{f(n)}$ is divergent if $\lim_{n\to\infty} f(n)=1$.

Series of Involutes

E 810 [1948, 159]. *Proposed by J. H. Butchart, Arizona State College*

Find the sum of the infinite series consisting of the following terms: the radius of the unit circle, the arc intercepted by the central angle x, the arc of the involute corresponding to this arc, the arc of the involute constructed on the first involute, and so on indefinitely.

Solution by Leo Moser, University of Manitoba. Let $S_n(x)$ be the length of the nth involute. It is easily shown that $S_1(x)=\int_0^x t\,dt$ and $S_n(x)=\int_0^x S_{n-1}(t)\,dt$. Hence $S_n(x)=x^{n-1}/(n+1)!$ and the required sum is e^x.

Also solved by W. G. Brady, D. M. Brown, Karl Itkin, D. W. Matlack, C. S. Ogilvy, R. A. Rosenbaum, F. Underwood, and the proposer.

Editorial Note. The relation $dS_n/dt=S_{n-1}$ is easily established by differential methods and may be found, *e.g.*, in Williamson, *Differential Calculus*, 1927, p. 304. Rosenbaum pointed out that $S_n(x)=x^{n+1}/(n+1)!$ appears in the solution of E 635 [1945, 161].

A given arc AB has two particular involutes, depending on whether end A or end B of an unwinding thread AB generates the curve. If we regard arc AB as a directed arc then the former case will lead to what we shall call the *direct involute* of arc AB, and the second case to what we shall call the *indirect involute* of arc AB. We now direct an involute by the direction in which it is traced by the moving end of the unwinding thread. If, in the given problem, we use the sequence of direct involutes we obtain the sum e^x. On the other hand, if we use the sequence of indirect involutes we obtain the sum $\tan x+\sec x$. Ogilvy found this second sum, which is immediately apparent from a figure.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4315. *Proposed by Albert Wilansky, Brown University*

Consider the Clairaut differential equation

$$y = px + f(p),$$

where $p=y'$ and f has a derivative. Prove that if f' is monotone the singular solution has exactly one point in common with any particular solution.

4316. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let there be given a skew quadrilateral having the sum of one pair of opposite sides equal to the sum of the other pair. Then, (1) there are infinitely many spheres tangent to all four sides of the quadrilateral, the locus of the centers being a straight line Δ, (2) the points of contact of any one of the spheres lie on a plane perpendicular to Δ, (3) the sides of the quadrilateral belong to a hyperboloid of revolution which envelopes all the spheres and has Δ as axis.

4317. *Proposed by Leo Moser, University of Manitoba*

Let G be an Abelian group and A a subset of order n, such that $a \in A$ implies $a^{-1} \in G-A$. Consider the n^2 (not necessarily distinct) elements of G of the form $a_i a_j$, a_i and a_j elements of A. Show that of these n^2 elements at most $\binom{n}{2}$ are elements of A.

4318. *Proposed by H. F. Sandham, Trinity College, Ireland*

Show that

$$\frac{1}{(\sinh \pi)^2} + \frac{1}{(2 \sinh 2\pi)^2} + \frac{1}{(3 \sinh 3\pi)^2} + \cdots$$
$$= \frac{2}{3}\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \cdots\right) - \frac{11\pi^2}{180}.$$

4319. *Proposed by Paul Erdös, Syracuse University*

If p is a prime greater than 3 and if $n = (2^{2p}-1)/3$, show that $2^n - 2$ is divisible by n.

SOLUTIONS

A Locus Problem

4170 [1945, 400]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The powers of the vertices A, B, C of a given triangle with respect to the circles (ω_1), (ω_2), (ω_3) are respectively (ka^2, kb^2, kc^2), (kb^2, kc^2, ka^2), (kc^2, ka^2, kb^2), where a, b, c are the lengths of the sides of ABC. Find the loci of the centers ω_i as k varies. (2) The circumcenter of ABC is the centroid of triangle $\omega_1\omega_2\omega_3$ which remains similar to itself. (3) The straight line $\omega_2\omega_3$ is perpendicular to the join of the centroid and Lemoine point of ABC.

*Solution by R. Bouvaist, Vincelles, Saône-et-Loire, France.** We choose triangle ABC as the triangle of reference in a system of normal coördinates. We let

$$C \equiv ayz + bzx + cxy, \qquad \Delta \equiv ax + by + cz.$$

Then $C=0$ and $\Delta=0$ are the equations of the circumcircle of ABC and of the line at infinity respectively. The equations of circles (ω_1), (ω_2), (ω_3) are then

$$C - k\Delta(a^3x + b^3y + c^3z)/abc = 0, \qquad (\omega_1) \tag{1}$$

$$C - k\Delta(ab^2x + bc^2y + ca^2z)/abc = 0, \qquad (\omega_2) \tag{2}$$

$$C - k\Delta(ac^2x + ba^2y + cb^2z)/abc = 0, \qquad (\omega_3) \tag{3}$$

The coördinates of the centers ω_1, ω_2, ω_3 are

$$\begin{aligned}
x_1 &= R\cos A + Rk(-a^3 + b^3\cos C + c^3\cos B)/abc,\\
y_1 &= R\cos B + Rk(a^3\cos C - b^3 + c^3\cos A)/abc,\\
z_1 &= R\cos C + Rk(a^3\cos B + b^3\cos A - c^3)/abc;\\
x_2 &= R\cos A + Rk(-ab^2 + bc^2\cos C + ca^2\cos B)/abc,\\
y_2 &= R\cos B + Rk(ab^2\cos C - bc^2 + ca^2\cos A)/abc,\\
z_2 &= R\cos C + Rk(ab^2\cos B + bc^2\cos A - ca^2)/abc;\\
x_3 &= R\cos A + Rk(-ac^2 + ba^2\cos C + cb^2\cos B)/abc,\\
y_3 &= R\cos B + Rk(ac^2\cos C - ba^2 + cb^2\cos A)/abc,\\
z_3 &= R\cos C + Rk(ac^2\cos B + ba^2\cos A - cb^2)/abc;
\end{aligned}$$

where R equals the circumradius of ABC. For varying k, these represent parametric equations of circumdiameters. The radical axes of the circumcircle and the circles (ω_1), (ω_2), (ω_3) have equations

$$\begin{aligned}
a^3x + b^3y + c^3z &= 0,\\
ab^2x + bc^2y + ca^2z &= 0,\\
ac^2x + ba^2y + cb^2z &= 0,
\end{aligned}$$

respectively. Hence ω_1, ω_2, ω_3 describe diameters perpendicular to their respec-

* Translated and checked by O. J. Ramler, Catholic University of America.

tive radical axes. As k varies from 0 to ∞, the triangle $\omega_1\omega_2\omega_3$ varies, beginning as a point, (R cos A, R cos B, R cos C), the circumcenter, and increasing in magnitude, its sides remaining parallel to themselves. Thus the circumcenter O is the center of homotheticity. Moreover, we have

$$x_1 + x_2 + x_3 = 3R\cos A, \qquad y_1 + y_2 + y_3 = 3R\cos B,$$
$$z_1 + z_2 + z_3 = 3R\cos C.$$

Hence O is also the center of gravity of the triangle $\omega_1\omega_2\omega_3$ for every value of k. Since the coördinates of the centers ω_1, ω_2, ω_3 given above are actual coördinates (and not relative) the point at infinity on the line $\omega_2\omega_3$ has coördinates,

$$\begin{aligned} x_2 - x_3 &= -a(b^2 - c^2) + b(c^2 - a^2)\cos C + c(a^2 - b^2)\cos B, \\ y_2 - y_3 &= a(b^2 - c^2)\cos C - b(c^2 - a^2) + c(a^2 - b^2)\cos A, \\ z_2 - z_3 &= a(b^2 - c^2)\cos B + b(c^2 - a^2)\cos A - c(a^2 - b^2). \end{aligned} \tag{4}$$

From (2) and (3) we find the equation of the radical axis for circles (ω_2) and (ω_3) to be

$$a(b^2 - c^2)x + b(c^2 - a^2)y + c(a^2 - b^2)z = 0. \tag{5}$$

Hence the point given by equations (4) is the point at infinity of a line perpendicular to (5), which joins the centroid to the Lemoine point of ABC. Therefore $\omega_2\omega_3$ is perpendicular to this line.

A Difference Sequence

4247 [1947, 232]. *Proposed by Leon Recht and Martin Rosenman, New York City*

The sequence

$$\{a\} = 1; 2; 1, 2; 2, 1, 2; 1, 2, 2, 1, 2; \cdots$$

is formed by writing down in succession sets of terms starting with 1 and such that every subsequent set is obtained from the preceding set by the substitution $1\to 2$; $2\to 1, 2$. Show that: (1) the $(n+1)$th* term a_n is

$$[k(n+1)] - [kn],$$

where $k=\frac{1}{2}(\sqrt{5}+1)$ and $[x]$ is the greatest integer not exceeding x: (2) Each set of terms in $\{a\}$, beginning with the third, is a repetition of the terms of the two preceding sets; (3) $\{a\}$ consists of sets of 2's separated by 1's, such that there are a_n 2's in the $(n+1)$th set of 2's.

Solution by E. P. Starke, Rutgers University. Let $c_1=1$, $c_2=2$, $c_3=3$, $c_4=5$, $\cdots$, $c_{n+1}=c_{n-1}+c_n$, be the Fibonacci sequence. Recalling familiar results we know that c_n/c_{n-1} is the nth convergent to the continued fraction development

* Corrected from the original statement where it appeared as the nth.

of k. We have also

$$(1) \qquad 0 < \frac{c_{2n+2}}{c_{2n+1}} - k < k - \frac{c_{2n+1}}{c_{2n}} < \frac{c_{2n+2}}{c_{2n+1}} - \frac{c_{2n+1}}{c_{2n}} = \frac{1}{c_{2n}c_{2n+1}},$$

$$(1') \qquad 0 < k - \frac{c_{2n+1}}{c_{2n}} < \frac{c_{2n}}{c_{2n-1}} - k < \frac{c_{2n}}{c_{2n-1}} - \frac{c_{2n+1}}{c_{2n}} = \frac{1}{c_{2n}c_{2n-1}},$$

$$[c_{2n}k] = c_{2n+1}, \qquad [c_{2n-1}k] = c_{2n} - 1.$$

Let u, v be any integers. Then if $u/v > k$, $c_{2n}u - c_{2n+1}v = 0$ if and only if $u = tc_{2n+1}$, $v = tc_{2n}$; $c_{2n}u - c_{2n+1}v = 1$ if and only if $u = c_{2n+2} \pm tc_{2n+1}$, $v = c_{2n+1} \pm tc_{2n}$; where t is an arbitrary integer and $n = 1, 2, 3, \cdots$. Thus for any $u > vk$ and for $c_{2n-1} < v < c_{2n+1}$, we have

$$c_{2n}u - c_{2n+1}v \geqq 2;$$

and therefore

$$u/v - c_{2n+1}/c_{2n} \geqq 2/c_{2n}v.$$

From (1)

$$k - c_{2n+1}/c_{2n} < 1/c_{2n+1}c_{2n} < 1/c_{2n}v,$$

and from these last two inequalities

$$u/v - k > 1/c_{2n}v.$$

If for u we take $[vk]+1$, we have

$$(2) \qquad vk - [vk] < 1 - 1/c_{2n}.$$

Now from (1′) we have

$$c_{2n} - 1/c_{2n} < c_{2n-1}k < c_{2n}.$$

Obviously,

$$[(v - c_{2n-1})k] < (v - c_{2n-1})k < [(v - c_{2n-1})k] + 1,$$

and by adding,

$$[(v - c_{2n-1})k] + c_{2n} - 1/c_{2n} < vk < [(v - c_{2n-1})k] + c_{2n} + 1.$$

From this we conclude

$$[vk] = c_{2n} + [(v - c_{2n-1})k],$$

because a value of $[vk]$ smaller by 1 would contradict (2). Replacing c_{2n} by $[(c_{2n-1})k]+1$ from (1′) we have

$$(3) \qquad [vk] = [c_{2n-1}k] + [(v - c_{2n-1})k] + 1.$$

Let $a_j=[(j+1)k]-[jk]$. By setting $v=c_{2n-1}+1,\ c_{2n-1}+2,\ \cdots,\ c_{2n+1}-1$ in (3) and subtracting the resulting equations, each from the following, we find

$$a_{c_{2n-1}} = [k] + 1 = 2$$

and

$$a_{w'} = a_w, \qquad w' = w + c_{2n-1}, \quad w = 1, 2, \cdots, c_{2n} - 2. \tag{4}$$

On the other hand, if $k>u/v$, a clearly analogous argument leads from (1) and (1′) through

$$vk - [vk] > 1/c_{2n-1}, \qquad c_{2n-2} < v < c_{2n}. \tag{2′}$$

$$[vk] = [c_{2n-1}k] + [(v - c_{2n-1})k], \tag{3′}$$

and finally to

$$a_{c_{2n-2}} = [k] = 1$$

and

$$a_{w'} = a_w, \qquad w' = w + c_{2n-2}, \qquad w = 1, 2, \cdots, c_{2n-1} - 2. \tag{4′}$$

We can combine (4) and (4′) as

$$a_{w'} = a_w, \qquad w' = w + c_n, \qquad w = 1, 2, \cdots, c_{n+1} - 2. \tag{5}$$

If therefore we separate the sequence $\{a\}$ into sets corresponding to the subscripts

$$\begin{array}{llll} 0; & 1; & 2,\ 3; & \cdots; \\ c_{2n-2}-1, & c_{2n-2}, & \cdots, & c_{2n-1}-2; \\ c_{2n-1}-1, & c_{2n-1}, & \cdots, & c_{2n}-2; \\ c_{2n}-1, & c_{2n}, & \cdots, & c_{2n+1}-2; \\ c_{2n+1}-1, & c_{2n+1}, & \cdots, & c_{2n+2}-2; \cdots; \end{array} \tag{6}$$

each set is seen by (5) to be exactly the terms of the two preceding sets taken in order. This shows that the sequences defined in (1) and (2) of the proposal are identical.

Note that the substitution

$$1 \to 2, \qquad 2 \to 1,\ 2 \tag{S}$$

changes the first set $a_0=1$ into the second, $a_1=2$, and the second, $a_1=2$, into the third, $a_2=1$ and $a_3=2$. Suppose (S) changes each set of (6) up to the jth into the succeeding set. Since it then changes the $(j-2)$th and the $(j-1)$th into the $(j-1)$th and the jth, it must also, because of the result of the previous paragraph, change the jth into the $(j+1)$th. Hence the sequence developed from $a_0=1$, $a_1=2$ by the use of (S) is the sequence $\{a_j\}$ of our previous discussion.

Let the integers s_j be defined by

$$s_j = j + [jk], \qquad j = 1, 2, 3, \cdots,$$

and note that

$$s_{j+1} = j + 1 + [(j+1)k] = s_j + 1 + a_j.$$

We prove that $a_n = 1$ if and only if n is one of the s_j.* This is readily seen to be equivalent to part (3) of the proposal because between the s_jth and the s_{j+1}th values of a (which are equal to 1) there are $s_{j+1} - s_j - 1 = a_j$ values which equal 2.

The proof follows by a slightly modified induction. Let c' represent c_{2n} and suppose

$$s_{c'+h} = s_{c'} + s_h \tag{7}$$

for a particular value of h. Then by (5), provided $h < c_{2n+1} - 1$, we have

$$\begin{aligned} s_{c'+h+1} &= s_{c'+h} + 1 + a_{c'+h} = s_{c'} + s_h + 1 + a_{c'+h} \\ &= s_{c'} + s_h + 1 + a_h = s_{c'} + s_{h+1}, \end{aligned}$$

and also

$$s_{c'+1} = s_{c'} + 1 + a_{c'} = s_{c'} + 2 = s_{c'} + s_1.$$

Hence (7) is true for $h = 1, 2, \cdots, c_{2n+1} - 2$. Now, since

$$s_{c'} = c_{2n} + [c_{2n}k] = c_{2n+2},$$

we have

$$s_{c'+h} = c_{2n+2} + s_h.$$

Similarly we may show

$$\begin{aligned} s_{c_{2n+1}+h} &= s_{c_{2n+1}} + s_h + 1, \\ &= c_{2n+3} - 1 + s_h + 1, \quad h = 1, 2, \cdots, c_{2n+2} - 2, \end{aligned} \tag{7'}$$

so that we may put

$$s' = s_{c_m+h} = c_{m+2} + s_h, \qquad h = 1, 2, \cdots, c_{m+1} - 2,$$

and by (5) we have

$$a_{s'} = a_{s_h}. \tag{8}$$

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* This theorem may be generalized as follows:

$$[(c_{2n} + v)k] - [c_{2n}k] = [vk] + 1$$

if and only if $v = c_{2n-1}j + c_{2n}[jk]$;

$$[(c_{2n+1} + v)k] - [c_{2n+1}k] = [vk]$$

if and only if $v = c_{2n}j + c_{2n+1}[jk]$, $j = 1, 2, \cdots$.

Thus, since every integer greater than 2 can be written as c_m+h with $h\leqq c_{m-1}\leqq c_{m+1}-2$, $m>2$, and since (by direct inspection) $s_1=2$ and $s_2=5$ give $a_s=1$, we conclude that $a_s=1$, for every s, as required.

We noted above that the number of a's between two successive values of a_s is always 1 or 2, but two consecutive a's cannot both be equal to 1 because $[(j+2)k]-[jk]\geqq(2k]=3$. Hence every a_s equals 1 and every other a equals 2 as required.

Editorial Note. C. D. Olds remarks that the above is a particular case of a problem originally proposed by Bernoulli. It was studied by A. Markoff [*Math. Ann.* 19, 1882, pp. 27–36] and more recently by J. V. Uspénsky [*Revista Union Mat. Argentina* 11, 1946, pp. 141–154, 165–183, 239–255 and 12, 1946, pp. 10–19]. These latter papers are listed in *Mathematical Reviews*, 8, 1947, pp. 5, 6, 443.

Convergence of Series

4250 [1947, 286]. *Proposed by Richard Bellman, Princeton University.*

If

$$s_n=\sum_{k=1}^{n}a_k,\qquad \sigma_n=\sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)a_k,$$

and

$$\sum_{n=1}^{\infty}|s_n-\sigma_n|^{\alpha}<\infty,$$

for some $\alpha>0$; prove that $\sum_{n=1}^{\infty}a_n$ is convergent.

Solution by N. J. Fine, Washington, D. C. Since $s_n-\sigma_n\to 0$, it is sufficient to prove that σ_n converges. Now

$$\sigma_n-\sigma_{n-1}=\frac{1}{n(n+1)}\sum_{k=1}^{n}ka_k=\frac{s_n-\sigma_n}{n}\cdot$$

Summing for $n=1, 2, \cdots, N$,

$$\sigma_N=\sum_{n=1}^{N}\frac{s_n-\sigma_n}{n}\cdot$$

If $\alpha\leqq 1$, convergence is immediate. If $\alpha>1$, Hölder's inequality yields

$$\sum_{n=1}^{N}\frac{|s_n-\sigma_n|}{n}\leqq\left\{\sum_{n=1}^{N}|s_n-\sigma_n|^{\alpha}\right\}^{1/\alpha}\left\{\sum_{n=1}^{N}n^{-\beta}\right\}^{1/\beta},$$

where $1/\alpha+1/\beta=1$. Since both factors converge, so does σ_N.

Also solved by R. C. Buck, W. P. DeWitt, G. G. Lorentz, Otto Szasz, J. G. Wendel, and the Proposer.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 W. 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Fundamentals of Statistics. By T. L. Kelley. Cambridge, Harvard University Press, 1947. 16+755 pages. $10.00.

Whoever has heard Professor Truman Lee Kelley knows how eloquently he speaks for those who enter the field of statistics not primarily from an interest in mathematics or in mathematical statistics but from some contact with an application of statistics to education, psychology, or the social sciences. The text is intended for such students; it makes interesting reading for anyone. The author with a distinguished career in statistics, psychology, and education has previously written a book on the subject "Statistical Method (1923)" and a set of well-known tables "The Kelley Statistical Tables" of which a revision promised for 1947 has presumably appeared. In the preface Professor Kelley states: "An alternative title for the book which would emphasize the point of view is Statistics, Its Philosophy and Method. The endeavor has been to place a great emphasis upon the logic and principles underlying the statistical study of phenomena." On the whole the author has been quite successful in his purpose. He has interwoven and interrelated the various topics and ideas into a closely knit fabric producing an over-all pattern of sound judgment, integrating the practical aspects with the many theoretical considerations involved. The text shows real scholarship and a very wide knowledge of both past and recent results. This has been done moreover with a minimum amount of mathematics.

The first ten chapters, the material for an elementary course, 398 pages, require little mathematics beyond college algebra, but in chapter seven the differential occurs, in chapter eight the slope, derivative, integral, and some trigonometry. The remaining four chapters, 225 pages, are advanced in nature. The highlights of the text besides those already indicated are: a discussion of the purposes and uses of statistics, the treatment of significant figures, the care in determining the size of the class interval and class limits in the frequency distribution, and the introduction of measures of variability before measures of central tendency, a bold and welcome departure. Along with this material the author introduces the standard error of the mean and clearly explains the concept of the number of degrees of freedom. This particular chapter, the sixth, illustrates the expert weaving of many strands into a related whole. The question of efficiency and consistency of a statistic is treated from the point of view of the critical ratio, the ratio of a population parameter to the standard error of a statistic estimating the parameter. This is done consistently and emphasis is placed on indicating the reliability of a statistic by computation of its standard error. Again in chapter seven, measures of central tendency are discussed not as isolated entities but in relation to other characteristics of the probability distri-

bution such as kurtosis and skewness. The discussion of the median and the mean is stimulating but, since the concepts of consistency and efficiency are not defined, it is based solely on intuitive reasoning. The chapter on the properties of the normal curve includes material seldom found elsewhere; another treats correlation from the point of view of regression, the bivariate probability function, and the analysis of variance. Here a gem deserves quotation "Galton's humility, after years of collection of data and subtle analysis of the same, in the face of the neat but not involved mathematical derivation, is worthy of note by the social scientists of this day who scoff at mathematical analysis." This enthusiasm for statistics is well communicated. In the same chapter the student is warned of the dangers in the indiscriminate use of biserial r, ϕ, and the tetrachoric coefficient of correlation. Chapter eleven furnishes an authoritative treatment of the reliability of a test, the tetrad difference, and various other topics in psychological testing. Multiple correlation is thoroughly discussed in chapter twelve, a modified Doolittle solution is given for regression planes as well as a solution of the general correlation problem by matrices. Chapter thirteen includes a study of periodicity, lead and lag, the approximate variance of a statistic for N large, moments of quotients, the Pearson system of curves, sampling from a bivariate normal function, and a substantial section, concise and clear on sequential analysis. Chapter fourteen introduces the student to mathematical topics beyond analytic geometry, such as matrices, determinants, moments of some discrete probability functions, the Gamma function, finite differences, and the Euler-Maclaurin sum formula. In chapter fifteen are selected references to statistical tables, tables of coefficients for Lagrangian interpolation, the normal curve, square and cube roots, and an outline of tables which occur in the text. Appendix A has the unusual feature of a mathematical background test in mathematics for elementary statistics with scoring keys, percentile norms, and reliability. The symbols used, a key to all formulas in the text, and a long list of references to original papers and texts constitute Appendix B. Appendix C is a correlation chart. Problems are given only in the first five chapters.

While a photo-offset process has been used, the results are quite clear. Misprints are very few. Formula 6:37 needs a parenthesis, similarly for 6:55, 6:58 should have $N-1$ in the denominator in place of $N+1$, and 7:08 should have $P_{.75}-P_{.25}$ in the denominator. It would be helpful to indicate that the formulas for the percentiles are merely linear interpolation formulas. In chapter five a generalized measure of variability is defined as

$$f = \frac{\Sigma \mid x_i - x_j \mid^m}{N(N-1)}, \qquad i, j = 1 \text{ to } N,\ i \neq j.$$

Then "This function, f, is worthy of investigation for different values of m, $\cdots$ but the writer has found the function rather intractable except for $m=2$." This function for $m=1$ is Gini's mean difference without repetition, and if divided by N^2 instead of $N(N-1)$ is Gini's mean difference with repetition. For $m=4$ the

expected value of $(x_i-x_j)^4$ would be $2(\mu_4-3\mu_2)$, where both μ_4 and μ_2 indicate population values. Thus for $m>2$, it cannot be claimed that f is a measure of variability.

A formula is stated for the standard error of the average deviation from the mean. It is rather useless, since to evaluate it would be difficult for any of the well known distributions. In the comparison of critical ratios for certain measures of skewness, all such measures should be consistent and estimate the same parameter. For example if α_3 in the population does not exist, it still may happen that the percentile measure of skewness may exist. While intuitional techniques are suggestive, they are not substitutes for rigorous mathematical demonstrations. The impression is created that if a curve has a high peak at the mode, then $\alpha_4>3$, certainly not true in general. One confused remark is the following: "Thus the sample mode, defining with maximum likelihood the parent mode, is a maximum likelihood statistic." The sample mode is usually very unreliable, seldom the solution of a maximum likelihood equation.

In chapter nine Paulson's formula is given for the normalization of the F distribution, in slightly different algebraic form, without a reference to its source, Annals of Mathematical Statistics, 1942. A convenient table is furnished by the author for its calculation. The results of Hotelling and Pabst would have strengthened the section on the coefficient of rank correlation. In the treatment of sequential analysis it should be clearer that the theory was developed almost solely by A. Wald. The books by A. Wald, P. Hoel, H. Cramer, and C. E. Weatherburn are not mentioned in the references, since they were published too recently for inclusion. To these should be added the text of S. S. Wilks.

The reviewer feels strongly that the Neyman-Pearson theory of testing hypotheses merited a very prominent position, a regrettable omission. Other omissions are the theory of confidence intervals and methods of sampling. More emphasis should have been placed on exact sampling distributions.

While a few deficiencies have been noted, this book fulfills a real need in the social sciences, psychology, and education. Both teachers and students will find it worthwhile as a text and for reference.

L. A. Aroian

Analytic Geometry. By D. S. Nathan and Olaf Helmer. New York, Prentice-Hall, Inc. 1947. 402 pages. $3.50.

This text contains 17 chapters with the following headings, in order: Coordinates on a Line; Rectangular Coordinates in the Plane; Angles; Loci; Lines; Circles; Coordinate Transformations; Conic Sections; Conic Sections in Arbitrary Position; Higher Plane Curves; Families of Curves; Parametric Representation; Polar Coordinates; Fitting of Empirical Data; Rectangular Coordinates in Space; Loci in Space; Planes and Lines. Thus 14 chapters are devoted to Plane Analytics and three to Solid Analytics.

Most of the main special features of this book are briefly indicated in the following seven paragraphs, in some of which our comments are added:

1. The discussion of topics is on the whole clear and for the most part fuller than is customary. In the first and third chapters, the discussions of directed line segments and angles seem unnecessarily involved.

2. The problems are plentiful in each chapter and are of great variety. They involve considerable newness, and the number of relatively difficult problems is larger than for most current textbooks on Analytics.

3. The last section of each chapter includes a summary and a list of review problems. These features seem worthwhile.

4. This book contains more figures than most other current texts on the subject—246 in all. These are on the whole well drawn and satisfactory. However, we would have preferred consistent labeling of the origin of coördinates.

5. Novelty of language is not infrequent in this text. For instances, "major axis" and "minor axis" of an ellipse are replaced, respectively, by "transverse axis" and "conjugate axis," as for a hyperbola; the term "discussion" of an equation is replaced by "analysis" of it; a certain rectangle which is customarily associated with a hyperbola is called its "guide rectangle"; and abbreviations that a reader should understand from context are introduced in several places without explanation.

6. The following elements of novelty of presentation seem noteworthy. The authors do not give a two-point form of equation of a straight line. In the chapter on Coordinate Transformations, the authors show how to remove the cross-product term from the general equation of the second degree (in x and y) before discussing the parabola, ellipse, or hyperbola. In the initial treatment of these conics, each conic is considered in just one (standard) position. The subject of polar coordinates is discussed for the first time in Chapter 13; Sections 86 and 88 of this chapter, which deal with rotations about the pole and intersections of curves, are exceptionally good. In the treatment of Solid Analytics, cylindrical and spherical coordinates are not used; nor are transformations of rectangular coordinates in space considered.

7. Formulas and Tables for Reference (pages 369–374) and Answers to Odd-Numbered Problems (pages 375–393) precede the Index (pages 395–402).

Now relative to corrections, aside from several nearly harmless misprints, we noted the following places where the authors' perspective seemed to need modification:

Page 15. The authors say "the nearer two points in the xy plane are to each other, the less will their abscissas, and likewise their ordinates, differ, and conversely." If the two points have the same abscissas (or ordinates), this quotation is clearly not quite appropriate.

Page 137. In discussing sections of a right circular cone by planes, the authors say "A circle appears as a mere special case of an ellipse (namely when $|VA'| = |BV'|$)." In their discussion, this equality is not sufficient to insure that the ellipse be a circle.

Page 249. In Example 1 here, the authors employ a conic with "arbitrary" eccentricity and then find the conic's equation in a form which can not represent

a circle with a positive radius. Thus the word "arbitrary" was mis-used; the case $e=0$, where e stands for eccentricity, should have been excluded.

Page 302. When k is negative (and $a=k/9$, $b=-4k/9$, $c=8k/9$), the three displayed equations in line 20 of this page are all incorrect; each one of them would be corrected by merely changing the sign of its right member. The authors' error seems to be due to the (false) assumption that $\sqrt{k^2}=k$ when k is negative.

We have no doubt but that this book can be used as a textbook in Analytics with satisfaction. However, in order to do this, we would feel it desirable to make lesson plans in advance which would surely include polar coordinates—perhaps at an early place in the course, so as to be able to use to advantage in many places the authors' Theorem 47 on polar rotations—and enough but not too many "hard" problems; and, for some courses, we would desire to add notes for one or two lectures on parts of Solid Analytics which this text omits.

H. A. SIMMONS

Analytic Geometry. By R. S. Underwood and F. W. Sparks. Boston, Houghton-Mifflin Co., 1947. 10+225 pages. $2.75.

This is a well written little book, clear in its explanations, and looks as if the student could be expected to read and absorb much information from it by himself. Too many text-books are so difficult that they degenerate merely to a source of homework problems. This book combines an excellent fund of exercises with clear exposition.

However, everything has its faults in this far from perfect world and this book is no exception.

First, there is a rather serious oversight in the development of equations of curves. The authors starting with the usual definitions of conic sections, for example, and simplifying (involving the squaring of both sides of an equation), arrive at the standard equations—thus proving that any point of the curve satisfies the equation, but they fail to retrace their steps and prove that any point satisfying the equation must be a point on the curve as defined. The two-edged meaning of "equation of a curve" is not brought out. However, a rather inadequate footnote on pp. 29–30, in connection with a much simpler locus problem, mentions that steps should be retraced but that an obvious reversal of steps is usually involved and hence is often omitted.

Secondly, the conic sections and their properties that are so elegantly studied by analytic methods, are neglected. The standard equations are exhibited, but the tangents to these curves are not considered at all, being brushed off with the statement that calculus is needed for their study. This is not a good excuse, since their slopes can be easily derived (by making the points of intersection of the secant coincide), or the formulas for slopes given without proof. It seems somewhat illogical to neglect conic sections and their tangents and yet spend considerable space on curve tracing and asymptotic behavior which do require calculus for full understanding. Moreover, in the discussion of the asymptotes of

a hyperbola, although the authors show the distance between the curve and its asymptote approaches zero, they cannot show (nor in fact do they mention) that the slope of the curve approaches that of the asymptote.

CHARLES WEXLER

NEW BOOKS RECEIVED

Rings and Ideals (The Eighth Carus Mathematical Monograph). By N. H. McCoy. LaSalle, Illinois, The Open Court Publishing Company, 1948. 12+216 pages. $3.00. (Members of the Mathematical Association of America may purchase one copy at $1.75, order to be placed with the Secretary-Treasurer.)

An Introduction to Pure Solid Geometry. Fourth Edition. By G. S. Mahajani. Poona, 1948. 12+106 pages. Rs3-8-0.

Handbook of Elementary Technical Mathematics. Revised Edition. By J. W. Greenwood and M. I. Chriswell, New York, Prentice-Hall, Inc., 1948. 6+186 pages. $2.10.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

Kappa Mu Epsilon, Pittsburg Kansas State Teachers College

Kansas Alpha Chapter of *Kappa Mu Epsilon* holds two initiation services each year, one near the close of the summer session, and one after the close of the first semester. About thirty-five new members are initiated each year at these services.

Six open meetings are held, at which topics of general mathematical interest are presented by guest speakers, faculty members, and students.

Topics presented by guest speakers were:

Mathematics in the secondary school, by Miss Jane Townsend, Principal of the Girard high school.

Vital Statistics, by D. E. Waggoner, Director of the Division of Vital Statistics for Kansas.

Topics presented by faculty and student members were:

Addition-subtraction logarithms, by Prof. F. C. German

The number 2 in mathematics, by James Hudson

Career fields for mathematicians, by Norval Phillips

The origin and meaning of mathematical terms, by Prof. J. A. G. Shirk

The scope of mathematics, by D. R. G. Smith.

A new conference room at the college provides adequate facilities for the public meetings to which all persons interested are cordially invited. Wives of veterans are finding these open meetings valuable in giving them a better understanding of the possible openings for their husbands after graduation.

Officers for 1947–48 are: President, James McCollam; Vice-President, James Hudson; Secretary, Betty Multhaup; Treasurer, Norval Phillips; Corresponding Secretary, Prof. J. A. G. Shirk; Sponsor, Dr. R. G. Smith.

Kappa Mu Epsilon, College of St. Francis

Meetings of the *Illinois Delta* Chapter of *Kappa Mu Epsilon* were held every month from October to May. At the various meetings the following papers were read:

An historical account of complex numbers, by Anne Whalen

Early forms of numerals and numerology, by Margaret Markiewicz

An historical account of fractions, by Sister M. Crescentia (visitor)

Introduction to the plane trigonometry of the Alamgest, by Marian Masters

Evolution of the symbolism in trigonometry, by Sister M. Claudia.

Other talks on book reports and recreational mathematics were given by members.

The officers for the year 1947–48 are: President, Elonore O'Connor; Vice-president, Betty Lanoue; Secretary, Jane Rourke; Treasurer, Mary Jean Lafond; Corresponding Secretary, Sister Rita Clare Flynn; Faculty Sponsor, Sister M. Claudia Zeller.

Pi Mu Epsilon, Michigan State College

The *Michigan Alpha* Chapter of *Pi Mu Epsilon* held eleven meetings in 1946–47, including Spring and Fall term picnics and a banquet in the Winter term. At three meetings a total of thirty one new members were initiated. Each initiate gave a brief talk on some mathematical riddle or fallacy or had to present a mathematical method for capturing lions. At most of the regular meetings two twenty minute talks were given by student members, after which refreshments were served. The following papers were presented:

The arbelos, by Eleanor Bessonen

The trimetric ruler, by Eunice Anderson

A problem in coin weighing, by Robert Jackson

Pathological functions, by Melvin LaVerne

Pathological curves, by Carolyn Trimm

A set of eight numbers, by Matha Hawley

Minimal tangents, by Nancy Pringle

A method for rapid calculation of Pi, by Mary Fuss

The five-color problem, by Phyllis Rowe.

Sixty members attended the annual banquet, and heard an address by Professor V. G. Grove on *A high school teacher of the nineteenth century*, in which he spoke of the life and work of the German mathematician, Karl Weierstrass.

The officers for 1946–47 were: President, Barbara Houston; Vice-President, Melvin LaVerne; Secretary-treasurer, Charles Costa; Membership Committee, Carolyn Trimm, Rosalie Manz, Phyllis Rowe; Faculty Sponsor, Prof. E. A. Nordhaus.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

NINTH CHRISTMAS CONFERENCE OF THE N.C.T.M.

The National Council of Teachers of Mathematics will hold its Ninth Christmas Conference at Ohio State University, Columbus, Ohio, on Wednesday and Thursday, December 29 and 30, 1948. Six sectional meetings will be held devoted to problems relating to the teaching of high school and junior college mathematics and the preparation and in-service training of teachers.

At the Thursday morning session, opportunities for discussion of important questions in mathematics teaching will be offered by attending one of seventeen discussion groups and clinics under the leadership of individual classroom teachers. Topics to be considered include questions involving the contents of courses in algebra and general mathematics, instructional and learning aids, guidance, tests, teaching of statistics in the high school and junior college, coordination of high school and college mathematics programs, and applications of mathematics in business and industry. The latest films and filmstrips will be shown at various periods during the two day conference.

Headquarters for the meeting will be in Baker Hall. Reservations for rooms in Baker Hall should be made by writing directly to Mr. Oscar Schaaf, Room 120 Arps Hall, Ohio State University, Columbus, Ohio not later than December 15, 1948.

A copy of the program may be obtained by writing to the National Council of Teachers of Mathematics, 212 Lunt Building, Northwestern University, Evanston, Illinois.

MECHANICS COLLOQUIUM AT ILLINOIS INSTITUTE OF TECHNOLOGY

The Mechanics Colloquium will consist of eight monthly lectures which will be held at Illinois Institute of Technology during the academic year 1948–1949.

Persons who desire an announcement of each session are asked to notify Mr. R. L. Janes, Illinois Institute of Technology, Technology Center, Chicago 16, Illinois.

PRELIMINARY ACTUARIAL EXAMINATIONS

The Actuarial Society of America and the American Institute of Actuaries offered prize awards to the nine undergraduates ranking highest on the combined score on Part 1 and Part 2 of the 1948 Preliminary Actuarial Examinations. The first prize of $200 was awarded to E. H. Larson, Massachusetts Institute of Technology. Additional prizes of $100 were awarded to each of the following: J. E. Brownlee, Haverford College; W. L. Farmer, University of Alabama; J. P. Fennell, Princeton University; B. F. Green, Jr., Yale University; Solomon Leader, Rutgers University; F. A. E. Pirani, University of Western Ontario; R. J. Semple, University of Toronto; C. A. Yardley, Dartmouth College.

The two actuarial organizations have authorized a similar set of nine prize awards for the 1949 Examinations on Part 2 which is a general mathematics examination covering algebra, trigonometry, coordinate geometry, differential and integral calculus. The closing date for applications is March 15, 1949.

Detailed information concerning the Examinations can be obtained from either of the following organizations: American Institute of Actuaries, 135 South La Salle Street, Chicago 3, Illinois; The Actuarial Society of America, 393 Seventh Avenue, New York 1, New York.

PERSONAL ITEMS

Professor E. F. Beckenbach of the University of California at Los Angeles represented the Association at the inauguration of President F. D. Fogg, Jr. of University of Southern California on June 11, 1948.

Professor Ruth M. Peters of St. Lawrence University was appointed a representative of the Association at the inauguration of President J. H. Davis of Clarkson College of Technology on October 8, 1948.

Dr. Miriam C. Ayer of Wellesley College has received an A.A.U.W. fellowship for study during 1948–49. She will return to Wellesley College in September, 1949 with the title of Assistant Professor.

Professor L. E. J. Brouwer of the University of Amsterdam has been elected to membership in the Royal Society of London.

Professor Daniel Buchanan was awarded an honorary degree of Doctor of Science by the University of British Columbia on the occasion of his retirement from the position of Dean of the Faculty of Arts and Science and Head of the Department of Mathematics.

Emeritus Professor Lennie P. Copeland of Wellesley College received an honorary degree of Doctor of Science from the University of Maine in June, 1948.

Professor Marston Morse of the Institute for Advanced Study has been elected a member of the Academia delle Scienze dell'Istituto di Bologna. Professor Morse has received recently an honorary degree of Doctor of Science from Kenyon College.

Professor Oswald Veblen of the Institute for Advanced Study has been elected to membership in the Accademia Nazionale dei Lincei.

Professor Hermann Weyl of the Institute for Advanced Study has been elected to honorary membership in the Calcutta Mathematical Society.

Albion College makes the following announcements: Professor H. D. Larsen has been appointed Head of the Department of Mathematics; Mr. Ralph Powers has been appointed to an instructorship.

At Stanford University a new Department of Statistics will be instituted this fall. Assistant Professor A. H. Bowker will serve as acting head of the department. Dr. M. A. Girshick, now research statistician for the government research project RAND at the Douglas Aircraft Company, has been appointed Professor of Statistics.

The University of Washington announces the following: Associate Professor C. M. Cramlet has been promoted to a professorship; Assistant Professor R. A. Beaumont has been promoted to an associate professorship; Professor T. G. Room of the University of Sydney has been appointed Visiting Professor for the Autumn Quarter, 1948; Dr. Edwin Hewitt of the University of Chicago has been appointed to an assistant professorship; Professor Nien-Yee Tang of Chias-Tung University has been appointed Lecturer; Dr. R. W. Ball of the University of Illinois, Mr. D. B. Dekker of the University of California have been appointed to instructorships.

Wellesley College announces the following appointments for 1948–49: Miss Zung-Nyi Loh as lecturer; Dr. Ilse Novak as instructor.

Professor C. B. Allendoerfer, who is on sabbatical leave from Haverford College for the academic year 1948–49, is at the Institute for Advanced Study.

Mr. J. W. Beach of Iowa State College of Agriculture and Mechanic Arts has been appointed to an associate professorship at the New Mexico School of Mines.

Mr. R. L. Beinert of Cornell University has been appointed to an instructorship at Hobart College.

Raj Chandra Bose has resigned as head of the graduate Department of Statistics of the University of Calcutta and has been appointed Professor of Mathematical Statistics at the University of North Carolina beginning in the winter of 1949.

Professor J. W. Bradshaw of the University of Michigan has retired with the title of Professor Emeritus.

Mr. G. P. Burns, who has been employed as a research physicist at the Naval Research Laboratory in Washington, has been appointed Assistant Professor of Physics at the Mary Washington College of the University of Virginia.

Associate Professor C. L. Buxton of Clarkson College of Technology has been made President of Paul Smith's College of Arts and Sciences.

Lieutenant P. F. Byrd, United States Army Air Corps, has been appointed to a professorship at Fisk University.

Assistant Professor R. L. Calvert of Utah State Agricultural College has been promoted to an associate professorship. He is on leave of absence for the year 1948–1949 and is resident at the University of Illinois.

Miss Louise H. Chin of the University of California has been appointed to an assistant professorship at the University of Arizona.

Dr. F. E. Clark has received an appointment as instructor at Tulane University.

Mr. Edward Craig has been appointed to an instructorship at Union College.

Professor J. C. Currie of Louisiana State University has accepted an appointment as associate professor at Alabama Polytechnic Institute.

Miss Alice C. Dean of Rice Institute has retired.

Assistant Professor Roy Dubisch of Triple Cities College, Syracuse University, has been appointed to an associate professorship at Fresno State College.

Mr. C. W. Dunnett has been appointed to an instructorship at the New York State Maritime Academy in Fort Schuyler.

Dr. W. F. Eberlein, who has been at the Institute for Advanced Study, has been appointed Assistant Professor of Mathematics at the University of Wisconsin.

Professor Beno Eckmann of the University of Lausanne has been appointed to a professorship at the Swiss Federal School of Technology in Zurich.

Miss Frances E. Falvey, formerly of Hollins College, has been appointed Dean of Women and Assistant Professor of Mathematics at James Millikin University.

Mr. Walter Fleming, lecturer at the University of Manitoba, has been appointed to an assistant professorship at Fort Hays Kansas State College.

Mr. M. K. Fort, Jr. of the University of Virginia has accepted an instructorship at the University of Illinois.

Mr. G. E. Gourrich has accepted a position as electrical engineer with the National Bureau of Standards.

Miss Helen Hand has been appointed to an instructorship at D'Youville College.

Dr. Frank Harary of the University of California has received an appointment as instructor at the University of Michigan.

Professor D. R. Hartree, Cambridge University has been granted a leave of absence to serve as acting director of the Institute for Numerical Analysis recently established at the University of California at Los Angeles as one of the operating units of the Applied Mathematics Laboratories, National Bureau of Standards.

Mr. T. R. Humphreys of Bergen Junior College has been appointed to an assistant professorship at New Jersey State Teachers College.

Professor W. R. Hutcherson, head of the Department of Mathematics and Astronomy of Berea College, has accepted a position as professor of mathematics at Northwestern State College, Natchitoches, Louisiana.

Mr. S. J. Jasper of the University of Kentucky has been appointed to an assistant professorship at Kent State University.

Assistant Professor J. R. F. Kent of the University of British Columbia has been appointed Associate Professor in Charge of Mathematics at Triple Cities College of Syracuse University.

Mr. Joseph Levitt of the University of Illinois is now Instructor in Physics at Pratt Institute.

Professor M. S. MacPhail of Acadia University, Canada, has been appointed to an associate professorship at Carleton College.

Professor A. W. McGaughey, Westminster College, has been appointed Associate Professor of Mathematics at Bradley University.

Associate Professor Josephine M. Mitchell of Oklahoma Agricultural and Mechanical College has received an appointment as assistant professor at the University of Illinois.

Assistant Professor Isaak Opatowski of the University of Michigan has accepted an appointment as research associate in mathematical biology at the University of Chicago.

Dean J. R. Overman of Bowling Green State University has retired from his position as Dean of the College of Liberal Arts but will continue as Professor of Mathematics.

Mr. I. B. Perrott of the College of Technology in Leicester has been appointed Lecturer at the University of Leeds.

Mr. O. L. Phillips, Louisiana State University, has been appointed Acting Head of the Department of Mathematics of Mississippi Southern College.

Mr. Peter Radkowski has accepted an instructorship at Washington and Jefferson College.

Professor R. F. Rinehart of Case Institute of Technology has been appointed Director of Planning Division, Research and Development Branch, Washington, D. C.

Mr. J. H. Ryser, University of Wisconsin, is now at the Institute for Advanced Study.

Mr. N. C. Scholomiti of De Paul University has been appointed to an instructorship at the University of Illinois, Navy Pier, Chicago.

Dr. J. P. Sholz, who has been employed as mathematician by the Bartol Research Foundation of Franklin Institute, has been named Instructor in Mathematics and German at Wilson College.

Professor Emeritus E. R. Sleight of Albion College has been appointed Visiting Lecturer in Mathematics at the University of Richmond.

Dean H. L. Slobin of the University of New Hampshire has retired.

Associate Professor C. G. Stipe of the Michigan College of Mining and Technology has been promoted to a professorship.

Dr. R. E. Street of the University of New Mexico has been appointed Acting Professor of Aeronautical Engineering at the University of Washington.

Dean Elijah Swift of the College of Arts and Sciences, University of Vermont, has retired.

Professor Mary E. Wells, Vassar College, has retired.

Dr. A. L. Whiteman of the Office of Chief of Naval Operations, Washington, D. C. has been appointed to an assistant professorship at the University of Southern California.

Mr. R. W. Young, Lehigh University, has accepted an instructorship at the University of Florida.

Associate Professor J. W. T. Youngs of Indiana University has been promoted to a professorship.

Professor W. B. Campbell of Philadelphia Textile Institute died on August 12, 1948.

Reverend Paul Muehlmann of West Baden College died on August 28, 1948. He had been a member of the Association for twenty-five years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTIETH SUMMER MEETING OF THE ASSOCIATION

The thirtieth summer meeting of the Mathematical Association of America was held at the University of Wisconsin, Madison, Wisconsin, on Monday and Tuesday, September 6–7, 1948, in conjunction with the summer meeting and colloquium of the American Mathematical Society, and meetings of the Institute of Mathematical Statistics, the Econometric Society, and Section A of the American Association for the Advancement of Science. About seven hundred and seventeen persons were in attendance at the meetings, including the following three hundred and twenty members of the Association:

C. R. Adams, Brown University
V. W. Adkisson, University of Arkansas
R. P. Agnew, Cornell University
A. A. Albert, University of Chicago
Florence E. Allen, University of Wisconsin
C. B. Allendoerfer, Institute for Advanced Study
K. J. Arnold, University of Wisconsin
W. L. Ayres, Purdue University
H. M. Bacon, Stanford University
R. H. Bardell, University of Wisconsin
J. J. Barron, Marshall College
R. C. F. Bartels, University of Michigan
Walter Bartky, University of Chicago
H. G. H. Bartram, University of Colorado
M. A. Basoco, University of Nebraska
Leon Battig, University of Wisconsin in Milwaukee
R. A. Baumgartner, Senior High School, Freeport, Illinois
Helen P. Beard, Newcomb College
H. M. Beatty, Ohio State University

E. F. BECKENBACH, University of California at Los Angeles
E. G. BEGLE, Yale University
J. H. BELL, Michigan State College
A. A. BENNETT, Brown University
DOROTHY L. BERNSTEIN, University of Rochester
LIPMAN BERS, Syracuse University
H. R. BEVERIDGE, Monmouth College
T. A. BICKERSTAFF, University of Mississippi
F. C. BIESELE, University of Utah
R. H. BING, University of Wisconsin
LOLLIE BELLE BIENVENU, Louisiana State University
MAY C. BLACKSTOCK, University of Tennessee
J. W. BRADSHAW, University of Michigan
RICHARD BRAUER, University of Toronto
R. W. BRINK, University of Minnesota
FOSTER BROOKS, Kent State University
M. C. BROWN, University of Kentucky
R. H. BRUCK, University of Wisconsin
H. E. BUCHANAN, Tulane University
R. C. BUCK, Brown University
P. B. BURCHAM, University of Missouri
F. J. H. BURKETT, Union College
L. P. BURTON, Waysata, Minneota
L. E. BUSH, College of St. Thomas
J. H. BUSHEY, Hunter College
JEWELL HUGHES BUSHEY, Hunter College
W. H. BUSSEY, University of Minnesota
NIKOLINE A. BYE, Central Michigan College of Education
S. S. CAIRNS, University of Illinois
W. D. CAIRNS, Oberlin College
R. H. CAMERON, University of Minnesota
C. C. CAMP, University of Nebraska
R. E. CARR, Michigan State College
W. B. CARVER, Cornell University
J. O. CHELLEVOLD, Lehigh University
R. V. CHURCHILL, University of Michigan
F. MARION CLARKE, University of Nebraska
HELEN E. CLARKSON, Creighton University
NATHANIEL COBURN, University of Michigan
C. J. COE, University of Michigan
H. J. COHEN, University of Wisconsin
B. H. COLVIN, University of Wisconsin
E. G. H. COMFORT, Illinois Institute of Technology
H. H. CONWELL, Beloit College
V. F. COWLING, Lehigh University
C. C. CRAIG, University of Michigan
A. B. CUNNINGHAM, West Virginia University
J. H. CURTISS, National Bureau of Standards
E. H. CUTLER, Lehigh University
J. A. DAUM, Texas A. & M. College
P. H. DAUS, University of California at Los Angeles
ROBERT DAVIES, University of Wisconsin
D. R. DAVIS, State Teachers College, Montclair, New Jersey
JAMES ELMER DAVIS, Drexel Institute of Technology
REV. L. A. V. DECLEENE, Catholic University
A. H. DIAMOND, Oklahoma A. & M. College
L. L. DINES, Northwestern University
H. L. DORWART, Washington and Jefferson College
D. M. DRIBIN, Army Security Agency
NELSON DUNFORD, Yale University
W. L. DUREN, JR., Tulane University
W. H. DURFEE, Dartmouth College
P. S. DWYER, University of Michigan
W. F. EBERLEIN, University of Wisconsin
P. D. EDWARDS, Ball State Teachers College
MARGARET C. EIDE, State Teachers College, River Falls, Wisconsin
SAMUEL EILENBERG, Columbia University
W. S. ERICKSEN, Forest Products Laboratory
R. L. ERICKSON, Purdue University
H. P. EVANS, University of Wisconsin
H. S. EVERETT, University of Chicago
G. M. EWING, University of Missouri
A. B. FARNELL, Princeton University
J. V. FINCH, Chicago, Illinois
C. H. FISCHER, University of Michigan
L. R. FORD, Illinois Institute of Technology
J. S. FRAME, Michigan State College
EVELYN FRANK, University of Illinois
R. E. GASKELL, Iowa State College
C. B. GASS, Wesleyan University
H. M. GEHMAN, University of Buffalo
B. E. GILLAM, Drake University
J. W. GIVENS, University of Tennessee
A. M. GLEASON, Harvard University
R. A. GOOD, University of Maryland
RUTH E. GOODMAN, Duquesne University
CORNELIUS GOUWENS, Iowa State College
A. A. GRAU, Grand Rapids, Michigan
L. M. GRAVES, University of Chicago
W. L. GRAVES, Kansas State College
EDISON GREER, Kansas State College
V. G. GROVE, Michigan State College
S. G. HACKER, State College of Washington
N. A. HALL, University of Minnesota

P. R. HALMOS, University of Chicago
E. S. HAMMOND, Bowdoin College
A. G. HANSEN, Cleveland, Ohio
E. G. HARRELL, State Teachers College, Platteville, Wisconsin
W. L. HART, University of Minnesota
G. E. HAY, University of Michigan
M. B. HASLAM, University of Buffalo
NOLA L. HAYNES, University of Missouri
C. T. HAZARD, Purdue University
ANNA S. HENRIQUES, University of Utah
FRITZ HERZOG, Michigan State College
E. H. C. HILDEBRANDT, Northwestern University
T. H. HILDEBRANDT, University of Michigan
EINAR HILLE, Yale University
J. J. L. HINRICHSEN, Iowa State College
D. L. HOLL, Iowa State College
D. G. HORVITZ, Iowa State College
A. S. HOUSEHOLDER, Clinton Laboratories
R. C. HUFFER, Beloit College
RALPH HULL, Purdue University
P. F. HULTQUIST, University of Wisconsin
M. GWENETH HUMPHREYS, Newcomb College
J. W. HURST, Montana State College
M. A. HYMAN, Naval Ordnance Laboratory
M. H. INGRAHAM, University of Wisconsin
W. C. JANES, Kansas State College
H. F. S. JONAH, Purdue University
G. K. KALISCH, University of Minnesota
WILFRED KAPLAN, University of Michigan
IRVING KAPLANSKY, Institute for Advanced Study
SAMUEL KARLIN, California Institute of Technology
M. W. KELLER, Purdue University
CLARIBEL KENDALL, University of Colorado
K. F. KENNEY, University of Wisconsin at Milwaukee
E. C. KIEFER, James Milliken University
J. R. KLINE, University of Pennsylvania
JOHN KRONSBEIN, Hoosier-Cardinal Corporation
R. E. LANGER, University of Wisconsin
LEO LAPIDUS, Michigan State College
E. H. LARQUIER, Spring Hill College
H. D. LARSEN, Albion College
C. G. LATIMER, Emory University
W. G. LEAVITT, University of Nebraska
H. L. LEE, University of Tennessee
SOLOMON LEFSCHETZ, Princeton University
WALTER LEIGHTON, Washington University
W. J. LEVEQUE, University of Colorado
HARRY LEVY, University of Illinois
F. A. LEWIS, University of Alabama
CHARLES LOEWNER, Syracuse University
W. S. LOUD, University of Minnesota
C. C. MACDUFFEE, University of Wisconsin
SAUNDERS MACLANE, University of Chicago
H. W. MARCH, University of Wisconsin
MORRIS MARDEN, University of Wisconsin at Milwaukee
A. M. MARK, University of Wisconsin
ANNA MARM, Bethany College
M. H. MARTIN, University of Maryland
W. T. MARTIN, Massachusetts Institute of Technology
C. W. MATHEWS, Washington University
KENNETH MAY, Carleton College
J. R. MAYOR, University of Wisconsin
R. B. MCCLENON, Grinnell College
DOROTHY MCCOY, Belhaven College
EDITH A. MCDOUGLE, University of Delaware
W. H. MCEWEN, University of Manitoba
W. R. MCEWEN, University of Minnesota
L. H. MCFARLAN, University of Washington
J. V. MCKELVEY, Iowa State College
MARTHA M. MCKELVEY, Ames, Iowa
A. E. MEDER, JR., Rutgers University
A. S. MERRILL, Montana State University
B. E. MESERVE, University of Illinois
H. J. MISER, Williams College
B. E. MITCHELL, Millsaps College
T. W. MOORE, U. S. Naval Academy
MAX MORRIS, Case Institute of Technology
DOROTHY J. MORROW, Civil Aeronautics Administration
D. S. MORSE, Union College
LEO MOSER, University of Manitoba
E. J. MOULTON, Northwestern University
A. L. NELSON, Wayne University
W. K. NELSON, University of Colorado
W. J. NEMEREVER, University of Michigan
C. V. NEWSOM, University of the State of New York
GEORGE OGAWA, University of Chicago
RUFUS OLDENBURGER, DePaul University
EMMA J. OLSON, Kent State University
F. W. OWENS, Pennsylvania State College
HELEN B. OWENS, Pennsylvania State College
J. C. OXTOBY, Bryn Mawr College
GORDON PALL, Illinois Institute of Technology
S. T. PARKER, Kansas State College
W. V. PARKER, University of Georgia

G. A. PARKINSON, University of Wisconsin at Milwaukee
P. M. PEPPER, University of Notre Dame
C. R. PERISHO, Nebraska Wesleyan University
H. P. PETTIT, Marquette University
C. R. PHELPS, Rutgers University
GEORGE PIRANIAN, University of Michigan
A. E. PITCHER, Lehigh University
GERTRUDE V. PRATT, Central Michigan College
G. B. PRICE, University of Kansas
A. L. PUTNAM, University of Chicago
E. D. RAINVILLE, University of Michigan
J. F. RANDOLPH, University of Rochester
RUTH B. RASMUSEN, Chicago City College
G. E. RAYNOR, Lehigh University
M. O. READE, University of Michigan
C. J. REES, University of Delaware
MINA S. REES, Office of Naval Research
P. V. REICHELDERFER, Ohio State University
IRMA MOSES REINER, University of Illinois
ERIC REISSNER, Massachusetts Institute of Technology
J. G. RENNO, JR., University of Wisconsin
C. N. REYNOLDS, West Virginia University
C. E. RHODES, Alfred University
P. R. RIDER, Washington University
R. F. RINEHART, Research & Development Board
FRED ROBERTSON, Iowa State College
H. A. ROBINSON, Agnes Scott College
G. F. ROSE, University of Wisconsin
ARTHUR ROSENTHAL, Purdue University
J. B. ROSSER, Cornell University
W. E. ROTH, Sampson College
E. H. ROTHE, University of Michigan
LULU L. RUNGE, University of Nebraska
ARTHUR SARD, Queens College
A. C. SCHAEFFER, Purdue University
ROBERT SCHATTEN, University of Kansas
HENRY SCHEFFÉ, Columbia University
E. D. SCHELL, U. S. Bureau of Labor Statistics
EDITH R. SCHNECKENBURGER, University of Buffalo
I. J. SCHOENBERG, University of Pennsylvania
K. C. SCHRAUT, University of Dayton
AUGUSTA SCHURRER, University of Wisconsin
NATHAN SCHWID, University of Wyoming
E. J. SCOTT, University of Illinois
W. T. SCOTT, Northwestern University
COL. W. E. SEWELL, Information and Education Division, U. S. Army
M. E. SHANKS, Purdue University
I. M. SHEFFER, Pennsylvania State College
L. W. SHERIDAN, College of St. Thomas
L. S. SHIVELEY, Ball State Teachers College
SISTER MARIOLA DOBBIN, Rosary College
SISTER MARY FELICE, Mount Mary College
SISTER MARY FERRER MCFARLAND, University of Notre Dame
M. F. SMILEY, Northwestern University
DOROTHY B. SMITH, University of Oklahoma
F. C. SMITH, College of St. Thomas
G. W. SMITH, University of Kansas
H. L. SMITH, Louisiana State University
D. W. SNADER, Galesburg University
ERNEST SNAPPER, University of Southern California
ANDREW SOBCZYK, Watson Laboratories
ELIZABETH S. SOKOLNIKOFF, University of Wisconsin
T. H. SOUTHARD, Wayne University
F. W. SPARKS, Texas Technological College
LT. W. K. SPEARS, Ft. Bliss, Texas
R. H. STARK, Northwestern University
H. E. STELSON, Michigan State College
C. F. STEPHENS, Morgan State College
R. P. STEPHENS, University of Georgia
B. M. STEWART, Michigan State College
M. H. STONE, University of Chicago
E. B. STOUFFER, University of Kansas
A. G. SWANSON, Gustavus Adolphus College
OTTO SZASZ, University of Cincinnati
A. H. TAUB, University of Washington
J. S. TAYLOR, University of Pittsburgh
H. P. THIELMAN, Iowa State College
J. M. THOMAS, Duke University
R. M. THRALL, University of Michigan
W. J. THRON, Washington University
JOHN TODD, King's College, London
LEONARD TORNHEIM, University of Michigan
J. I. TRACEY, Yale University
G. R. TROTT, University of Mississippi
P. L. TRUMP, University of Wisconsin
A. W. TUCKER, Princeton University
BRYANT TUCKERMAN, Cornell University
J. W. TUKEY, Princeton University
P. M. TULLIER, Loyola University
H. L. TURRITTIN, University of Minnesota
J. I. VASS, University of Wisconsin
JOHN VON NEUMANN, Institute for Advanced Study
R. W. WAGNER, Oberlin College
G. L. WALKER, Purdue University
R. J. WALKER, Cornell University

J. L. WALSH, Harvard University
J. A. WARD, University of Georgia
K. W. WEGNER, Carleton College
MARIE J. WEISS, Newcomb College
R. L. WESTHAFER, New Mexico College of A & M
A. E. WHITFORD, Alfred University
P. M. WHITMAN, Johns Hopkins University
L. R. WILCOX, Illinois Institute of Technology
R. L. WILDER, University of Michigan
J. E. WILKINS, JR., American Optical Company
S. S. WILKS, Princeton University
R. L. WILSON, University of Tennessee
R. M. WINGER, University of Washington
LOUISE A. WOLF, University of Wisconsin
F. H. YOUNG, Oregon State College
J. W. T. YOUNGS, Indiana University
J. L. ZEMMER, JR., University of Wisconsin
M. A. ZORN, Indiana University

Rooms for members of the mathematical organizations and their families were provided in Adams Hall and Tripp Hall, and meals were served in the cafeteria in Van Hise Hall. Tea was served by the ladies of the Department of Mathematics of the University of Wisconsin on Tuesday afternoon on the lawn beside Lake Mendota. That evening a concert of chamber music was presented in the Music Hall Auditorium by the *Pro Arte* quartet of the University of Wisconsin consisting of Rudolph Kolisch, first violin; Albert Rahier, second violin; Bernard Milofsky, viola; and Ernst Friedlander, cello.

An opportunity was given on Tuesday evening and again on Thursday evening to visit the Washburn Astronomical Observatory. A film on "Solar Explosions" was shown and visitors were permitted to inspect the telescope of the Observatory.

Wednesday afternoon was left open for recreation. After the group picture was taken in the Stock Pavilion, members could choose between a boat ride around Lake Mendota, a visit to the U. S. Forest Products Laboratory, or a visit to the Headquarters of the U. S. Armed Forces Institute. On Thursday evening, a picnic was held at Picnic Point, a scenic spot about a mile distant from the dormitories.

A dinner for members of the mathematical organizations was held on Wednesday evening in the Crystal Ball Room of the Hotel Loraine. Dean Walter Bartky was the toastmaster, and Dean M. H. Ingraham welcomed the guests on behalf of the University of Wisconsin. Short talks were given by Professor J. L. Walsh, Dr. C. V. Newsom, and Professor P. S. Dwyer on behalf of the Society, the Association, and the Institute. In conclusion, Dean Bartky expressed the thanks of the visitors for the excellent work of the local members of the Committee on Arrangements, especially Professor H. P. Evans, Chairman, and Professors B. H. Colvin and Elizabeth S. Sokolnikoff. A formal resolution to the same effect prepared by Professor G. B. Price was adopted at the meeting on Thursday afternoon.

The sessions of the American Mathematical Society began on Tuesday afternoon and continued through Friday morning. The colloquium lectures on "Representations of Groups and Rings" were given by Professor Richard Brauer of the University of Toronto. On Thursday afternoon, Professor J. W. T. Youngs of Indiana University delivered an address entitled "Topological meth-

ods in the theory of Lebesgue area." On Wednesday morning the Society held a joint session with Section A of the A.A.A.S. at which time Professor R. L. Moore of the University of Texas gave his retiring address as Vice-President of the A.A.A.S. and Chairman of Section A on the subject of "Spirals."

Sessions of the Institute of Mathematical Statistics and the Econometric Society were held from Tuesday morning through Friday morning. A joint session on stochastic processes was held on Wednesday and Thursday was devoted to a symposium on the theory of games. On Friday morning there was a joint session on sequential estimation.

The Mathematical Association held its sessions on Monday afternoon and Tuesday morning in Room 272 of Bascom Hall. President L. R. Ford presided at the first session and Vice-President C. B. Allendoerfer at the second session. The program was arranged by a committee consisting of W. L. Ayres, Chairman, R. H. Bing, and D. H. Lehmer.

FIRST SESSION OF THE ASSOCIATION

"Mathematical problems associated with the use of laminates in aircraft," by Professor H. W. March, University of Wisconsin and U. S. Forest Products Laboratory.

"Definition of angles in higher dimensional spaces," by Professor S. S. Cairns, University of Illinois.

"An experiment in teaching large classes in mathematics," by Professor H. F. S. Jonah, Purdue University.

SECOND SESSION OF THE ASSOCIATION

"Exterior ballistics of artillery rockets," by Professor J. B. Rosser, Cornell University.

"What is homotopy?" by Professor Samuel Eilenberg, Columbia University.

"Enrichment of the mathematical curriculum for juniors and seniors," by Professor M. H. Stone, University of Chicago.

MEETING OF THE BOARD OF GOVERNORS

The Board met at 7:30 p.m. on Monday in the Lounge of Slichter Hall. Of the thirty-five members of the Board, twenty-four were present. Following are some of the more important items of business transacted.

The Board approved the appointment of L. L. Dines in place of G. C. Evans as a member of the Nominating Committee for 1948.

It was voted to hold a meeting of the Association in June 1949 at Rensselaer Polytechnic Institute, Troy, New York, in conjunction with the annual meeting of the American Society for Engineering Education. The regular summer meeting of the Association will be held in Boulder, Colorado, in September 1949. The Board also voted to hold no meeting of the Association in the summer of

1950, since the International Congress of Mathematicians will be held at Harvard University at that time.

Caroline A. Lester and L. J. Green were elected Associate Editors of the MONTHLY and the resignation of R. F. Rinehart as an Associate Editor was accepted.

The Board voted as a general policy to pay an honorarium to authors of Slaught Memorial Papers at the rate of one dollar per printed page.

Effective December 1, 1948, the prices of all Carus Monographs are to be raised to $1.75 for members and to $3.00 for copies purchased through the Open Court Publishing Company.

An appropriation of $1000 was voted from the General Fund of the Association toward the expenses of the International Mathematical Congress to be held in 1950.

The Board voted that the Association accept the invitation to be officially represented on the Policy Committee for Mathematics by the appointment of three members. The president was authorized to appoint temporary representatives to serve until the next annual meeting of the Board.

It was agreed that the Association should resume the direction of the Putnam Prize Competition and that the President be authorized to appoint a Director of Putnam Competition for a term of five years.

The Board voted unanimously to recommend to the Association that W. B. Carver be elected an honorary life member. At a brief business meeting of the Association held on Tuesday morning, this recommendation was approved and Professor Carver became the eighth person to be honored by election to honorary life membership.

MEETING OF SECTION SECRETARIES

A meeting of secretaries of the Sections of the Association was held at 7:30 p.m. on Tuesday in the Lounge of Slichter Hall. Of the twenty-five Sections, twenty-one were represented by their secretaries or other officers.

Associate Secretary Edith R. Schneckenburger sketched the history and activities of the sections from the founding of the Association. A statistical table giving data on membership and potential membership of each section was distributed.

Professor C. C. MacDuffee spoke on the work of the Committee for the Coordination of Studies in Mathematical Education, of which he is chairman.

A general discussion followed concerning the relations of the central office of the Association to the section secretaries. Matters of finance, procurement of speakers, and election of sectional governors were discussed for possible future action by the Board of Governors.

It is expected that similar meetings of section secretaries will be held at future meetings of the Association.

H. M. GEHMAN, *Secretary-Treasurer*

APRIL MEETING OF THE OHIO SECTION

The thirty-second annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, on Saturday, April 3, 1948. Professor H. S. Pollard, chairman of the Section, presided at the morning and afternoon sessions.

Ninety-two persons registered attendance, including the following seventy-two members of the Association: H. H. Alden, W. E. Anderson, Max Astrachan, Grace M. Bareis, I. A. Barnett, H. M. Beatty, Theodore Bennett, W. D. Berg, J. B. Brandeberry, Foster Brooks, V. B. Caris, F. E. Carr, A. B. Carson, C. W. Cassel, B. B. Clark, Florentina M. Clinton, Wayne Dancer, R. C. Davis, R. H. Downing, B. B. Dressler, P. L. Evans, H. E. Fettis, B. E. Gatewood, Landis Gephart, B. C. Glover, L. J. Green, Ralph Hafner, Marshall Hall, R. G. Helsel, E. D. Jenkins, M. L. Johnson, Margaret E. Jones, K. D. Kelly, L. C. Knight, Jr., H. W. Kuhn, L. L. Lowenstein, R. H. Marquis, Margaret Mauch, S. W. McCuskey, H. E. Menke, Sister M. Mercedes, E. J. Mickle, L. H. Miller, C. C. Morris, Max Morris, Emma J. Olson, H. C. Parrish, C. G. Peckham, H. S. Pollard, Tibor Rado, S. E. Rasor, R. F. Rinehart, Louis Ross, S. A. Rowland, K. C. Schraut, C. E. Sealander, Samuel Selby, Ruth B. Smyth, W. H. Spragens, V. C. Stechschulte, C. W. Topp, E. P. Vance, R. H. Van Voorhis, R. W. Wagner, E. H. Wang, D. R. Whitney, F. B. Wiley, Alberta Wolfe, W. D. Wood, G. E. Woodson, Jr., C. H. Yeaton, Marie M. Yeaton.

The following officers were elected for the coming year: Chairman, R. H. Marquis, Ohio University; Secretary-Treasurer, Foster Brooks, Kent State University; Member of Executive Committee, L. H. Miller, Ohio State University; Member of Program Committee, I. A. Barnett, University of Cincinnati. The next regular meeting of the Section is scheduled to be held at the Ohio State University, Columbus, Ohio, on Saturday, April 2, 1949.

The following papers were presented:

1. *The mathematics major*, by Professor H. S. Pollard, Miami University.

The scientific developments which have taken place during and since the war have tended to emphasize the application of mathematics to engineering and science. There may be a tendency to regard the mathematics major as a potential engineer or scientist, and to confine his training to such topics as he will find of "practical" value in preparing him in the shortest time for further technical training.

The point of view of this paper is that the mathematics major is a composite of a number of primary interests and vocational aims, and that to stream-line courses for one particular group of students, particularly in a small institution where provision cannot be made for a number of different groups, is a fairly certain way of losing the interest of other students. Acceleration for accelerations's sake is not always desirable. The traditional subject matter and methods of mathematics still have value. In spite of the recent growth which mathematics has experienced in conjunction with the physical sciences, historically mathematics belongs to the liberal arts and is not a mere technical tool of the sciences. Mathematics is more than a collection of manipulative skills; it is a body of concepts and methods that constitute a way of thinking.

2. *A report on a recent questionnaire regarding mathematics in Ohio colleges*, by Professor Wayne Dancer, University of Toledo.

Most of the 35 colleges reporting encounter students with inadequate preparation in mathematics to the extent that they feel obliged to give courses in elementary and intermediate algebra, and in plane and solid geometry.

The most popular courses on the Junior-Senior level area, in order: Differential Equations, Theory of Equations, Advanced Calculus, Higher Algebra, and Mathematics of Statistics. Nearly half of the colleges give a four (semester) hour course in analytic geometry, including some solid analytics. The first course in calculus meets ordinarily 4 or 5 times a week. Twenty of the colleges teach differential and integral calculus alternately in a "unified" course. The author tabulated the topics which are most commonly covered in a course in Advanced Calculus.

It was found that most colleges are now giving special consideration to those non-technical students planning to take not more than a year of college mathematics. Very few colleges explicitly require mathematics for a bachelor's degree, but the majority do specify the study of mathematics or science. Half of the colleges reporting graduate fewer than six mathematics majors per year. The average number of hours required for a major is slightly in excess of 28, the minimum being 24, and the maximum 36.

3. *A coil winding problem*, by C. L. Emmerich, University of Cincinnati, introduced by Professor I. A. Barnett.

The problem is that of finding the manner of winding a coil having a fixed resistance R and maximum number of turns N. This may be formulated as an isoperimetric problem in the calculus of variations. It is found that the wire diameter is proportional to the square root of the distance from the coil axis.

4. *A solution of the problem of rapid scanning radar antenna*, by Professor R. F. Rinehart, Case Institute of Technology.

The problem of designing a method of focusing electromagnetic energy by electrical means is considered. The so-called rapid scanning problem in radar antenna is essentially the problem of designing a method for the efficient focusing of electromagnetic energy into a parallel beam by electrical means, and of providing for the rapid variation of the direction of this beam. The question reduces to the following problem in differential geometry: Determine a surface which contains a circle and a straight line segment (called the aperture) with respect to which the following two conditions are satisfied: (1) The circle is a geodesic circle relative to some point of the segment as center; (2) All geodesics joining *any* fixed point of the circle to all the points of the line segment cut the line segment at a fixed angle. Earlier efforts at solution of this problem were reviewed, and a solution in the form of a certain surface of revolution was described. This solution satisfactorily fulfills the requirements with the exception that the required line segment is a virtual, rather than an actual, aperture.

5. *Equations and loci in polar coördinates*, by Professor R. W. Wagner, Oberlin College.

This paper appeared in the June-July number of this MONTHLY, 1948.

6. *Matric diophantine equations*, by Professor I. A. Barnett, University of Cincinnati, and Professor C. W. Mendel, University of Illinois (introduced by Professor Barnett).

The authors consider equations of the form $X^k+Y^k=Z^k$, where the symbols X, Y, Z stand for

matrices whose elements may be zero, positive, or negative integers, and where multiplication is interpreted as matric multiplication. A general method of solution is given for the equation $X^2+Y^2=Z^2$ where the matrices are of order n, with Y and Z commutative, and X non-singular. For matrices of order 2, necessary and sufficient conditions for a solution are obtained when $k=2, 3, 4, 5$. These conditions are given in terms of the traces and determinants of the matrices involved. The equation $X^2+Y^2=I$ is fully discussed.

7. *Prospects in projective geometry*, by Professor Marshall Hall, The Ohio State University.

Professor Hall's paper was a one hour address by invitation of the Program Committee. The fundamental axioms of geometry are of three types: incidence, order, and metric axioms. Projective geometry is the study of the axioms of incidence. Ordinarily analytic geometry is thought of as giving coördinates to points in terms of distances to axes, and as such a part of metric geometry. Van Staudt first discovered in 1856 that a purely projective construction of coördinates could be made. In this construction the theorem of Desargues plays a central role. The theorem of Desargues is equivalent on the one hand to the existence of coördinates from a skew field, and on the other to the possibility of embedding a projective plane into a space of higher dimensions. The theorem of Pappus (or Pascal) implies the theorem of Desargues and also the commutative law of multiplication. Recent work has established more far-reaching relations between projective geometries and many branches of algebra, such as lattice theory, alternative rings, groups, and the theory of loops.

FOSTER BROOKS, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-second Annual Meeting, Columbus, Ohio, December 31, 1948.

Thirty-first Summer Meeting, Boulder, Colorado, September, 1949.

The following is a list of the Sections of the Association with dates of future meetings insofar as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN

ILLINOIS, Bradley University, Peoria, May 13–14, 1949

INDIANA, University of Notre Dame, Spring, 1949

IOWA, Drake University, Des Moines, April 15–16, 1949

KANSAS

KENTUCKY

LOUISIANA-MISSISSIPPI, University of Mississippi, Oxford, Spring, 1949

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK

MICHIGAN

MINNESOTA

MISSOURI

NEBRASKA, Lincoln, May, 1949

NORTHERN CALIFORNIA, San Francisco, January 29, 1949

OHIO, Ohio State University, Columbus, April 2, 1949

OKLAHOMA

PACIFIC NORTHWEST, Oregon State University, Corvallis, Spring, 1949

PHILADELPHIA, Philadelphia, November 27, 1948

ROCKY MOUNTAIN, Colorado School of Mines, Golden, April, 1949

SOUTHEASTERN, University of Alabama, University, March 18–19, 1949

SOUTHERN CALIFORNIA, John Muir College, Pasadena, March 12, 1949

SOUTHWESTERN

TEXAS, Denton, Spring, 1949

UPPER NEW YORK STATE, University of Buffalo, May, 1949

WISCONSIN, Lawrence College, Appleton, May 14, 1949

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

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ANALYSIS

GEOMETRY

PEDAGOGY, HISTORY

MATHEMATICAL NOTES

Edited by E. F. BECKENBACH, University of California at Los Angeles

CLASSROOM NOTES

Edited by C. B. ALLENDOERFER, Haverford College

PROBLEMS AND SOLUTIONS

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CORRECTIONS

Numbers in black face type refer to problems, those in light face to pages.

AUTHORS

Numbers refer to pages, blackface type indicating a problem solved and solution published; italics, a problem solved, but the solution not published; ordinary type, a problem proposed.

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Numbers in black face type refer to problems, those in light face to pages.

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Edited by EDITH R. SCHNECKENBURGER, University of Buffalo

GENERAL INFORMATION

PERSONAL INFORMATION

The following persons presented papers at meetings of the Association and its Sections.

Personal Mention. This section contains the names of officers of the Association and its Sections, persons mentioned in the Department of News and Notices, and those conducting the business of the Association. The list does not include names of new members or of attendants at meetings.

NECROLOGY

REPORTS AND ANNOUNCEMENTS OF THE ASSOCIATION AND ITS SECTIONS

MEETINGS AND ANNOUNCEMENTS OF THE ASSOCIATION

MEETINGS OF ITS SECTIONS

THE AMERICAN MATHEMATICAL MONTHLY

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THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

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ON THE AVERAGES OF THE DIVISORS OF A NUMBER

OYSTEIN ORE, Yale University

Let n be some integer and let

$$n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$$

be its decomposition into prime factors. As usual we shall denote the *number* of divisors of n by $\nu(n)$ and the *sum* of the divisors by $\sigma(n)$ so that

$$\nu(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_r + 1),$$

and

$$\sigma(n) = \sum_{d/n} d = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \cdots \frac{p_r^{\alpha_r+1} - 1}{p_r - 1}.$$

The various means of the divisors may be computed easily from the formulas that follow. For the *arithmetic mean* one has

$$A(n) = \frac{\sigma(n)}{\nu(n)} \tag{1}$$

and for the *geometric mean* the result is

$$G(n) = \sqrt[\nu(n)]{\prod_{d/n} d} = \sqrt{n}. \tag{2}$$

Finally the harmonic mean is defined by

$$\frac{1}{H(n)} = \frac{1}{\nu(n)} \cdot \sum_{d/n} \frac{1}{d}$$

and since

$$n \sum_{d/n} \frac{1}{d} = \sum_{d/n} \frac{n}{d} = \sum_{d/n} d$$

it follows that

$$H(n) = \frac{n\nu(n)}{\sigma(n)}. \tag{3}$$

The combination of the three formulas (1), (2), and (3) gives

$$H(n) \cdot A(n) = n = G(n)^2;$$

in other words, *the geometric mean of the divisors is the geometric mean of the arithmetic and harmonic means of the divisors.*

One may now ask in which cases these means may be integers. For the geometric mean this is trivial: *The geometric mean of the divisors is an integer only for square numbers.*

For the other two means the problem is by no means simple. We may observe first that both the arithmetic and harmonic means have the multiplicative property

$$A(a\cdot b) = A(a)\cdot A(b), \qquad H(a\cdot b) = H(a)\cdot H(b)$$

provided a and b are relatively prime. Thus one may only look for the *primitive* integral means, that is, such numbers that they are not the product of relatively prime factors, each of which has an integral arithmetic or harmonic mean.

Let us consider the arithmetic mean in a few special cases. For an odd prime p one has

$$A(p) = \frac{p+1}{2}$$

and this is always integral, while $p=2$ is an exception since $A(2)=3/2$ is not integral. From the multiplicative property we conclude that every number which is the product of different odd primes has an integral arithmetic mean of divisors while in the case of different prime factors including 2 the mean is only integral if one of the primes is of the form $4k-1$.

One may consider also when the powers of a prime may have an integral arithmetic mean of divisors. Since

$$A(p^\alpha) = \frac{p^{\alpha+1}-1}{(p-1)(\alpha+1)},$$

this is a problem closely connected with the solution of the congruence

$$x^k \equiv 1 \pmod{k}.$$

We shall not go into details about this problem. In certain cases the arithmetic mean is integral; for instance,

$$A(5^3) = 39, \qquad A(5^5) = 651.$$

It can be shown that no power of 2 can have an integral arithmetic mean.

The corresponding problems for the harmonic mean seem more interesting. In certain simple cases it can be established easily that the harmonic mean of the divisors cannot be integral. We mention first:

For the power of a prime the harmonic mean is not integral.

If namely $n=p^\alpha$ then

$$H(p^\alpha) = \frac{p^\alpha(\alpha+1)}{p^\alpha+p^{\alpha-1}+\cdots+p+1}.$$

Here p^α is relatively prime to the denominator and since $\alpha+1$ is a number less

than the denominator the harmonic mean cannot be integral.

Another observation is:

When $n \neq 6$ is the product of different prime factors the harmonic mean cannot be integral.

We write

$$n = p_1 p_2 \cdots p_r$$

where the primes are arranged in increasing order, and one finds

$$H(n) = \frac{p_1 p_2 \cdots p_r}{(p_1 + 1) \cdots (p_r + 1)} \cdot 2^r.$$

Let us assume first that n is odd, and let us write

$$H(n) = \frac{p_1 \cdots p_r}{\dfrac{p_1 + 1}{2} \cdots \dfrac{p_r + 1}{2}}.$$

Here the denominator contains at least r prime factors, and since p_r is not among them the quotient cannot be integral. Next let $p_1 = 2$ so that

$$H(n) = \frac{2^{r+1} \cdot p_2 \cdots p_r}{3 \cdot (p_2 + 1) \cdots (p_r + 1)} = \frac{4 \cdot p_2 \cdots p_r}{3 \cdot \dfrac{p_2 + 1}{2} \cdots \dfrac{p_r + 1}{2}},$$

and this expression can only be integral if $p_2 = 3$ so that

$$H(n) = \frac{2 p_3 \cdots p_r}{\dfrac{p_3 + 1}{2} \cdots \dfrac{p_r + 1}{2}}.$$

Since none of the $r-2$ factors in the denominator are equal to p_r we conclude that each of them must be equal to one of the other prime factors in the numerator. But the assumption

$$\frac{p_3 + 1}{2} = 2$$

or $p_3 = 3$ was already excluded.

A result of some interest is the following:

A perfect number has an integral harmonic mean of divisors.

Since a perfect number is defined by the property that

$$\sigma(n) = 2n$$

it follows that

$$H(n) = \frac{n\nu(n)}{\sigma(n)} = \frac{\nu(n)}{2}.$$

When n is an even perfect number it has the form $n=2^\alpha \cdot p$ so that $\nu(n) = (\alpha+1)\cdot 2$ and $H(n)=\alpha+1$ is integral. When we have an odd perfect number it is known that one of the exponents in the prime factor decomposition is odd so that $\nu(n)$ is even also in this case.

By means of the tables of J. W. L. Glaisher giving the values of $\sigma(n)$ and $\nu(n)$ one can determine rather easily the values of n below 10000 for which $H(n)$ is integral. One finds the following table:

TABLE I

n	$H(n)$	$A(n)$
$6=2\cdot 3$	2	3
$28=2^2\cdot 7$	3	$9\frac{1}{3}$
$140=2^2\cdot 5\cdot 7$	5	28
$270=2\cdot 3^3\cdot 5$	6	45
$496=2^4\cdot 31$	5	$99\frac{1}{5}$
$672=2^5\cdot 3\cdot 7$	8	84
$1638=2\cdot 3^2\cdot 7\cdot 13$	9	182
$2970=2\cdot 3^3\cdot 5\cdot 11$	11	270
$6200=2^3\cdot 5^2\cdot 31$	10	620
$8128=2^6\cdot 127$	7	$1161\frac{1}{7}$
$8190=2\cdot 3^2\cdot 5\cdot 7\cdot 13$	15	546

It would be of interest to make a systematic examination of such numbers up to higher numerical limits. This requires an extension of Glaisher's Number-divisor tables, a project which would be desirable also in connection with other number theoretical investigations.

There are, however, various methods by means of which one can construct new numbers with integral harmonic means from given ones. Let us suppose that n has this property so that

$$n\cdot\nu(n) = a\cdot\sigma(n), \qquad a = H(n).$$

We multiply both sides by some number k relatively prime to n and write $n_1=k\cdot n$. Then one finds

$$H(n_1) = \frac{n_1\cdot\nu(n_1)}{\sigma(n_1)} = \frac{a\cdot k\cdot\nu(k)}{\sigma(k)}\cdot$$

Thus if the right-hand expression is an integer the number n_1 will also have an integral harmonic mean. This may again be examined by means of Glaisher's tables. The values of a in the last column of the table given above lead successively to the following numbers:

TABLE II

n	$H(n)$	n	$H(n)$
$2\cdot3^2\cdot7\cdot13\cdot17$	17	$2^3\cdot3^3\cdot5^2\cdot31\cdot53$	53
$2\cdot3^2\cdot5\cdot7\cdot13\cdot29$	29	$2^6\cdot13\cdot127$	13
$2^3\cdot3\cdot5^2\cdot31$	15	$2^6\cdot3^2\cdot13\cdot127$*	27
$2^3\cdot3^3\cdot5^2\cdot31$	27	$2^6\cdot3^2\cdot5\cdot13\cdot127$	45
$2^3\cdot3^3\cdot5^2\cdot17\cdot31$	51	$2^6\cdot3^2\cdot13\cdot17\cdot127$	51
$2^3\cdot5^2\cdot19\cdot31$	19	$2^6\cdot3^2\cdot5\cdot13\cdot17\cdot127$	85
$2^3\cdot5^2\cdot7^2\cdot19\cdot31$	49	$2^6\cdot3^2\cdot5\cdot13\cdot29\cdot127$	87
$2^2\cdot5^2\cdot7^2\cdot13\cdot19\cdot31$	91	$2^6\cdot3^2\cdot13\cdot53\cdot127$	53
$2^3\cdot3\cdot5^2\cdot29\cdot31$	29	$2^6\cdot3^2\cdot5\cdot13\cdot89\cdot127$	89
$2^3\cdot5^2\cdot19\cdot31\cdot37$	37		

Table I shows that there are eleven numbers below 10,000 with integral harmonic mean for the divisors. Among these are the four perfect numbers below this limit. One verifies simply that an even perfect number cannot have an integral arithmetic mean of the divisors. However, if these are excluded from Table I the remaining numbers have the property that also their arithmetic means of divisors are integral. One might be led to the conjecture that this would be a general property and the computation of Table II was executed with this in mind. However, the number marked by an asterisk proved to be a counter example. It is not perfect and its arithmetic mean of divisors is not integral. A much more interesting conjecture, however, appears from our numerical computations, as an extension of the famous conjecture for perfect numbers, namely, that *a number with integral harmonic mean of divisors must be even.*

POLYGONS WITH TWO EQUIANGULAR POINTS

ARTHUR WORMSER, Riverside, Illinois

1. Introduction. In 1816 A. L. Crelle [1] discovered that in every triangle ABC there are two points G and G' such that $\measuredangle GAB=GBC=GCA=G'BA=G'CB=G'AC=\omega$. After H. Brocard [2] rediscovered this feature in 1875, he and other mathematicians found a great number of related facts [3]. Since then the points G, G' and angle ω have been called Brocard's points and Brocard's angle. Interest in the subject remained at a high level for many years, and in 1886 R. Tucker [4] found that Moebius' harmonic quadrangle [5] has a similar property. Although this type of geometry of the triangle fascinated quite a few, all efforts failed to construct or calculate polygons with more than four sides having two equiangular points [6] as defined above. In 1930 K. Hagge [7] pub-

lished a lecture in which he gave a history of the problem, and hinted at a construction for one class of these polygons without giving any lead how he had arrived at his guess.

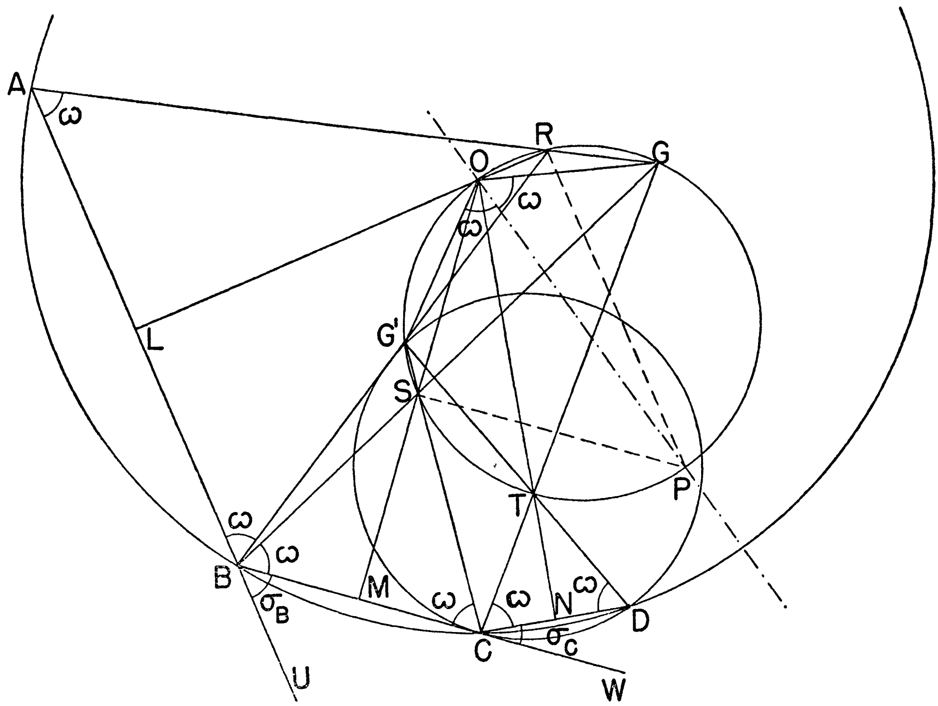

FIG. 1

2. Construction of nomogons. Let AB and BC in Figure 1 be two consecutive sides of the polygon to be found and their exterior angle $\sigma_B = CBU$. Then Brocard's angle ω determines two equiangular points [8] G and G' at the intersections of AR, BS and BR, CS, respectively. The direction of a third side CD of the polygon follows from the requirement that $\angle GCD = \omega$, and in order to make $\angle G'DC = \omega$, D has to be on a circle with ω as an inscribed angle intercepting CG'. By the same construction one vertex after the other could be found from the given two sides and two angles, and this rigid character of the polygons under discussion has caused me to call them nomogons.

While R and S are the vertices of isosceles triangles with bases AB and BC respectively, T is the vertex of isosceles triangle CTD. The intersection O of the perpendicular bisectors LR and MS of the sides AB and BC is the center of a circle passing through A, B, and C. Now, because $\measuredangle RG'S$, ROS, and RGS equal σ_B or $\pi - \sigma_B$, points O, G, G', R, and S are on a circle. Further, because $\measuredangle G'RG$, $G'SG$, and $G'TG$ equal 2ω or $\pi - 2\omega$, point T is on the same circle as the five other points. Let us assume that the perpendicular bisectors MS and NT intersect

in a point O' different from O; then $\measuredangle SG'T$, SGT, and $SO'T$ are equal to $\sigma_C = DCW$ or $\pi - \sigma_C$. Therefore O' is also on the same circle, and since it is not identical with either S or T, it is identical with O. It follows that CD and all following sides of the nomogon are chords to the circle through ABC, and every nomogon is an inscribed polygon. It will be shown below how it is determined whether or not the nomogon closes after one or more circuits of the circle.

The circle $OGG'RST \cdots$ corresponds to Brocard's circle in the triangle, and the other end P of its diameter through O is the counterpart of Lemoine's point [9]. Because $\measuredangle GSO = G'SO = \pi/2 - \omega$, $\angle G'OG = 2\omega$, and both the arcs and the chords OG and OG' are equal respectively; it follows that $\measuredangle POG = POG' = \omega$.

3. The circumcircle and inscribed conic. When the circumscribed circle, one equiangular ray GA, and Brocard's angle are given, the nomogon is readily found as a chain of chords. If A moves on the circle, these chords envelop a curve whose equation with O as origin, OP as positive axis of ordinates, $OA = r$, and $OG = f$, is found thus: Let the slope of lines AB and AG be θ and $\theta - \omega$, respectively. Since AG is a line through $G(f \sin \omega, f \cos \omega)$ and AB is a line through $A(x_A, y_A)$, their equations are

$$(y - f \cos \omega) \cos (\theta - \omega) = (x - f \sin \omega) \sin (\theta - \omega)$$
$$(y - y_A) \cos \theta = (x - x_A) \sin \theta$$

respectively. We use the first equation and the equation $x^2 + y^2 = r^2$ of the circumscribed circle as simultaneous equations to determine x_A and y_A, and substitute them in the second equation. Thus the equation of AB is

$$y = x \tan \theta + f \cos \omega \pm \frac{\sin \omega}{\cos \theta} \sqrt{r^2 - f^2 \cos^2 \theta}.$$

From $\partial y / \partial \theta = 0$ we find the parametric equations of the envelope:

$$x = \pm \frac{r^2 \sin \omega \sin \theta}{\sqrt{r^2 - f^2 \cos^2 \theta}}, \qquad y = f \cos \omega \pm \frac{(r^2 - f^2) \sin \omega \cos \theta}{\sqrt{r^2 - f^2 \cos^2 \theta}},$$

and the elimination of θ produces the single equation

$$\text{(1)} \qquad \frac{x^2}{r^2 \sin^2 \omega} + \frac{(y - f \cos \omega)^2}{(r^2 - f^2) \sin^2 \omega} = 1.$$

This inscribed conic, defined as a feature of the triangle, is known as Brocard's ellipse. Its foci are identical with the equiangular points, but it is an ellipse only if these points are inside the circumcircle, that is, $f < r$. If $f > r$, it is a hyperbola.

By determining the intersections of the conic with the circumscribed circle one finds two double points:

$$\text{(2)} \qquad x = \pm \frac{r}{f} \sqrt{f^2 - r^2 \cos^2 \omega}, \qquad y = \frac{r^2 \cos \omega}{f}$$

which means that the inscribed conic is tangent to the circumscribed circle in two points. The points of tangency are real if $f > r \cos \omega$, they coincide if $f = r \cos \omega$, and they are imaginary if $f < r \cos \omega$. At the same time $OP = f/\cos \omega$ becomes greater than, equal to, or less than r, respectively.

From the two basic facts just proved that every nomogon is inscribed into a circle and that it is circumscribed about a conic which touches the circumcircle in two points, the complete solution of the problem of polygons with two equiangular points can be deduced. There are various chains of deductions, each offering a different aspect of our problem.

In the language of hyperbolic non-euclidean geometry and using the circumcircle as absolute, the inscribed conic may be (**A**) a proper circle, (**B**) a horocycle, (**Ca**) an ordinary equidistant curve, or (**Cb**) an ultra-infinite equidistant curve. Each of these circles generates a different class of nomogons which can also be defined by the position of the equiangular points with relation to the center of the circumscribed circle and to each other. For, if one considers f and ω variable, producing an infinite number of nomogons, the conditions $f = r$ and $f = r \cos \omega$ indicate those nomogons whose equiangular points are on the circumcircle and on a limiting circle passing through O and tangent to the circumcircle, respectively. If the equiangular points are on the circumcircle, both the inscribed conic and the nomogon degenerate into the same straight line, and so we have the following four classes of nomogons:

(**A**) G and G' inside the limiting circle, P inside the circumcircle, all sides of finite length.

(**B**) G and G' on the limiting circle, P on the circumcircle, the vertices of the nomogons approach one point.

(**C**) G and G' outside the limiting circle, P outside the circumcircle, the vertices of the nomogons approach two points;

- (**a**) G and G' outside the limiting circle but inside the circumcircle, the vertices of the nomogons approach the two points from both sides without alteration;
- (**b**) G and G' outside the circumcircle, the vertices of the nomogons approach the two points alternating from one side of each point to the other.

From the fact that $OP = f/\cos \omega$ it follows easily that Brocard's circle meets the circumcircle in the intersections of the axis with the absolute, that is, in two imaginary points in case (**A**), in a double point in case (**B**), and in two real points in case (**C**). On the other hand, the equation of the axis being $y = r^2 \cos \omega/f$, we have $y \times OP = r^2$ and, therefore, Lemoine's point P is the center of the non-euclidean circles. Hence the vertices of the nomogon are produced by equal hyperbolic motions whose center is Lemoine's point P; the motion is a rotation in case (**A**), a parallel displacement in case (**B**), a translation in case (**Ca**), and a glide-reflection in case (**Cb**). In the last case the sides of the nomogon touch the two branches of the ultra-infinite equidistant curve alternately.

When four consecutive vertices $ABCD$ of the nomogon are given, the center P is found as Brianchon's point of the circumscribed hexagon $ACCBBD$. The following vertices of the nomogon are then readily constructed by making $\measuredangle DPE, EPF, \cdots = APB$ and, therefore, the nomogon may be described as a regular polygon whose consecutive sides are parallel.

Using the language of projective geometry, the circumcircle and the inscribed conic, because of their double tangency, may be projected simultaneously into concentric circles. Therefore, the tangents to the inscribed conic will cut projective non-involutory ranges on the circumcircle, and there will be a projectivity T on the circle transforming A to B, B to C, and so on. We have case (**A**), (**B**), or (**C**) according as projectivity T is elliptic, parabolic, or hyperbolic, and subclasses (**a**) or (**b**) according as it is direct or opposite, that is, preserves or reverses the sense.

4. The cross ratio k. The fact that the cross ratio of four consecutive vertices of a nomogon is constant, which follows from projectivity T, can be derived directly from the basic figure by a short trigonometric calculation. With α, β, γ designating inscribed angles with their vertex V on the circle $ABCD$ and intercepting AB, BC, CD, the Law of Sines expressed for GB in triangles GAB and GBC results in the formula

$$\cot(\beta+\gamma) = \cot\beta - \cot(\alpha+\beta) - \cot\omega \tag{3}$$

which allows us to calculate point D when besides A, B, and C, the angle ω is given, or vice versa. Eliminating ω from this formula taken for two consecutive quadruples, we get:

$$\cot(\gamma+\delta) + \cot\beta = \cot(\alpha+\beta) + \cot\gamma \tag{4}$$

which yields a fifth vertex without knowledge of ω. Formula (4) can readily be rewritten in the form,

$$\frac{\cot\beta - \cot(\beta+\gamma)}{\cot\beta + \cot\alpha} = \frac{\cot\gamma - \cot(\gamma+\delta)}{\cot\gamma + \cot\beta} \tag{5}$$

and this expression which has the same value k for every consecutive quadruple of vertices, can be written:

$$k = \frac{\sin(BVA)}{\sin(CVA)} : \frac{\sin(BVD)}{\sin(CVD)} = (BCAD) = (CDBE) = \cdots. \tag{6}$$

Thus we have the cross ratio

$$k = (BCAD) = \frac{\cot\beta - \cot(\beta+\gamma)}{\cot\beta + \cot\alpha}. \tag{7}$$

If solved for $\cot(\beta+\gamma)$, this formula reads:

$$\cot(\beta+\gamma) = (1-k)\cot\beta - k\cot\alpha \tag{8}$$

and by substituting $\cot\beta-\cot(\beta+\gamma)$ from formula (3) we find:

$$\cot\omega = k(\cot\alpha + \cot\beta) - \cot(\alpha+\beta). \tag{9}$$

These two formulas allow us to compute the fourth vertex and Brocard's angle, respectively, when three consecutive vertices are required to generate a nomogon characterized by a certain cross ratio k.

On the other hand, it follows from formula (9) that to every value of k there is a corresponding value of ω, and because it is also true that D is uniquely determined by cross ratio k, we conclude that every chain of chords with a constant cross ratio of its intercepting inscribed angles is a nomogon.

Among the nomogons with a certain cross ratio k, the one which has its equiangular points in the center of the circumscribed circle and which, therefore, always belongs to class (A), has equal sides; if in formula (7) we make $\alpha=\beta=\gamma=\alpha_0$, we find

$$k = \frac{1}{4\cos^2\alpha_0} \qquad \left(0 \leqq \alpha_0 \leqq \frac{\pi}{2}\right) \tag{10}$$

and since the number n of sides per circuit is

$$n = \frac{\pi}{\alpha_0} \tag{11}$$

the two formulas allow us to find the number n generated by a particular value $\infty > k > \frac{1}{4}$; or vice versa we can find the k resulting in a given number $2 \leqq n \leqq \infty$ of sides per circuit. Remembering that both the cross ratio k and the number n of sides of a polygon are projective properties, we find that any triangle is characterized by $k=1$, any quadrangle by $k=\frac{1}{2}$, the two kinds of pentagons by $k=\frac{1}{2}(3\pm\sqrt{5})$, any hexagon by $k=\frac{1}{3}$, and so forth. If n is irrational or transcendent the nomogon does not close; if it is rational ($n=u/v$, u and v positive integers, $v \leqq u/2$), the nomogon closes after v circuits and has a total number u of sides until it closes. In the classes other than (A) the number of sides per circuit is infinite; yet, each nomogon is characterized by a definite value of k. This is the solution of the old problem of finding a nomogon with a required number of sides.

5. Some properties of the nomogons. It is now but a question of predilection how far one wants to investigate the properties of the nomogons, and a question of space how many of our results we may mention here.

(a) In Figure 1, the parallels RP, SP to the sides AB, BC through the vertices of the similar isosceles triangles ARB, BSC intersect in Lemoine's point P. Therefore, if one keeps AB, BC unchanged and varies ω and thereby k and the number of sides, the locus of P is a straight line through B such that

$$\cot ABP = \cot\alpha - 2\cot(\alpha+\beta). \tag{12}$$

The fact that this line is identical with the so-called symmedian in the triangle

[**10**] and with the diagonal in Moebius' quadrangle explains why the latter can be used to construct the other [**11**].

(b) In Figure 2, O, P, G, G', A, B, and R have the same meaning as in Figure 1; AG is reflected in GO into $A'G$, and BG' is reflected in OP into $B'G$. Putting $\angle AGG'=\phi$, we find $\angle G'GA'=2OGA+AGG'=\pi-(2\omega+\phi)$ and $\angle G'GB'=GG'B=G'BU+G'UB=2\omega+\phi$; therefore, $A'GB'$ is a straight line while the segments $A'G$, GB' equal AG, $G'B$, respectively. With GP produced to K and

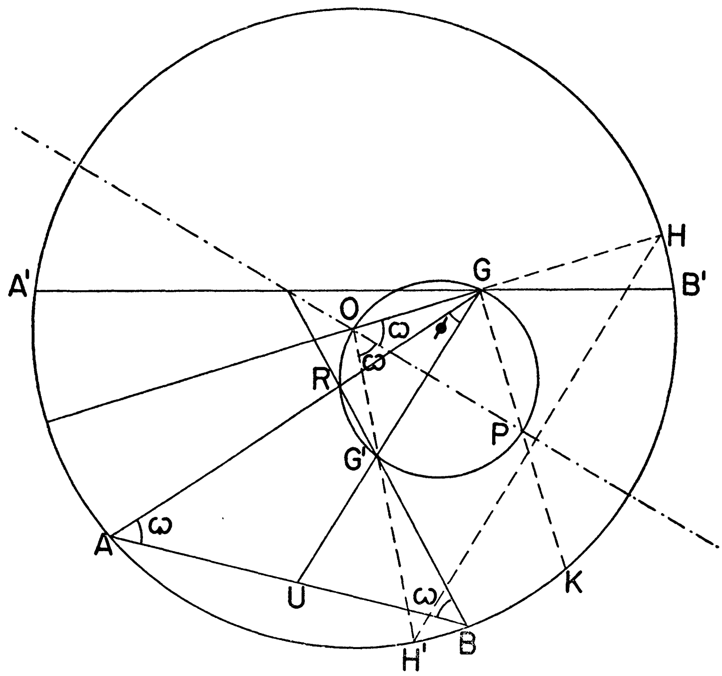

FIG. 2

$OG=f$ we get $AG\times BG'=\overline{GK}^2=-(f^2-r^2)$, and this equation states that the rectangle formed by the two equiangular rays associated with any one side of the nomogon equals the negative value of the power of the equiangular points with respect to the circumscribed circle, and is therefore invariant; it is independent of any other condition, including even angle ω and what it implies.

(c) The Law of Sines applied to triangles ABG and BCG' yields:

$$\frac{AG}{AB}=\frac{\sin(\alpha+\beta+\omega)}{\sin(\alpha+\beta)}; \qquad \frac{BG'}{BC}=\frac{\sin\omega}{\sin(\alpha+\beta)}. \tag{13}$$

We divide the product $AG\times BG'$ found from these formulas by the square $4r^2\sin^2\omega$ of chord HH' intercepted by OG and OG' produced, and find the value of the quotient,

$$(14)\qquad \frac{AG \times BG'}{HH'^2} = \frac{\cot \omega + \cot (\alpha + \beta)}{\cot \alpha + \cot \beta} = \frac{\cot \beta - \cot (\beta + \gamma)}{\cot \alpha + \cot \beta},$$

which equals k according to formula (7). Substituting $r^2 - f^2$ for $AG \times BG'$, we get

$$(15)\qquad k = \frac{r^2 - f^2}{4r^2 \sin^2 \omega},$$

a formula which defines the cross ratio k for a given position of the equiangular points with relation to the circumscribed circle and axis OP. For class (**A**) we find $\infty \geqq k > \frac{1}{4}$ as before, for class (**B**) $k = \frac{1}{4}$, for class (**Ca**) $\frac{1}{4} > k > 0$, and for class (**Cb**) $0 > k \geqq -\infty$.

(d) Formula (15) solved for f is

$$(16)\qquad f^2 = r^2(1 - 4k \sin^2 \omega).$$

This is the equation in polar coördinates with axis OP of the locus of the equiangular points for any given k. The pencil of curves covers the whole plane and, therefore, any two points equally distant from O can be chosen for equiangular points. The locus for the quadrangle is a lemniscate, for nomogons of class (**B**) it consists of two limiting circles.

(e) For the triangle and quadrangle it is known that $\cot \omega = \cot \alpha + \cot \beta + \cot \gamma$ and $\cot \omega = \frac{1}{4}(\cot \alpha + \cot \beta + \cot \gamma + \cot \delta)$, respectively. From formula (8), with α and γ exchanged, and formula (9) we eliminate $\cot(\alpha + \beta)$ and get $\cot \omega = k(\cot \alpha + 2 \cot \beta + \cot \gamma) - \cot \beta$. Let the nomogon be closed and its total number of sides be u as defined above; then we add u such equations, one for each of the u sets of three successive subtended angles, and divide by u. We find

$$(17)\qquad \cot \omega = \frac{4k - 1}{u} (\cot \alpha + \cot \beta + \cot \gamma + \cdots).$$

With $k = 1$, $u = 3$, and with $k = \frac{1}{2}$, $u = 4$ this general formula yields the above equations for the triangle and the quadrangle, and for the hexagon with $k = \frac{1}{3}$, $u = 6$, for instance, we find that the coefficient is $1/18$.

6. Properties of nomogons explained by modern geometry. In the following paragraphs we propose to show another approach to our problem which explains the basic properties of the nomogons by the methods of modern geometry and goes far in deducing even their particular geometrical properties by such methods.

Similar to our procedure in Figure 1, we assume that in Figure 3 three vertices ABC and Brocard's angle ω are given. We produce AG and BG to A' and B', respectively, on the circumscribed circle, and find that arc $BA' = CB'$ because they are intercepted by equal inscribed angles. By subtracting the common part CA', we get $A'B' = BC$. We produce CG to C' on the same circle and rotate $B'C'$ about O back into CD. The angle of rotation $B'OC = 2B'BC = 2\omega$.

When the procedure is repeated, two polygons $ABC \cdots$ and $A'B'C' \cdots$ result which are (a) congruent, (b) perspective with center G, and (c) rotated about O by angle 2ω.

It has yet to be proved that (d) the inscribed polygons have a second equiangular point to make them nomogons, and also, in order to ensure complete generality, a proof is needed that (e) all vertices of the two nomogons are necessarily on the same circle. Locate N' on the circle so that $N'A'B'C' \cong ABCD$. Then, since $\angle BB'A' = BAA' = \omega$ and arc $BN' + N'A' = AB + BN'$, $\angle AA'N' = A'AB = \omega$. When triangle $A'B'G$ is rotated through the angle 2ω about O, it

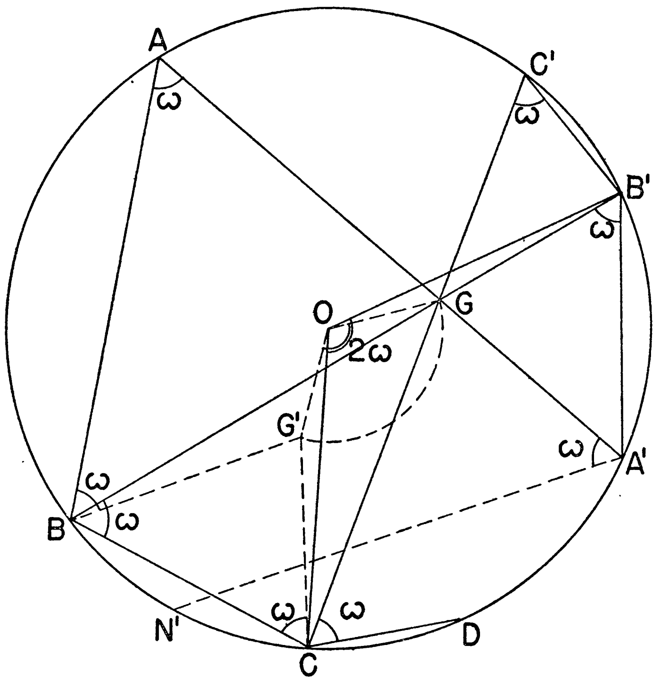

FIG. 3

will coincide with triangle BCG', and similarly the others. This proves not only the existence of the second equiangular point but also that $GO = G'O$ and $\angle GOG' = 2\omega$. On the other hand, $\angle CC'B'$ which intercepts the same arc as $\angle COB' = 2\omega$, equals ω only if C' is on the same circle as points C, B, and B'. Hence D and all following vertices of nomogon $ABC \cdots$ lie on circle ABC. The following discussions are based on the five facts thus proved.

The pairs of points AA', BB', $CC' \cdots$ are in involution because through projection from G point $A \to A' \to A$, $B \to B' \to B$, $\cdots$. The same correspondence is produced by a transformation T which transforms A into B, B into C, $\cdots$ and a rotation R by 2ω which transforms B into A', C into B', $\cdots$. Thus we have

$$I = RT, \quad (RT)^2 = 1. \tag{18}$$

If RT is an involution I, $R^{-1}T^{-1}$ is also an involution I' with center G'. Thus the fact that every inscribed polygon with one equiangular point has another equiangular point is deduced from general principles.

Because both I and R are projective transformations, transformation T is also projective. Thus it is proved that the vertices of a nomogon are found by iterated linear transformation and that, therefore, the cross ratio of four consecutive vertices is constant. The period of transformation T determines the number of sides of the nomogon.

It is well known that the formula

$$\frac{w'-a'}{w'-b'} : \frac{c'-a'}{c'-b'} = \frac{z-a}{z-b} : \frac{c-a}{c-b}, \tag{19}$$

where all numbers are complex, and the cross ratio on each side is real, represents a linear transformation of a circle into itself when three pairs of points aa', bb', cc' are given; it becomes an algebraic expression of transformation T if we put $a'=b$, $b'=c$, and $c'=d$.

Rotation R may be written

$$w = w'e^{2i\omega} \tag{20}$$

and thus the product RT is found to be an involution ($\alpha+\delta=0$) if

$$e^{2i\omega} = \frac{b(d-c)(c-a) - a(d-b)(c-b)}{b(d-c)(c-a) - c(d-b)(c-b)}. \tag{21}$$

This formula allows us to calculate the angle ω corresponding to four consecutive vertices. On the other hand, we have

$$\frac{d-a}{d-b} : \frac{c-a}{c-b} = 1 - k \tag{22}$$

and hence

$$d = \frac{c(a-b) + kb(c-a)}{(a-b) + k(c-a)} \tag{23}$$

which, substituted in formula (21), yields:

$$e^{2i\omega} = \frac{a(a-b)(c-b) + kb(c-a)^2}{c(a-b)(c-b) + kb(c-a)^2}. \tag{24}$$

Thus we are able to calculate a nomogon with a given number of sides when three consecutive vertices are given, and also to determine the angle ω which generates this nomogon. Note that formulas (21) to (24) accomplish the same as formulas (3), (7), (8), and (9).

Because involution I' with center G' is generated from involution I with center G by a rotation R^{-1}, it follows that $R^{-1}(G)=G'$. On the other hand, $RT(G)$

$=G$ implies that $T(G)=R^{-1}(G)$ and hence generates G'. Thus G is transformed into G' both by the euclidean rotation R^{-1} about O and by the non-euclidean rotation T about P. We conclude that G' is symmetrical to G with respect to axis OP and $\angle OPG=G'OP=\omega$. A transformation $RT(P)=P'$ determines a line PP' which, like any other line connecting a pair of conjugate points produced by means of involution I, passes through point G. On the other hand, $RT(P)=R(P)$ and therefore $PO=P'O$ and, considering that $\angle GOP$ has just been proved to equal ω, $\angle GOP'=\omega=GOP$. From the last three statements it follows that OGP and, of course, $OG'P$ are euclidean right angles and that the equiangular points G and G' are on a circle with diameter OP. It also follows that P is Lemoine's point.

Let K be one imaginary circular point at infinity. A line through K and $T^{-1}(K)$ is tangent to the inscribed conic. But because $R(K)=K$, we have $RT(T^{-1}(K))=K$ and, since a line through the conjugate pair $T^{-1}(K)$ and $RT(T^{-1}(K))$ passes through G, tangents to the conic from the imaginary circular points pass through G and G' and therefore, according to the usual projective definition, the equiangular points are foci of the inscribed conic.

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2. Nouvelles Annales de Mathématiques. Deuxième Série, Tome 14. 1875. Page 192. Question 1166: Trouver, à l'intérieur d'un triangle rectiligne ABC, un point O tel que les angles OAB, OBC, OCA soient égaux. Page 286. Solution by M. C. Chadu mentioning the second equiangular point and giving formulas for ω, the equiangular rays and so on.

3. Mostly in mathematical journals like: Comptes rendus de l'Association Française pour l'Avancement des Sciences, Mathésis, and many others as quoted in the Enzyklopädie der Mathematischen Wissenschaften. Vol. III.1.2.: Neuere Dreiecksgeometrie by G. Berkhan and W. Fr. Meyer, p. 1181.

4. Some properties of a quadrilateral in a circle, the rectangles under whose opposite sides are equal. Reprints from Educational Times, 1886. 44, 125–135.

5. (Crelles) Journal f.d. reine u. angewandte Mathematik. Vol. 52. 1856. Pages 239–241: Harmonische Lage von vier Punkten (Harmonic position of four points).

6. The word, isogonal, might have been preferable, but it is used for another conception (Enzykl. d. Math. Wissensch. as quoted in ref. 3), p. 1218). The expressions, nomogon, equiangular ray, and limiting circle, have also been coined by the author.

7. Die Grundlagen der Brocardschen Geometrie des Dreiecks und die Erweiterung auf das Vieleck (The basis of Brocard's geometry of the triangle and its generalization for the polygon). Jahresbericht d. Mathematiker-Vereinigung. 39. Heft 1/4. Pages 81–88.

8. Instead of the usual way of distinguishing the two equiangular points according to the clockwise or anticlockwise rotation $GA\rightarrow AB$, I have found it more convenient to distinguish them according to the lettering or numbering of the vertices.

9. Mathésis II, 1882, p. 40. Solution to question 68 by E. Lemoine. Point P is defined as the point for which $x^2+y^2-z^2$ is a minimum if x, y, z are the distances from the sides of a triangle. The solution is $x:-a=y:-b=z:c$.

10. Mathésis V, 1885. Lemoine, Propriétés diverses du cercle et de la droite de Brocard.

11. A very elegant construction of Moebius' quadrangle and accordingly of the symmedian has been devised by E. Eckhardt (Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht 36, p. 417).

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

G. W. MACKEY, Harvard University

The following results of the eighth annual William Lowell Putnam Mathematical Competition held March 20, 1948, have been determined in accordance with the rules of the Competition agreed to by the representatives of the Mathematical Association and the trustees of the William Lowell Putnam Intercollegiate Memorial Fund. The contestants were known to the grader only by number.

The first prize, four hundred dollars, is awarded to the Department of Mathematics of Brooklyn College, Brooklyn, New York. The members of the team were L. Geller, M. Hausner, S. Schwartzman; to each of these a prize of forty dollars is awarded.

The second prize, three hundred dollars, is awarded to the Department of Mathematics of the University of Toronto, Toronto, Ontario, Canada. The members of the team were G. F. D. Duff, J. M. Kennedy, I. A. Rosenberg. To each of these a prize of thirty dollars is awarded.

The third prize, two hundred dollars, is awarded to the Department of Mathematics of Harvard University, Cambridge, Massachusetts. The members of the team were W. Stinespring, D. L. Yarmush, A. Zemach; to each of these a prize of twenty dollars is awarded.

The teams of the Departments of Mathematics of the College of the City of New York and McGill University, Montreal, Quebec, Canada, tied for fourth place. The fourth prize of one-hundred dollars, will be divided equally between the two schools. The members of the team of the College of the City of New York were D. J. Newman, M. Davis, E. Fischer. The members of the team of McGill University were H. Gonshor, B. A. Rattray, and D. D. Patterson. To each of these a prize of five dollars is awarded.

The six* persons ranking highest in the examination, named in alphabetical order, were: G. F. D. Duff, University of Toronto; L. Geller, Brooklyn College; H. Gonshor, McGill University; R. L. Mills, Columbia University; D. J. Newman, College of the City of New York; E. L. Whitney, University of Alberta. Each of these will receive a prize of forty dollars.

The four succeeding persons ranking highest in the examination, named in alphabetical order, were: J. Ehrman, University of Pennsylvania; M. Gerstenhaber, Yale University; M. Hausner, Brooklyn College; W. F. Stinespring, Harvard University. To each of these a prize of twenty dollars is awarded.

The following teams, named in alphabetical order, won honorable mention: University of Alberta, Edmonton, Alberta, Canada, the members of the team being I. A. Lesk, Mrs. L. Parks, E. L. Whitney; Carnegie Institute of Technology, Pittsburgh, Pennsylvania, the members of the team being J. F. Nash, D. L.

* Here two men tied for fifth place; hence the first six instead of the usual first five.

Wallace, H. F. Weinberger; Columbia University, New York, New York, the members of the team being R. L. Feldmann, C. F. Langley, R. L. Mills; University of Pennsylvania, Philadelphia, Pennsylvania, the members of the team being J. Ehrman, M. E. Hamstrom (Miss), and W. J. Turanski; Yale University, New Haven, Connecticut, the members of the team being L. Carteret, M. Gell-mann, and M. Gerstenhaber.

Eleven individuals were given honorable mention. The names are listed in alphabetical order. M. Davis, College of the City of New York; M. Djorup (Miss), Ursinus College; M. Gutterman, College of the City of New York; J. F. Nash, Carnegie Institute of Technology; J. B. Patterson, University of Toronto; B. A. Rattray, McGill University; I. A. Rosenberg, University of Toronto; S. Schwartzman, Brooklyn College; R. Spinelli, College of the City of New York; H. F. Weinberger, Carnegie Institute of Technology; D. L. Yarmush, Harvard University.

The following is a list of all colleges and universities which entered teams in the competition. The list, in alphabetical order, is: University of Alberta, University of British Columbia, Brooklyn College, Carleton College, Carnegie Institute of Technology, City College of the College of the City of New York, College of St. Thomas, Columbia University, Harvard University, Howard University, Macalester College, Massachusetts Institute of Technology, McGill University, University of Michigan, University of Oregon, University of Pennsylvania, Queen's College (Flushing, N. Y.), Queen's University (Kingston, Ontario), Swarthmore College, Texas Technological College, University of Toronto, Ursinus College, United States Naval Academy, and Yale University.

The following additional colleges and universities entered individual contestants only: Holy Cross College, Loyola College (Montreal), Oklahoma Agricultural and Mechanical College, Rutgers University, and the University of Saskatchewan.

A total of 120 undergraduates representing 29 institutions took part in the competition.

Participants in the competition were given the following lists of problems.

PART I. THREE HOURS

(Answer the questions in any order and by any method. Show all your work in logical sequence, and indicate your answers clearly. No tables or other books may be used.)

1. What is the maximum of $|z^3-z+2|$, where z is a complex number with $|z|=1$?

2. Two spheres in contact have a common tangent cone. These three surfaces divide the space into various parts only one of which is bounded by all three surfaces; it is "ring-shaped." Being given the radii of the spheres, r and R, find the volume of the "ring-shaped" part. (The desired expression is a rational function of r and R.)

3. Let $\{a_n\}$ be a decreasing sequence of positive numbers with limit 0 such that

$$b_n = a_n - 2a_{n+1} + a_{n+2} \geqq 0$$

for all n. Prove that

$$\sum_{n=1}^{\infty} nb_n = a_1.$$

4. Let D be a plane region bounded by a circle of radius r. Let (x, y) be a point of D and consider a circle of radius δ and center at (x, y). Denote by $l(x, y)$ the length of that arc of the circle which is outside D. Find

$$\lim_{d\to 0} \frac{1}{\delta^2} \int\int_D l(x, y)dxdy.$$

5. If $x_1, \cdots, x_n$ denote the n-th roots of unity, evaluate

$$\pi(x_i - x_j)^2 \qquad (i < j).$$

6. Answer either (a) or (b):

(a) A force acts on the element ds of a closed plane curve. The magnitude of this force is $r^{-1}ds$ where r is the radius of curvature at the point considered, and the direction of the force is perpendicular to the curve; it points to the convex side. Show that the system of such forces acting on all elements of the curve keep it in equilibrium.

(b) Show that

$$x + \frac{2}{3}x^3 + \frac{2}{3}\frac{4}{5}x^5 + \frac{2}{3}\frac{4}{5}\frac{6}{7}x^7 + \cdots = \frac{\arcsin x}{\sqrt{1-x^2}}.$$

PART II. THREE HOURS

(Answer the questions in any order and by any method. Show all your work in logical sequence, and indicate your answers clearly. No tables or other books may be used.)

1. Let $f(x)$ be a cubic polynomial with roots x_1, x_2 and x_3. Assume that $f(2x)$ is divisible by $f'(x)$ and compute the ratios $x_1:x_2:x_3$.

2. "A penny in a corner." A circle moves so that it is continually in contact with all three coordinate planes of an ordinary rectangular system. Find the locus of the center of the circle.

3. If n is a positive integer, prove that

$$[\sqrt{n} + \sqrt{n+1}] = [\sqrt{4n+2}],$$

where $[x]$ denotes as usual the greatest integer not exceeding x.

4. Let $\min(x, y)$ denote the smaller of the numbers x and y. For what λ's does the equation

$$\int_0^1 \min(x, y) f(y) dy = \lambda f(x)$$

have continuous solutions which do not vanish identically in (0, 1)? What are these solutions?

5. The pairs of numbers (a, b) such that $|a+bt+t^2| \leqq 1$ for $0 \leqq t \leqq 1$, fill a certain region in the (a, b)-plane. What is the area of this region?

6. Answer *either* (a) or (b):

(a) Let V_1, V_2, V_3 and V denote four vertices of a cube. V_1, V_2 and V_3 are next neighbors of V, that is, the lines VV_1, VV_2 and VV_3 are edges of the cube. Project the cube orthogonally onto a plane (the z-plane, the Gaussian plane) of which the points are marked with complex numbers. Let the projection of V fall in the origin and the projections of V_1, V_2 and V_3 in points marked with the complex numbers z_1, z_2 and z_3, respectively. Show that $z_1^2+z_2^2+z_3^2=0$.

(b) Let a_{ij} be a determinant in which each diagonal element exceeds in absolute value the sum of the absolute values of the other elements of its row, that is $|a_{ii}| > |a_{i1}| + |a_{i2}| + \cdots + |a_{i,i-1}| + |a_{i,i+1}| + \cdots + |a_{in}|$. Show that the determinant is not equal to zero. (Consider the corresponding system of linear homogeneous equations.)

MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California and Institute for Numerical Analysis

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THREE CUBIC LOCI

F. C. GENTRY, Arizona State College

The points of contact of the inscribed circle with the sides of a triangle are the vertices of the pedal triangle of the incenter. If these points are joined respectively to the corresponding vertices of the given triangle, the lines thus obtained are concurrent in the Gergonne point of the triangle. The lines joining the corresponding vertices of a triangle and the pedal triangle of its circumcenter are concurrent in the median point. We seek the locus of a point y such that the lines joining the vertices of its pedal triangle to the corresponding vertices of the given triangle may be concurrent in a point z, and the locus of z.

Let A_i, a_i and α_i ($i=1, 2, 3$), be respectively the vertices, the lengths of the sides, and the angles of a general triangle. Assume a normal trilinear system of coordinates with $A_1A_2A_3$ as reference triangle.

The pedal triangle of $y:(y_1, y_2, y_3)$ has the vertices [1]:

$$P_i:(0,\ y_j + y_i \cos \alpha_k,\ y_k + y_i \cos \alpha_j) \qquad (i, j, k = 1, 2, 3;\ i \neq j \neq k).$$

The lines joining these points respectively to the vertices $A_i:(1, 0, 0)$ are concurrent if the coordinates of y satisfy the equation

$$\sum y_i(y_j^2 - y_k^2)(\cos \alpha_i - \cos \alpha_j \cos \alpha_k) = 0. \tag{1}$$

This is the well-known cubic of Darboux [2]. It is a member of the anallagmatic pencil of cubics obtained by requiring that a point P and its isogonal conjugate with respect to the triangle be collinear with a variable point of the Euler line [3].

The cevian through A_i and z meets A_jA_k in $Q_i:(0, z_j, z_k)$. Perpendiculars to the sides of the triangle at Q_1, Q_2 and Q_3 are concurrent if the coordinates of z satisfy the equation:

$$\sum a_i^2 z_i^2(z_j \cos \alpha_j - z_k \cos \alpha_k) = 0. \tag{2}$$

This is the cubic of Lucas [2].

The coordinates of z are given in terms of those of y by the transformation:

$$\begin{aligned} \rho z_1 &= (y_3 + y_1 \cos \alpha_2)(y_1 + y_2 \cos \alpha_3) \\ \rho z_2 &= (y_2 + y_1 \cos \alpha_3)(y_3 + y_2 \cos \alpha_1) \\ \rho z_3 &= (y_3 + y_2 \cos \alpha_1)(y_3 + y_1 \cos \alpha_2). \end{aligned} \tag{3}$$

This is essentially a quadratic Cremona transformation of the plane into itself. Its fundamental points are the points at infinity on the altitudes of the triangle through A_1 and A_2 and the symmetric of A_3 as to the circumcenter O. Since all of these points lie on the curve (1), each of the principal curves, the lines joining them in pairs, meets the cubic in one further point, which is therefore the analogue of the corresponding fundamental point of the inverse transformation:

$$\begin{aligned} \rho' y_1 &= z_1z_3 - z_2(z_3 \cos \alpha_3 - z_1 \cos \alpha_1) \\ \rho' y_2 &= z_2z_3 - z_1(z_3 \cos \alpha_3 - z_2 \cos \alpha_2) \\ \rho' y_3 &= z_3^2 - (z_1 \cos \alpha_1 - z_3 \cos \alpha_3)(z_2 \cos \alpha_2 - z_3 \cos \alpha_3). \end{aligned} \tag{4}$$

The fundamental points of (4) are the vertices A_1 and A_2 and the ex-median point opposite A_3. They also lie on the curve (2) so that the principal curves of (4) meet the cubic in one variable point each, which therefore corresponds to the fundamental point of (3). There is therefore a one-to-one correspondence, without exception, between the points of (1) and the points of (2).

The lines joining the points of contact of the excircles, and the sides to

which they are relative, to the opposite vertices of the given triangle are concurrent in the Nagel point N. The points of contact are the feet of perpendiculars dropped from the excenters on the corresponding sides of the triangle, which are also known to be concurrent. The excenters, themselves, are the so-called associated points of the incenter. In a manner similar to that outlined above, it may easily be shown that if a point y traces the 17-point cubic:

$$\sum a_i y_i^2 (a_j y_k - a_k y_j) = 0, \tag{5}$$

then the perpendiculars let fall from the associated points of y on the corresponding sides of the triangle are concurrent in a point of the cubic of Darboux (1) and the lines joining the feet of these perpendiculars to the opposite vertices of the triangle are concurrent in a point of the cubic of Lucas (2). The 17-point cubic is a member of the anallagmatic pencil mentioned above [4].

In each case the transformation and its inverse are quadratic. The fundamental points lie on the curve concerned so that a one-to-one correspondence is set up between the two curves.

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ANALLAGMATIC CUBICS

J. M. FELD, Queens College, New York

In a paper which appeared recently in this MONTHLY [4], C. E. Noble discussed some properties of two cubic curves which remain invariant under the isogonal transformation

$$x_1' : x_2' : x_3' = x_2 x_3 : x_3 x_1 : x_1 x_2, \tag{1}$$

where the x_i are trilinear coordinates of a point in the plane. The cubics discussed by Noble have the equations

$$\alpha_1 x_1 (x_2^2 - x_3^2) + \alpha_2 x_2 (x_3^2 - x_1^2) + \alpha_3 x_3 (x_1^2 - x_2^2) = 0 \tag{A}$$

and

$$\alpha_1 x_1 (x_2^2 + x_3^2) + \alpha_2 x_2 (x_3^2 + x_1^2) + \alpha_3 x_3 (x_1^2 + x_2^2) + \alpha_4 x_1 x_2 x_3 = 0. \tag{B}$$

Although these cubics have received considerable attention in connection with the geometry of the triangle (many references are given by Goormaghtigh [3] and Patterson [6]), it appears to be generally overlooked that all non-singular cubics can, by a suitable choice of the fundamental triangle, be reduced both

to form (A) and to form (B). Singular cubics can be reduced to form (B).

It was proved by Ogura [5] that a cubic C_3 invariant under (1) either passes once through each vertex of the fundamental triangle or else it passes twice through one vertex and once through another. In the former case the equation of C_3 assumes form (A) or form (B). Ogura's results remain valid when the x_i are assumed to be projective coordinates rather than trilinear. The present writer [1, 2] then showed that it is possible, in an infinite number of ways, to choose the fundamental triangle so that the equation of any non-singular C_3 takes either form (A) or form (B), and that therefore every non-singular C_3 is an anallagmatic cubic with respect to infinitely many triangles. Singular cubics, since they can be reduced to form (B), are, of course, also anallagmatic with respect to suitably chosen fundamental triangles.

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ON ANALLAGMATIC CUBICS

R. GOORMAGHTIGH, Bruges, Belgium

In his article on anallagmatic cubics (this MONTHLY, vol. 55, 1948, p. 7) C. E. Noble has given several interesting geometric interpretations of the cubic

$$\sum LX(Y^2 + Z^2) + KXYZ = 0$$

in trilinear coordinates.

A remarkable interpretation of the kind has been mentioned by Bouvaist in MATHESIS, 1927, p. 211: *the locus of a point whose pedal circle as to a triangle has a constant power as to a given point is an anallagmatic cubic of the considered type.*

When the cubic splits into a straight line and its isogonal conic, we find the well known theorem of Lemoyne, according to which the pedal circles of the points of a straight line as to a triangle have a constant power as to a fixed point, the orthopole of that line with respect to the triangle.

Bouvaist's paper also contains, p. 214, the geometric interpretation of the cubic as locus of the foci of inscribed conics tangent to a given straight line; this is Theorem 4 in Noble's article and also Theorem 6, as the inscribed conics tangent to a straight line have their centers on another straight line.

ON THE ALTITUDES OF THE TRIANGLE AND OF THE TETRAHEDRON

V. Thébault, Tennie, Sarthe, France

1. Let A_1, B_1, C_1 be the points which divide the altitudes AA', BB', CC' of a triangle $T \equiv ABC$ in the same ratio

$$A_1A : A_1A' = B_1B : B_1B' = C_1C : C_1C' = m : n.$$

Consider masses ka^2, kb^2, kc^2 proportional to the squares a^2, b^2, c^2 of the sides BC, CA, AB, applied at A, B, C and A_1, B_1, C_1. The centroid of these six masses coincides with that of the masses

$$k(m+n)a^2, \qquad k(m+n)b^2, \qquad k(m+n)c^2 \tag{1}$$

applied at the feet A', B', C' of the altitudes of T. But the centroid of masses ka^2, kb^2, kc^2 applied at A, B, C and that of the masses (1) applied at A', B', C' coincide with the Lemoine (symmedian) point K of T [1]. The same holds for the centroid of masses ka^2, kb^2, kc^2 applied at A_1, B_1, C_1. If A_1, B_1, C_1 are on a line Δ, then K is on Δ [2]. There are two lines Δ_1, Δ_2 which divide the altitudes of triangle T in the same ratio. These lines Δ_1, Δ_2 intersect at the Lemoine point K of triangle T [3].

2. Let the points A_1, B_1, C_1, D_1 divide the altitudes AA', BB', CC', DD' of an arbitrary tetrahedron $T \equiv ABCD$ in the same ratio

$$A_1A : A_1A' = B_1B : B_1B' = C_1C : C_1C' = D_1D : D_1D' = m : n.$$

If we apply at the vertices A, B, C, D, and at the points A_1, B_1, C_1, D_1, masses $k\mathfrak{A}^2$, $k\mathfrak{B}^2$, $k\mathfrak{C}^2$, $k\mathfrak{D}^2$ proportional to the squares of the areas of the faces BCD, CDA, DAB, ABC, the centroid of these eight masses coincides with that of the masses

$$k(m+n)\mathfrak{A}^2, \qquad k(m+n)\mathfrak{B}^2, \qquad k(m+n)\mathfrak{C}^2, \qquad k(m+n)\mathfrak{D}^2$$

applied at the feet A', B', C', D' of the altitudes of T. On the other hand, we know that the centroid of masses $k\mathfrak{A}^2$, $k\mathfrak{B}^2$, $k\mathfrak{C}^2$, $k\mathfrak{D}^2$ located at A, B, C, D coincides with the point such that the sum of the squares of its distances from the planes of the opposite faces is a minimum (first Lemoine point K of T) [4]. Furthermore, we have shown that the centroid of masses proportional to the squares of the areas of the faces applied at the feet of the altitudes of a tetrahedron coincides also with the point K [5]. The centroid of the masses $k\mathfrak{A}^2$, $k\mathfrak{B}^2$, $k\mathfrak{C}^2$, $k\mathfrak{D}^2$ applied at A_1, B_1, C_1, D_1 coincides therefore with the first Lemoine point K of T.

If the points A_1, B_1, C_1, D_1 are in the same plane (P), then K is on (P). There are three planes (P_1), (P_2), (P_3) which divide the altitudes of an arbitrary tetrahedron in the same ratio [6]. We have there the following result.

THEOREM. *The three planes* (P_1), (P_2), (P_3) *which divide the altitudes of an arbitrary tetrahedron in the same ratio meet at the first Lemoine point of the tetrahedron.*

Hence in this respect there is complete analogy between the triangle and the tetrahedron.

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E.F.B.

CLASSROOM NOTES

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A REMARK ON INVOLUTORY FUNCTIONS

JOHN ACZÉL, Budapest, Hungary

We say that a function is involutory if it is identical with its inverse function; that is if $y=f(x)$ implies $x=f(y)$. For example $y=a/x$; $y=x$; $y=c-x$; $y=x/(x-1)$ are involutory functions. The graphs of the involutory functions are obviously symmetric with respect to the straight line $y=x$. Therefore we can produce the most general involutory function by drawing an arbitrary curve

in one half of the plane divided by the straight line $y=x$ and by reflecting our curve about this straight line into the other half plane. In the remarks to follow I shall give three analytic ways each of which enables us to construct the most general involutory function.

I. Let $\phi\ (x, y)$ be an arbitrary symmetric function. Then every solution $y=f(x)$ of the equation $\phi(x, y)=0$ is evidently an involutory function. If $y=f(x)$ is multi-valued, it is necessary to consider all branches of the function, for a single branch may not be involutory. For example, if $\phi(x, y)=\sin x+\sin y$, the particular solution $y=x+\pi$ is not involutory, but the solution $y=x\pm n\pi$ is involutory. Similar remarks apply to the solution of equations given in II and III below. Not only does this procedure give an involutory function, but all involutory functions can be constructed in this fashion. For if $y=f(x)$ is involutory, then $x=f(y)$; and $f(x)+f(y)-(x+y)=0$; and $\phi(x, y)=f(x)+f(y)-(x+y)$ is symmetric. For example the involutory function $y=x/(x-1)$ is a solution of the equation $x/(x-1)+y/(y-1)-(x+y)=0$. It is also, of course, the solution of other symmetric equations such as: $x+y-xy=0$.

In a similar fashion let $\phi(x, y)$ be antisymmetric in x and y. Then any solution of $\phi(x, y)=0$ is involutory, and any involutory function $y=f(x)$ satisfies the equation $\phi(x, y)=f(x)-f(y)-(x-y)=0$.

II. Let $F(x, y)$ be an arbitrary function of two variables. Then $F(x, y)+F(y, x)=0$ is a symmetric function and $F(x, y)-F(y, x)$ is antisymmetric. From I it follows that any solution of $F(x, y)+F(y, x)=0$ or of $F(x, y)-F(y, x)=0$ is involutory, and that any involutory function satisfies both of these relations if we put $F(x, y)=y-f(x)$.

III. Let $\phi(z)$ be any function of a single variable. Then $\phi(x)+\phi(y)$ is a symmetric function and $\phi(x)-\phi(y)$ is antisymmetric. From I it follows that any solution of $\phi(x)+\phi(y)=0$ or of $\phi(x)-\phi(y)=0$ is involutory, and that any involutory function satisfies both of these equations if we put $\phi(x)=x+f(x)$ in the first case and if we put $\phi(x)=x-f(x)$ in the second case.

A THEOREM IN ELEMENTARY MATHEMATICS

A. D. WALLACE, Tulane University

The following definition is adopted from a textbook on algebra: An extraneous root of $f(x)=0$ is a number which is not a root but which satisfies an equation derived from $f(x)=0$ by permissible operations.

Our purpose is to the prove the

THEOREM: *Any number not a root of* $f(x)=0$ *is an extraneous root.*

Proof: The equation $(x-t)f(x)=0$ is derived from $f(x)=0$ by a permissible operation.

We have (following the author of the textbook) failed to define "equation" and "permissible operation." Such definitions are beyond the scope of this paper. Their significance will, however, be apparent to any thoughtful reader.

In closing we express the hope that our theorem will, in some measure, indicate the importance to be attached to the concept of extraneous root.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 841. *Proposed by C. W. Trigg, Los Angeles City College*

(a) Can any sealed rectangular envelope, after a single straight cut, be folded into two congruent tetrahedra? Will the position of the cut affect the size of the tetrahedra?

(b) How should the cut be made to make the total number of folds and unfolds a minimum?

(c) What should be the relative dimensions of the envelope in order that the tetrahedra be regular?

E 842. *Proposed by H. L. Lee, University of Tennessee*

Determine cubic functions $f(x)$ for which $f(x)=0$ and $f'(x)=0$ have rational roots.

E 843. *Proposed by P. D. Thomas, Washington, D. C.*

Show that the difference in the radii of the two spheres, one equivalent in area, the other equivalent in volume, to an oblate ellipsoid of revolution is of the fourth order in the eccentricity of the generating ellipse.

SOLUTIONS

A Dart Game

E 811 [1948, 248]. *Proposed by H. D. Larson, Albion College, Michigan*

A, B, and C participate in a novel dart game, the targets consisting of three small balloons marked A, B, and C, respectively. At each turn one dart is thrown, the order of the turns being determined in advance by drawing lots. As soon as a balloon is hit and destroyed, the owner of that balloon is eliminated from the game. The balloons are placed in such a manner that there is no danger of destroying a balloon by a dart aimed at another balloon. It is known by all participants that A can hit a balloon 4 out of 5 times, B 2 out of 5 times, and C 2 out of 5 times; this knowledge is used by each player to his best advantage. What is each contestant's chance of winning the game?

Solution by Leo Moser, University of Manitoba. In a two man game, say E *vs.* F, with chance of hit on aim p_1 and p_2 respectively, the probability of E winning if he has first shot is

$$\sum_{i=0}^{\infty} p_1[(1-p_1)(1-p_2)]^i = p_1/[1-(1-p_1)(1-p_2)].$$

It is clear that when all three men are in the game the best strategy for A and B is to try to eliminate each other while C does his best to miss. Thus there are only two essentially different orders of firing, A, B, C and B, A, C, and these have equal probability.

For A to win he must first win a two man game with B and then a two man game with C in which C has first shot. The case for B is similar. Taking these facts into account we find that the chances of A, B, and C winning are 1596/4807 891/4807, 2320/4807, respectively. One observes that the poorest shot has the best chance of winning!

Essentially this problem (somewhat different formulation and different probabilities) is given in Kinnaird, *Encyclopedia of Puzzles and Pastimes*, p. 246. However, the solution there given is incorrect. It is quite likely that certain variations of this problem might have application to economics.

Also solved by Karl Itkin, Roger Lessard, Albert Shaw, John Walker and the proposer.

Lessard, Shaw, and Walker proceeded on the assumption that when all three men are in the game the best strategy for B and C is to try to eliminate A while A's best strategy is to try to eliminate B. Under this assumption they found that the chances of A, B, and C winning are approximately 0.266, 0.304, 0.430. C still has the best chance of winning, but his chances are not so great as under the assumption that when all three men are in the game he should do his best to miss.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4320. *Proposed by V. F. Ivanoff, San Francisco*

Let $f(s, \phi)=0$ be the intrinsic equation of a simple, closed curve; s is the length of arc of the curve, measured from a fixed point P_0 to any point P, and ϕ is the angle between the tangent lines at P and P_0. Find the area enclosed by the curve.

SOLUTIONS

Tangent Circles

4253. [1947, 286]. *Proposed by G. T. Williams, Cambridge, Massachusetts*

Given two tangent unit circles, C_1 and C_2, and their common external tangent T. A third circle, C_3, is drawn tangent to C_1, C_2 and T; C_4 is then drawn tangent to C_1, C_2, and C_3; and so on, each C_{j+1} being tangent to C_1, C_2, and C_j. Find the total area of the aggregate of circles, $C_1, C_2, C_3, \cdots$.

Solution by J. Certaine, Howard University, Washington, D. C. Let r_k be the radius of C_{k+2} and let b_k $(k=2, 3, \cdots)$ be the distance between the point of contact of the circles C_1 and C_2 and the point of contact of C_{k+1} and C_{k+2}. $b_1=1$ is the distance from the point of contact of the circles C_1 and C_2 to their common tangent. Then from a figure we have

$$(1 + r_k)^2 = (b_k - r_k)^2 + 1 \quad \text{or} \quad r_k = b_k^2/2(b_k + 1).$$

But $b_{k+1}=b_k-2r_k=b_k/(b_k+1)$. It is easily verified by induction that $b_k=1/k$, whence

$$r_k = \frac{1}{2k(k+1)} = \frac{1}{2}\left[\frac{1}{k} - \frac{1}{k+1}\right].$$

Therefore the total area is

$$A = 2\pi + \pi \sum_{k=1}^{\infty} r_k^2.$$

Now $r_k^2=\{1/k^2+1/(k+1)^2-2/k(k+1)\}/4$. From the known series

$$\sum_{k=1}^{\infty} 1/k(k+1) = 1, \qquad \sum_{k=1}^{\infty} 1/k^2 = \pi^2/6,$$

there result finally

$$\sum_{k=1}^{\infty} r_k^2 = \{\pi^2/6 + (\pi^2/6 - 1) - 2\}/4, \qquad A = (\pi^3 + 15\pi)/12.$$

Also solved by Norman Anning, W. G. Brady, G. Y. Cherlin, N. J. Fine, Free Jamison, Ou Li, Leo Moser, G. A. Williams, and the Proposer. See also *A Family of Integers and a Theorem on Circles* (this MONTHLY, 1947, 534–536), by Williams and Browne.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Rings and Ideals (The Eighth Carus Mathematical Monograph). By N. H. McCoy. LaSalle, Illinois, The Open Court Publishing Company, 1948. 12+216 pages. $3.00. (Members of the Mathematical Association of America may purchase one copy at $1.75, order to be placed with the Secretary-Treasurer).

This book is an extremely carefully written exposition of the fundamentals of the theory of commutative rings. The central core of this topic is the arithmetical theory of prime and primary ideals. The author not only gives a simple presentation of Krull's results, but also provides many interesting glimpses into other branches of our knowledge of rings. Let us turn, then, to a more detailed consideration of the contents of his book.

The first four chapters comprise a leisurely introduction and include a discussion of polynomial rings, ideals, homomorphisms and isomorphisms, imbedding of a ring into a ring with unit element, and rings of quotients. Many elementary examples of rings are introduced and used to illustrate the general considerations. (This excellent practice is continued throughout the monograph.) The use of the unexplained word *indeterminate* in the definition of *polynomial* and the introduction of *ideals* prior to *homomorphisms* were the only points which seemed inconsistent with the high level of careful exposition maintained elsewhere in these chapters.

The remainder of the book treats selected topics from the theory of commutative rings. A basis for a theory of prime ideals is deduced in Chapter V by means of the Maximum Principle of Zorn. The discussion turns to structure theory in Chapter VI where a commutative ring is characterized as a subdirect sum of fields if and only if its Jacobson radical is zero. This result is applied to Boolean rings, to rings in which $x^p=x$ and $px=0$ for some prime p and all x, and to the regular rings of von Neumann in Chapter VII. Rings of matrices with elements in a commutative ring with unit are treated in Chapter VIII. This theory is due to the author and concerns the relations between the null ideal and the characteristic ideal of a given matrix. Chapter VIII also contains an economical development of the properties of determinants and resultants in a commutative ring with unit element based on the results of the usual theory for fields. Primary ideals and a generalization of the fundamental theorem of arithmetic to commutative rings which satisfy the ascending chain condition are studied in the last chapter. There is an indication at the close of the importance of this theory of ideals in algebraic geometry. A brief bibliography and an index of terms is appended.

The author has made every effort to lead the reader gradually from the very

definition of a ring to the useful and advanced results of his last chapter. There is no question in this reviewer's mind of the great value of this monograph as a text for graduate students. It is true that no attempt is made to cover a wide variety of topics—only those topics are studied which fall naturally into the logical pattern selected. But this is the virtue of the book, since the student is presented with material which is sufficiently general to be suggestive and *useful* in his own efforts at creative work and which is, at the same time, not so difficult as to require an unreasonable amount of time for assimilation.

If the expository powers of this reviewer could match those of the author, it would now be clear that the Eighth Carus Monograph is a welcome addition to the literature of algebra and is truly a report of some of "the best thoughts and keenest researches" in this field.

M. F. SMILEY

Projective and Analytical Geometry. By J. A. Todd. New York, Pitman Publishing Company, 1948. 10+289 pages. $4.50.

This book was written as a textbook for Part II of the Mathematical Tripos at Cambridge. The treatment throughout is in terms of homogeneous real or complex coordinates. Extensive study is made not only of conics and quadric surfaces but also of the twisted cubic. An elementary knowledge of matrices is presupposed and the chapter on collineations is concerned with reduction of matrices to canonical form, and includes the relevant material on elementary divisors. Since the projective properties of geometric figures are those invariant under the transformations of the projective group, in an elegant and natural way the theory of invariants is introduced and related to geometric properties. For example: Given two homogeneous ternary quadratic forms S and S^1, the condition that there exist triangles inscribed in the conic $S^1=0$ which are self-polar with respect to $S=0$ is given by the vanishing of an invariant of S and S^1.

Almost no synthetic proofs are given. The modern theory of introduction of coordinates and the role of the Theorem of Desargues and the Theorem of Pappus are omitted. Reference is made to modular coordinate fields, but none to non-commutative division rings as coordinates.

MARSHALL HALL, JR.

NEW BOOKS RECEIVED

Ebene und Sphärische Trigonometrie. By H. Athen. (Bücher der Mathematik und Naturwissenschaften.) Wolfenbutteler, Wolfenbuttel and Hannover, 1948. 112 pages.

Analytische Geometrie. By W. Blaschke. (Bücher der Mathematik und Naturwissenschaften.) Wolfenbutteler, Wolfenbuttel and Hannover, 1948. 152 pages.

Vektorrechnung. By H. Athen. (Bücher der Mathematik und Naturwissenschaften.) Wolfenbutteler, Wolfenbuttel and Hannover, 1948. 2+90 pages.

Tables of the Bessel Functions of Fractional Order. Vol. I. Prepared by the Computation Laboratory of the National Bureau of Standards. New York, Columbia University Press, 1948. 42+413 pages. $7.50.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activites, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS 1947–48

Mathematics Club, University of Dayton

Biweekly meetings were held at which members presented the following papers:

Bessel functions and their application to frequency modulation, by Daniel Groszewski

A novel trigonometric reduction rule, by J. F. Schell

Dimensional analysis, by P. F. Swift

Atomic models, by H. J. Eichel

Impossibility of trisection of all angles by ruler and compasses, by E. J. Freeh

Hyperbolic functions and their uses, by Thomas Hanlon

Calculus of variation, by C. L. Keller

Mathematical logic, by Frank Levin

Cardan's solution of a cubic equation, by Robert Luthman

Controversy concerning the invention of the calculus, by Thomas Beckert.

The members attended a lecture at the University of Cincinnati given by Professor G. Birkhoff of Harvard University, who spoke on *The effects of small vibrations on incompressible liquids*, and assisted in the Fourth Annual Colloquium of the National Mathematics Honor Society of Secondary Schools at Mother of Mercy High School in Cincinnati, Ohio.

Preceding the annual dinner, Professor R. V. Churchill of the University of Michigan spoke on *Applications of the Laplace transformation.* The Dean of Science Award was conferred upon P. F. Swift for his paper presented during the first semester. The award for the second semester was conferred upon C. L. Keller. The Alumni Calculus Prize to the member of the sophomore class who made the highest achievement in a competitive integral calculus examination was presented to G. T. Ryan.

At commencement the Mathematics Club Alumni Awards of Excellence in advanced mathematics were given to C. L. Keller of the senior class and upon Robert Luthman of the junior class.

Officers during the year were: President, E. J. Freeh; Vice-President, P. F. Swift; Secretary, G. D. Moon; Treasurer, John Quinlisk; Publicity Secretary, Daniel Groszewski; Faculty advisor, Dr. K. C. Schraut.

Mathematics Club, Berea College

The mathematics majors of Berea College, under the supervision of Professors W. R. Hutcherson, G. G. Roberts, D. W. Pugsley and Ruth Porter,

presented on March 20, 1948, a mathematics and astronomy exhibit as a part of the annual Science Open House.

By the use of interesting displays, mechanical devices, graphs and charts, the students planned and demonstrated the following exhibits:

Applications of calculus, by James Howard and Robert Rogers
Mechanical exhibit of derivative, by Robert Lufburrow
Mathematics Club activities, by Sarah Ann Hutcherson and Rosa Lee Case
Statistics, by Nancy Furry, Wanda Hammons, and Delpha Davis
Computing machines, by Betty Shaffer, Nella Walker, and June Puckett
Astronomy and surveying instruments, by Prof. G. G. Roberts and Jack Hale
Conic sections and interesting solids, by Russel Dean and Lee Wickline
Telescope, by Professor D. W. Pugsley
Slide rule, by Charles Hibbits and Andre Rieben
Puzzles, by Jo Crotchfield, Jacky Hopper, and "Doc" Stevens
Concentrations of liquids, by Earl and Ora Lee Skeen
Mechanical drawing, by Ernest Graham and Richard Parker.

As a result of the students' enthusiasm, the Mathematics Club was organized to further promote interest in mathematics and astronomy in the college. At the first meeting the constitution was adopted by the twenty-five charter members.

The officers elected were: President, Lee Wickline; Vice-president, Wanda Hammons; Secretary-treasurer, Earl Skeen; Faculty advisor, Dr. W. R. Hutcherson.

Mathematics Club, Iowa State College

The Mathematics Club of Iowa State College held six regular meetings during the academic year, 1947–48. The purpose of the club, which is sponsored by the *Iowa Alpha* Chapter of *Pi Mu Epsilon*, is to promote interest in mathematics among the freshman and sophomore students of the college. Refreshments were served and the following programs were given at these meetings:

Mathematics in life insurance, by G. A. Harper, vice-president of Bankers Life Insurance Co., Des Moines, Iowa.

Mathematical quiz show, with H. D. Block as master of ceremonies.

Pi, its mathematical uses and origin. Three short papers by J. E. K. Smith, Robert Miller, and William Ritts.

Mathematics in other countries, talks given by student and faculty representatives from India, Russia, Sweden and Poland.

Mechanical aids to computation

Astronomy—A challenge from the heavens, by Prof. Fred Brandner of Iowa State College.

Officers of the club include: Vice-director of Pi Mu Epsilon in charge of the Mathematics Club, Ruth Royer; Program Chairman, Ruth Rychnovsky; Refreshment Committee, Charles Hopper, Norman Burch; Publicity Chairman, James Mairs; Faculty Advisors, Fred Robertson, C. H. Lindahl, R. J. Lambert.

NEWS AND NOTES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

CONFERENCES AT OHIO STATE UNIVERSITY

Two conferences, sponsored by the Department of Mathematics and the Graduate School of Ohio State University, will be held in Columbus on December 27, 1948. One conference will pertain to surface area theory and one to group theory. Information can be obtained by writing to the Department of Mathematics.

PERSONAL ITEMS

Professor Emil Artin of Princeton University has been appointed to one of the Sigma Xi national lectureships for 1948–49.

Professor Elie Cartan, University of Paris, and Professor Marston Morse, Institute for Advanced Study, have been awarded the honorary degree of Doctor of Science by the University of Pisa.

Dr. H. M. MacNeille, chief of the Fundamental Research Branch, Division of Research, United States Atomic Energy Commission, has been awarded the President's Certificate of Merit in recognition of his services as a member of the resident staff of the London Mission of the Office of Scientific Research and Development from February, 1944 to February, 1946. Other mathematicians who have received the Certificate of Merit for outstanding service to the Government are: Dr. Marston Morse, Dr. A. H. Taub, Dr. Oswald Veblen, Dr. S. S. Wilks.

Professor Tibor Radó has been named a corresponding foreign member of the Academy of Science of Bologna, Italy.

Boston University announces the following: Assistant Professor R. N. Johanson has been promoted to an associate professorship; Mr. J. E. Alman, Dr. J. B. Giever, and Dr. F. J. Scheid have been appointed to instructorships.

At Catholic University of America the following appointments to instructorships have been made: Dr. Paul Nesbeda, formerly at the Institute for Advanced Study, and Mr. R. N. Huckins of Suffolk University.

Colorado Agricultural and Mechanical College announces the promotions of Associate Professor H. T. Guard to a professorship and Assistant Professor M. L. Madison to an associate professorship.

Columbia University announces: Associate Professor J. A. Northcott has been promoted to a professorship; Mr. E. R. Kolchin and Mr. Howard Levi have been promoted to assistant professorships; Mr. J. A. Zilber has been appointed Lecturer; Mr. John Rausen has been appointed Assistant.

Cornell University announces the following: Professor W. H. J. Fuchs of the

University of Liverpool is serving as visiting associate professor for the current academic year; Mr. M. S. Donsker of the University of Minnesota and Dr. Christine Williams of Yale University have been appointed to instructorships; two graduate students, Samuel Goldberg and G. M. Wing, received Atomic Energy Fellowships for 1948–49.

At Florida State University: Associate Professor W. H. Spragens of Georgia State Woman's College has been appointed to an associate professorship; Mr. J. L. Bagg, Western Maryland Teachers College, and Miss Frances Dean, graduate student at University of Alabama, have been appointed to instructorships; Miss Margaret Cockrell, formerly teaching assistant, is now employed as mathematician by the Shell Oil Company.

Hastings College announces: Professor R. M. McDill has become Professor Emeritus; Mr. Everett Lowry, formerly of the Veteran's Administration in Nebraska, has received an appointment in the Mathematics Department.

At Indiana University: Professor V. Hlavaty, formerly of Charles University, Prague, has been appointed Visiting Professor; Professor Eberhard Hopf is employed on the ONR project; Professor D. Gilbarg attended New York University last summer in connection with the ONR project work; Professors W. Gustin and T. Y. Thomas attended California Institute of Technology during the summer in connection with the ONR project work.

Iowa State College makes the following announcements: Assistant Professor Bernard Vinograde has been promoted to an associate professorship; Dr. R. A. Griffin has been appointed to an assistant professorship; Mr. R. D. Branstetter, Mrs. M. L. Butler, Mr. R. F. Deniston, Miss Mary B. Lieberknecht, Miss Ruth Royer, and Mrs. C. C. Scott have been appointed to instructorships. Also, Dr. B. R. Seth of Hindu College, Delhi, India will be Visiting Professor of Applied Mathematics during the year 1949; he will give courses in elasticity and hydrodynamics during the Winter and Spring Quarters.

Kent State University announces: Assistant Professor John Kaiser has been promoted to an associate professorship; Associate Professor Foster Brooks is on leave of absence for the year and is connected with the Research and Development Board of the National Military Establishment, Washington, D. C.

At Iowa State Teachers College the following appointments to instructorships have been made: Miss Dorothy DeWitt and Mr. Robert Lankton.

Massachusetts Institute of Technology announces the following: Associate Professor Witold Hurewicz has been promoted to a professorship; Dr. Walter Pitts has been appointed Research Associate in Mathematics and Dr. R. P. Boas, Jr. has been appointed Lecturer. Professors Warren Ambrose and Norman Levinson are on leave of absence this year on Guggenheim fellowships; Professor Ambrose is spending the year at the Institute for Advanced Study and Professor Levinson is at the University of Copenhagen. Professor Henry Wallman, who is on leave of absence, is Visiting Professor of Teletechnics at the Chalmers Institute of Technology in Gothenburg, Sweden.

Miami University reports: Professor H. S. Pollard, formerly acting head, is now Head of the Department of Mathematics; Dr. R. W. Bryant, graduate student at the University of Alabama, has been appointed to an instructorship; Miss Kathryn B. Aldrich, Mr. R. P. Bacon, Mr. R. H. Johnston and Mr. G. E. Witter are newly appointed graduate assistants; Professor Emeritus W. E. Anderson has retired.

Michigan State College announces the following appointments: Dr. L. M. Kelly of the University of Missouri, Dr. Mary H. Payne of the University of Detroit, and Dr. M. D. Springer of the University of Illinois to assistant professorships; Miss Betty S. Grossman, Miss Roslyn Hurwitz, and Mr. Leo Lapidus to instructorships.

At Michigan State Normal College: Miss Mildred J. Brannon, graduate assistant at the University of Wisconsin, has been appointed to an assistant professorship; Assistant Professor R. G. Peterson is now in charge of visual education.

The following promotions have been made at Mississippi State College: Associate Professor Arthur Ollivier to a professorship; Assistant Professor S. B. Murray to an associate professorship; Miss Monica Goen, Mr. A. C. Grimes, and Mr. M. M. Temple to assistant professorships.

Ohio State University reports: Associate Professors Marshall Hall, H. B. Mann, and P. V. Reichelderfer have been promoted to professorships; Assistant Professors H. H. Alden, R. G. Helsel, and E. J. Mickle have been promoted to associate professorships; Instructor R. R. Whitney has been promoted to an assistant professorship; Dr. O. W. Rechard has been appointed to an instructorship.

Oklahoma Agricultural and Mechanical College makes these announcements: Professor J. C. C. McKinsey has been granted a leave of absence to participate in the Rand Project at Douglas Aircraft, Santa Monica, California; Associate Professor P. E. Lewis has been granted a leave of absence for teaching and advanced study at North Carolina State; Assistant Professor R. R. Reynolds has been granted a leave of absence for advanced study at Harvard University; Dr. Nachman Aronsazjn has been appointed Visiting Professor for the period October 18 to December 18; Dr. M. L. Richards has been appointed Instructor and Research Assistant.

Rutgers University announces the following promotions: Instructors Erwin Biser, L. M. Court, and L. M. Rauch to assistant professorships.

Texas State College for Women reports: Professor Harlan C. Miller has been appointed Director of the Department of Mathematics; Miss Jean Beal, formerly graduate assistant at Oklahoma Agricultural and Mechanical College, has been appointed to an instructorship; Dean F. V. White has retired with the title of Professor Emeritus.

The University of California at Los Angeles announces the following promotions and appointments: Associate Professor P. G. Hoel has been promoted

to a professorship; Assistant Professors J. W. Green and W. T. Puckett have been promoted to associate professorships; Instructors Leonard Greenstone and Alfred Horn have been promoted to assistant professorships; Dr. Robert Steinberg, University of Toronto, and Dr. E. G. Straus, formerly of the Institute for Advanced Study, have been appointed to instructorships.

At the University of Delaware: Associate Professor G. C. Webber has been promoted to a professorship; Dr. R. F. Jackson, formerly a research mathematician with York Refrigerating Company, York, Pennsylvania, has been appointed Assistant Professor; Mr. Morris Neuman of the University of Pennsylvania has been appointed Instructor.

The University of Denver announces the promotions of Professor A. J. Lewis to the Chairmanship of the Department of Mathematics and Assistant Professor K. L. Noble to the associate chairmanship.

The University of Detroit makes the following announcements: Associate Professor L. E. Mehlenbacher has been made Professor and Director of the Department of Mathematics; Professor Jerzy Lubelfeld of the Polish Naval College and Polish Technical Training Center, Okehampton, Devon, England has been appointed Assistant Professor: Assistant Professor Emily C. Pixley of Wayne University has been appointed to an assistant professorship; Assistant Professor G. E. Markle has been granted a leave of absence for graduate study at Harvard University.

Dr. H. A. Arnold has been appointed to an instructorship at the University of California at Davis.

Miss Laura Blakeley, who has been teaching at Dayton Senior High School, Kentucky, is now an instructor at Armstrong Junior College.

Miss Margaret Bootz has been appointed to an instructorship at Marietta College.

Associate Professor H. W. Brinkmann of Swarthmore College has been promoted to a professorship.

Mr. Paul Brock of Hunter College has accepted a position as mathematician with the Reeves Instrument Company of New York City.

Associate Professor D. M. Brown of Central Michigan College has been appointed Supervisor of the Applied Mathematics Group, Aeronautical Research Center, University of Michigan.

Professor Emeritus W. D. Cairns of Oberlin College is serving as part-time lecturer in mathematics at California Institute of Technology for the year 1948–49.

Assistant Professor R. H. Cole of the University of Western Ontario has been appointed Research Associate at Princeton University.

Mr. R. A. Deutsch of the Triborough Bridge and Tunnel Authority has accepted a position as electrical engineer with the Kellex Corporation of New York City.

Professor Alexander Dillingham of the United States Naval Academy has retired.

Dr. Mary P. Dolciani, who has been at the Institute for Advanced Study, has been appointed to an instructorship at Vassar College.

Mr. E. J. Downie of Colgate University has been promoted to an assistant professorship.

Mr. R. P. Eddy has joined the Applied Mathematics Section of the Mechanics Division, Naval Ordnance Laboratory, Silver Spring, Maryland.

Assistant Professor A. B. Farnell of the University of Colorado has been appointed Lecturer at Princeton University.

Miss Virginia I. Felder of Odessa Junior College has been appointed to an assistant professorship at Mississippi Southern College.

Assistant Professor Landis Gephart of the University of Dayton has accepted a position as mathematician with the United States Air Force.

Mr. M. T. Goodrich, Keene Teachers College, New Hampshire, has been promoted to an assistant professorship.

Mr. Arthur Grad has accepted an appointment as mathematician with the Mathematics Branch of the Office of Naval Research at Washington, D. C.

Mr. G. B. Haggerty of Rhode Island State College has been promoted to an assistant professorship.

Associate Professor F. F. Helton of Central College has been promoted to a professorship.

Assistant Professor Manuel Herschdorfer, Seton Hall College, has been promoted to a professorship.

Mr. R. J. Howerton of the University of Denver has been appointed Instructor in Mathematics and Physics at Regis College.

Associate Professor S. A. Jennings of the University of British Columbia has been promoted to a professorship

Dr. C. M. Jensen, University of Minnesota, has been appointed to an assistant professorship at Augustana College.

Professor B. W. Jones of Cornell University has accepted a professorship at the University of Colorado.

Mr. H. P. Kean has been appointed to an associate professorship at McMurry College.

Assistant Professor Fred Kiokemeister, Mount Holyoke College, has been promoted to an associate professorship.

Professor W. I. Layton of Austin Peay State College has accepted an appointment as associate professor at Alabama Polytechnic Institute.

Assistant Professor A. T. Lonseth of Northwestern University has been appointed to an associate professorship at Oregon State College.

Assistant Professor Kenneth May, Carleton College, has been promoted to an associate professorship.

Assistant Professor P. E. Meadows of Washington and Lee University has been appointed to an associate professorship at Carroll College.

Professor L. I. Mishoe of Delaware State College has been appointed to an associate professorship of physics at Morgan State College.

Associate Professor E. W. Montroll of the University of Pittsburgh has been promoted to a professorship.

Mr. M. R. Moore has been appointed to an instructorship at Bowling Green State University.

Dr. F. M. Morgan, former headmaster of Clark School, Hanover, New Hampshire, has been appointed Treasurer of Clark School Foundation.

Dr. E. N. Nilson, who has been serving as an aeronautical engineer, Research Division, United Aircraft Corporation, has been appointed to an assistant professorship at Trinity College.

Assistant Professor S. T. Parker of Kansas State College has been promoted to an associate professorship.

Professor A. W. Philips of Kansas State Teachers College has been appointed to a professorship at Eastern Washington College of Education.

Assistant Professor D. H. Rock of Iowa State College is now engaged in research work with the United States Army Air Force, Washington, D. C.

Mr. C. T. Rodney of Sampson College has been appointed Assistant Professor at New York State Teachers College, Buffalo.

Dr. R. H. Stark of Northwestern University is now employed at Los Alamos Scientific Laboratory, New Mexico.

Assistant Professor J. C. Stewart, Lawrence College, has been promoted to an associate professorship.

Assistant Professor Feodor Theilheimer of Trinity College has joined the Naval Ordnance Research Laboratory.

Dr. D. L. Thomsen, Jr. of Haverford College has been promoted to an assistant professorship.

Assistant Professor Eleanor B. Walters, Delta State Teachers College, has been promoted to an associate professorship.

Associate Professor C. P. Wells, who is on leave of absence from Michigan State College, is Lecturer at Brown University.

Assistant Professor G. N. Wollan of Sampson College has been appointed to an associate professorship at North Georgia College.

Dr. Marie A. Wurster of Temple University has been promoted to an assistant professorship.

Dr. Walter Mayer, associate in Mathematics at the Institute for Advanced Study, died on September 10, 1948.

Professor W. A. Wilson of Yale University died on September 21, 1948. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE NINTH ANNUAL WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Professor L. E. Bush of the College of St. Thomas, St. Paul 1, Minnesota, has been appointed Director of the William Lowell Putnam Mathematical Competition by President L. R. Ford for the five-year period 1949–1953. At the recent meeting of the Board of Governors in Madison, Wisconsin, it was voted that the Association should resume direction of the Putnam Competition and the appointment of a Director was authorized. This competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is open to undergraduate students in universities and colleges of the United States and Canada who have not received a degree.

The ninth annual Putnam Competition will be held on Saturday, March 26, 1949. The examination will consist of two parts of three hours each. The questions will be taken from the fields of calculus (elementary and advanced) with applications to geometry and mechanics not involving techniques beyond the usual applications, higher algebra (determinants and theory of equations), elementary differential equations, and geometry (advanced plane and solid analytic geometry). Any college or university wishing to enter a team or individual contestants may secure an application blank from Professor L. E. Bush, 112 Albertus Magnus Hall, College of St. Thomas, St. Paul 1, Minnesota, by a postcard request. All applications must be filed with the Director not later than March 1, 1949. If three candidates are presented from a college or university, they are to constitute a team; if more than three are presented from any one college or university, the team of three must be named on the application. Fewer than three from one college or university may compete as individuals.

The examination may be given at any place where a team, or at least three candidates, can be assembled. Exceptions to this rule may be made by the Director in cases where it would entail unusual inconvenience to a contestant. Sealed copies of the examinations will be sent to the supervisor of the examination in time for the examination day and are not to be opened before the hour set. At the supervisor's first opportunity after the afternoon examination the books are to be sent by registered mail or by express to the Director, who will forward them to a qualified reader chosen by the Association.

The prizes to be awarded to the departments of mathematics of the institutions with the winning teams are \$400, \$300, \$200, and \$100, in the order of their rank. In addition, there will be prizes of \$40, \$30, \$20 and \$10 awarded to the members of these teams according to the rank of the team; a prize of \$50 to each of the five highest contestants and a prize of \$20 to each of the succeeding five highest contestants. Each of the winners will receive a suitable medal. Honorable mention will be given to several teams next in order after the four

winning teams and to several individuals next in order after the ten individual winners. For further encouragement of the Competition, there will be awarded at Harvard University (or at Radcliffe College in the case of a woman) an annual \$1500 William Lowell Putnam Prize Scholarship to one of the first five contestants, this to be available either immediately or on the completion of the student's undergraduate work.

Reports on the eight previous competitions and examination questions will be found in this MONTHLY for May, 1938, 1939, 1940, 1941, 1942, October, 1946, August–September, 1947, and in this issue.

APRIL MEETING OF THE IOWA SECTION

The thirty-fifth annual meeting of the Iowa Section of the Mathematical Association of America was held at Parsons College, Fairfield, Iowa, on Friday and Saturday, April 16 and 17, 1948, in conjunction with the Iowa Academy of Science. Professor H. P. Thielman, Chairman of the Section, presided.

The attendance was seventy, including the following twenty-nine members of the Association: E. W. Anderson, E. L. Canfield, E. W. Chittenden, Marian E. Daniells, W. M. Davis, R. E. Gaskell, B. E. Gillam, Cornelius Gouwens, F. S. Harper, Gertrude A. Herr, E. H. C. Hildebrandt, J. J. L. Hinrichsen, L. A. Knowler, O. C. Kreider, C. E. Langenhop, Walter Lyche, R. B. McClenon, F. M. McGaw, J. V. McKelvey, E. E. Moots, E. N. Oberg, Fred Robertson, R. M. Robinson, E. R. Smith, L. W. Swanson, H. P. Thielman, Henry Van Engen, Roscoe Woods, E. A. Zubay.

The following officers were elected for the coming year: Chairman, Professor W. M. Davis, Cornell College; Vice-Chairman, Professor B. E. Gillam, Drake University; Secretary, Professor Fred Robertson, Iowa State College.

Professor Hildebrandt delivered an invited address at 4:00 P.M. on Friday. Five of the following papers were read Friday afternoon, and the remainder Saturday morning.

1. *A use of matrices in accounting*, by Professor E. S. Allen, Iowa State College, introduced by the Secretary.

Let an organization have n service departments $S_1, \cdots, S_n$, acquiring respective surpluses $s_1, \cdots, s_n$; let the fraction of s_i attributable to aid given by S_a be l_{ai}, the fraction attributable to producing department P_α be $m_{\alpha i}$ ($\sum_{a=1}^{n} l_{ai} + \sum_{\alpha=1}^{\nu} m_{\alpha i} = 1$). If the producing departments $P_1, \cdots, P_\nu$ are to receive the total surplus $s_1 + \cdots + s_n$, an equitable distribution will be

$$P = M(1 - L)^{-1}S,$$

P, M, L, S being the matrices of the corresponding lower case letters.

2. *Simultaneous reduction of quadratic forms*, by Professor B. Vinograde, Iowa State College, introduced by the Secretary.

This paper will appear in the *Proceedings of the Iowa Academy of Science.*

3. *Linear recurrence relations between iterated kernels*, by C. E. Langenhop, Iowa State College.

A kernel of an integral equation, whose iterates satisfy a linear relation

$$a_1K_1(x, y) + a_2K_2(x, y) + \cdots + a_nK_n(x, y) = 0, \qquad a \leqq x, y \leqq b,$$

with $a_1 \neq 0$, is a solution of the integral equation $K(x, y) = \int_a^b H(x, t)K(t, y)dt$, where $H(x, t)$ is another linear combination of the iterates. If $K(x, t)$ is bounded, $H(x, t)$ is bounded, and there are then only a finite number of linearly independent $K(x, y_i)$ where $a < y_i < b$, $i = 1, 2, \cdots, n$. Thus $K(x, y)$ must be of the special form $\sum_{i=1}^n \alpha_i(x)\,\beta_i(y)$. For a kernel of this form it is known that the Fredholm determinant $D(\lambda)$ is a polynomial. The converse is not true, but if the Fredholm first minor $D(x, y; \lambda)$ of a bounded $K(x, y)$ is a polynomial in λ, and if $D(\lambda)$ is a polynomial of the same degree, then the iterated kernels satisfy a linear relation as above, and $K(x, \beta)$ is of the special form. If $K(x, y)$ is continuous and symmetric, then it is sufficient for the $D(\lambda)$ merely to be a polynomial that $K(x, y)$ to be of the special form given above.

4. *A third order irregular boundary value problem*, by H. E. Ellington, State University of Iowa, introduced by the Secretary.

Consider the differential system

$$\frac{d^3u}{dx^3} + [\rho^3p(x) + q(x)]u = 0,$$

with boundary conditions $u(0) = u'(0) = u(\pi) = 0$. The transformation $u(x) = h(x)\,R(z)$, $z = \phi(x)$ will reduce the equation to the form

$$\frac{d^3R}{dz^3} + \phi(z)\frac{dR}{dz} + \left[\frac{\phi'(z)}{2} + \psi(z)\right]R + \rho^3R = 0$$

with the boundary conditions $R(a) = R'(a) = R(\pi) = 0$. The characteristic solutions are

$$R_n(z) = -2e^{\rho_n(z-a)/2}\left[\cos\left\{\frac{\pi}{3} + \frac{\sqrt{3}}{2}\rho_n(z-a)\right\} + \delta_n(z)\right]$$

where $|\delta_n(z)| < \delta$ if $n < N$ and $a \leqq z \leqq \pi$.

5. *Construction of parabolas*, by Masihur Rahman, Iowa State College, introduced by the Secretary.

Mr. Rahman showed ways of constructing parabolas and (or) discovering some of their properties by use of compass and straight edge. As one illustration he found the point of intersection of a parabola and a line through the focus with given direction, without the usual procedure of constructing the parabola.

6. *Instructional aids for the teaching of junior college mathematics*, by Professor E. H. C. Hildebrandt, Northwestern University.

Professor Hildebrandt exhibited a number of charts and models. He discussed devices of interest to both mathematics majors and other students. After the address the audience inspected the models and discussed their use with Professor Hildebrandt.

7. *An approximate solution of a differential equation by difference equations*, by Spencer Macy, Iowa State College, introduced by the Secretary.

The theory of approximation by difference equations developed by R. G. D. Richardson is extended to a linear self-adjoint equation

$$[p(x)u''(x)]'' + [q(x)u'(x)]' + [r(x) + \nu k(x)]u(x) = 0 \tag{1}$$

where γ is a parameter. Quadratic forms Q_1 and Q_2 can be found such that $Q_1 = \gamma Q_2$ for the minimum value of Q_1. The set of homogeneous linear equations derived by minimizing Q_1 with respect to

$u(x_i)$ with the constraint $Q_2=1$ is used to find a linear form corresponding to the integral equation equivalent to (1). A method of minimization of Q_1 due to G. H. Shortley is used on the equation of a stiff vibrating string to find the first and second characteristic values.

8. *Mapping of a region on the unit circle by orthogonal polynomials*, by R. N. Goss, Iowa State College, introduced by the Secretary.

In 1921 Szego (*Math. Zeits.*, vol. 9, pp. 218–270) proved that a region in the complex plane bounded by a closed continuous rectifiable curve is mapped on the unit circle $|G(t)| \leq 1$ by the function

$$G(t) = \lim_{n\to\infty} \left\{ \frac{A}{K_n(a, a)} \int_a^t [K_n(a, z)]^2 dz \right\}$$

where $K_n(a, z)$ depends upon a sequence of orthogonal polynomials associated with the bounding curve, and A is a constant. The generality of its application suggested use of the method in the mapping of airfoils. To test the practical characteristics of the mapping method, a region bounded by the minor arcs of two intersecting circles of different radii was chosen, with numerical data compatible with airfoil dimensions. It was found that for this type of region the successive approximations converge slowly, and involve increasingly laborious computation at each stage. It was concluded on the basis of this and other tests that the method can be employed profitably only for regions for which no simpler mapping function is known, and where efficient computing equipment is available.

9. *A note on the relaxation method*, by Professor R. E. Gaskell, Iowa State College.

The relaxation method was applied to a linear system of equations for purposes of illustration, and then to non-linear systems. Some difficulty in convergence may be encountered in non-linear systems. It may be overcome if the residues are not completely liquidated; that is, if the corrections are damped.

10. *A class of integral identities*, by W. H. Marlow, State University of Iowa, introduced by the Secretary.

Mr. Marlow considered families of definite integrals related to the Gamma function by means of the Laplace transform and other transformations.

11. *On the theory of limits in general analysis*, by Professor E. W. Chittenden, State University of Iowa.

The Peano limit of a function on a directed system with values in a topological space is defined and used to establish theorems about iterated limits.

12. *Complete systems of invariants of the cyclic groups of equal order and degree* (by title), by Dr. C. W. Strom, Washington, D. C., introduced by the Secretary.

This paper will be published in the *Proceedings of the Iowa Academy of Science*.

13. *A two-dimensional potential problem with applications to soil drainage*, by Professor Don Kirkham, Iowa State College, introduced by the Secretary.

Professor Kirkham discussed the flow of water in equally spaced rain tiles of uniform depth. These tiles were embedded in one soil medium lying over another of different permeability. He illustrated by sand over clay and vice-versa. He gave a solution to the problem under special conditions.

14. *A general finite difference summation formula*, by P. M. Bailey, State University of Iowa, introduced by the Secretary.

Where the general term $\phi(x)$ of a series satisfies the relation $\phi(x)=f(x+h)-f(x)=\Delta_h f(x)$, various sums of the series are obtainable. The two following formulae giving the definite and indefinite sums of the $\phi(x)$ series, span r, in terms of the function $f(x)$ were presented:

$$\sum_{a=a}^{\alpha+n(-\nu)r}{}_r\ \phi(x) = \sum_{a=0}^{(h/r)-1}{}_1\ \Delta_n f(\alpha + xr)$$

$$\sum{}_r\ \phi(x) = \sum_{k=0}^{(h/r)-1}{}_1\ f(x + kr).$$

The formulae apply when h/r is integral. As an example, the first formula was applied to the summation of a reciprocal factorial function.

FRED ROBERTSON, *Secretary*

APRIL MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The twenty-fifth annual meeting of the Louisiana-Mississippi Section was held at Southwestern Louisiana Institute, Lafayette, Louisiana, on Friday and Saturday, April 23–24, 1948. Professor Z. L. Loflin, Vice-Chairman for Louisiana, presided at the Friday afternoon session, and Professor T. A. Bickerstaff, Vice-Chairman for Mississippi, presided at the Saturday morning session.

There were one hundred six persons registered, including the following thirty-five members of the Association: T. A. Bickerstaff, Ann Buchanan, H. E. Buchanan, Virginia Carlton, J. C. Currie, Margaret R. Davis, W. L. Duren, Katherine S. Foote, W. W. Gandy, L. M. Garrison, F. C. Gentry, M. E. Mills, A. L. Gilmore, W. C. Griffith, R. V. Guthrie, J. A. Hardin, H. T. Karnes, C. G. Killen, Z. L. Loflin, J. W. McClimans, Dorothy McCoy, A. C. Maddox, B. E. Mitchell, C. V. Newsom, I. C. Nichols, R. L. O'Quinn, P. K. Rees, F. A. Rickey, H. F. Schroeder, H. L. Smith, V. B. Temple, J. F. Thomson, B. A. Tucker, P. M. Tullier, A. D. Wallace.

The following officers were elected for one year: Chairman, W. L. Duren, Tulane University; Vice-Chairmen: Eleanor Walters, Delta State Teachers College, and Peter Tullier, Loyola University; Secretary-Treasurer, F. A. Rickey, Louisiana State University. The 1949 meeting will be held at the University of Mississippi, Oxford, Mississippi.

The following papers were presented at the Friday afternoon session:

1. *The inversion of certain integrals*, by Professor D. B. Sumner, Louisiana State University. (Introduced by Professor P. K. Rees.)

Inversion formulas obtained by Widder and Pollard for the generalized Stieltjes transform are used in a modified form to obtain inversion formulas in the form of a contour integral, which is a generalization of the original result of Stieltjes. The argument is based on a new generalization of Cauchy's "singular integral." The method is capable of extension to a large class of integral equations, in particular, to the convolution transform.

2. *Subpedal curves*, by Professor V. B. Temple, Louisiana College.

The author first defined a pedal curve as the locus of the vertex of a right angle, and gave, as examples, the circle, the ellipse, and the witch of Agnesi. He showed four other pedal curves, namely, the differential axle, the racquets, the cane knife, and the boomerang. He then defined a subpedal curve as the vertex of a right angle projected by the vertex of a right angle. As examples of subpedal curves he showed a few drawings and their equations from his collection. These included, the egg, the fig, the pumpkin, and the longhorns. He further showed that the cissoid of Diocles is a subpedal curve and gave its general equation as $X^2+Y^2=abY^2/(aY+bX)$, where a and b are, respectively, the X- and Y-intercepts of the straight line directrix.

3. *Mathematical topics for college freshmen*, by Professor S. W. Shelton, Northwestern State College. (Introduced by the secretary.)

4. *Dimension and compacting*, by Professor A. D. Wallace, Tulane University.

Herein "space" will mean "normal Hausdorff space." A space X will be said to have dimensional type S provided that for each closed subset A of X and each map $f:A\rightarrow S$ there is an extension g of f mapping X into S. For simplicity assume that S is an absolute neighborhood retract. It was shown by Alexandroff and Hurewicz that (assuming X separable metric) X has the dimensional type of the n-sphere if and only if its dimension is at most n. This result has recently been extended by Alexandroff, Dowker and Hemmingsen without the assumption that X be separable metric. They also showed that the dimensions of X and βX (the Chech-Stone-Tychonoff compacting of X) are equal. In this note it is shown that: In order that X have dimensional type S it is necessary and sufficient that βX have dimensional type S.

5. *The solution of Killing's matrix equations*, by Professor M. E. Gillis, Blue Mountain College.

In Killing's equation $UV\mp(V+1)U$, U and V being matrices, V is given and is assumed to be in the Jordan classical canonical form. Then the most general solution U is exhibited. This form of U has certain submatrices which are zero, while the non-zero submatrices are triangular with all diagonal elements equal. Certain theorems about solutions of the equations were stated.

6. *A condition for the oscillation of the solutions of a system of homogeneous linear differential equations*, by Mr. C. P. Gadsden, Tulane University. (Introduced by Professor W. L. Duren.)

Consider the system (1) $dx/dt=Ax+By$, $dy/dt=Cx+Dy$, where A, B, C, D are continuous functions of t defined on the interval (2) $0\leqq t<\infty$. In the case of constant coefficients A, B, C, D it is known that if $(A-D)^2+4BC<0$, then every solution, $[x(t), y(t)]$, of (1) oscillates; that is, the functions $x(t)$ and $y(t)$ change sign continually as $t\rightarrow\infty$. The question may be asked whether a similar condition can be given for the case in which A, B, C, D are not necessarily constants. The following theorem provides an affirmative answer to this euqestion.

Theorem: Let $x(t)$, $y(t)$ be any non-trivial solution of (1). Suppose that for all t and (2) we have $(A-D)^2+4BC<-m<0$, $|B|<M$ and $|C|<M$, where m and M are constants. Then both $x(t)$ and $y(t)$ change sign on the interval $t_1<t<t_1+\Delta t$, t_1 arbitrary on (2), provided Δt is taken sufficiently large.

7. *Recent problems in the training and preparing of mathematics teachers*, by Professor C. R. Trott, University of Mississippi. (Introduced by Professor T. A. Bickerstaff.)

8. *Twenty-five years with the Louisiana-Mississippi Section of the M. A. of A.*, by Professor H. F. Schroeder, Louisiana Polytechnic Institute.

Professor Z. T. Gallion of Southwestern Louisiana Institute presided at a joint banquet with the Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics that was held at Oak Grove Inn Friday evening. Professor C. V. Newsom of Oberlin College, Oberlin, Ohio, was guest speaker for both organizations at the banquet and again on Saturday morning. His addresses were:

1. *Relationship of the Association and the National Council.*

After reviewing recent developments in the field of mathematical education, Mr. Newsom emphasized that many urgent problems could be solved only by the cooperation of college and secondary teachers. In particular, he suggested a study of the entire mathematics curriculum from the first grade to graduate school in the light of new mathematical knowledge that is available, the needs of modern science, and the teaching problems introduced as a result of mass education.

2. *Mathematics and our culture.*

This paper presented some points of view in regard to the significance of mathematics in man's attempt to comprehend his environment. Mr. Newsom expressed the idea that more of the philosophical contributions of mathematics should be a part of all elementary courses in mathematics.

F. C. GENTRY, *Secretary.*

CALENDAR OF FUTURE MEETINGS

Thirty-second Annual Meeting, Columbus, Ohio, December 31, 1948.

Thirty-first Summer Meeting, Boulder, Colorado, September, 1949.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Bradley University, Peoria, May 13–14, 1949
INDIANA, University of Notre Dame, Spring, 1949
IOWA, Drake University, Des Moines, April 15–16, 1949
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, University of Mississippi, Oxford, Spring, 1949
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May, 1949
NORTHERN CALIFORNIA, San Francisco, January 29, 1949
OHIO, Ohio State University, Columbus, April 2, 1949
OKLAHOMA
PACIFIC NORTHWEST, Oregon State College, Corvallis, Spring, 1949
PHILADELPHIA
ROCKY MOUNTAIN, Colorado School of Mines, Golden, April, 1949
SOUTHEASTERN, University of Alabama, University, March 18–19, 1949
SOUTHERN CALIFORNIA, John Muir Junior College, Pasadena, March 12, 1949
SOUTHWESTERN
TEXAS, Denton, Spring, 1949
UPPER NEW YORK STATE, University of Buffalo, April 30, 1949
WISCONSIN, Lawrence College, Appleton, May 14, 1949

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